SIAST Palliser Campus

Mathematics
MAT 201

Lecture Notes and Examples

Unit 3
Applications of the Definite Integral

by Alan Dill and Robert G. Petry

published by the Department of Mathematics, SIAST Palliser Campus

History

- Original document produced in 2011 entitled "Applications of the Definite Integral" written by Alan Dill and Robert G. Petry. Published by the Department of Mathematics, SIAST Palliser Campus. A transparent copy of this document is available via
  http://www.campioncollege.ca/about-us/faculty-listing/dr-robert-petry

1 Area Between Curves

The area bounded by the vertical lines $x = a$ and $x = b$ and the curves $y_1 = f_1(x)$ on the bottom and $y_2 = f_2(x)$ on top may be written

$$\boxed{A = \int_a^b (y_2 - y_1)\, dx \qquad (1)}$$

To find the area between curves by integration, follow the steps given on page 892 of the text.

It is also a good idea to do a rough estimate of the area by sketching a rectangle or triangle and estimating what fraction the given area is of the area of the rectangle or triangle.

Examples

1. Find the area bounded by the curve $y = 3x^2$, the x-axis and the line $x = 3$.
2. Find the first-quadrant area bounded by the curve $y = x^2 + 2$, $x = 0$ and $y = 4$.
3. Find the area bounded by the curves $y = 9 - x^2$ and $y = 2x + 1$.
4. Find the area bounded by the curves $y = 4x$ and $y = x^3$.
5. Find the area bounded by the curve $y = x^3 - 3$ and the lines $x = 2$, $y = -1$, and $y = 3$.

Additional Examples (Text, p. 900)

- Problem 32
- Problem 36

Reading:
Sec. 31-3, pp. 892-899
Problems:
Ex. 3, p. 899 # 1-23 (odd), 31, 33, 35

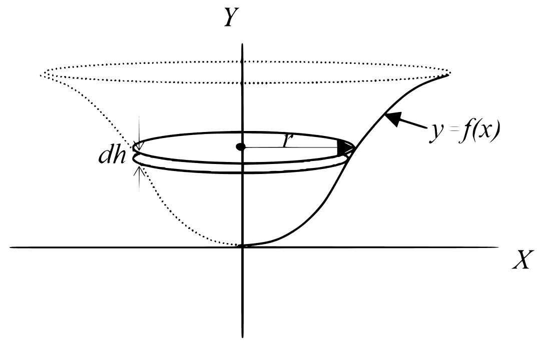
2 Volume of Solids of Revolution

When an area bounded by two or more curves is rotated about a horizontal or vertical axis, it sweeps out a solid of revolution. The volume of such a solid can be found by one or more of the following methods.

2.1 Disk Method

$$dV = \pi r^2 \, dh \quad (2)$$

$$V = \int_a^b \pi r^2 \, dh \quad (3)$$



Note that, for a specific problem, r and h must be expressed in terms of x and y.

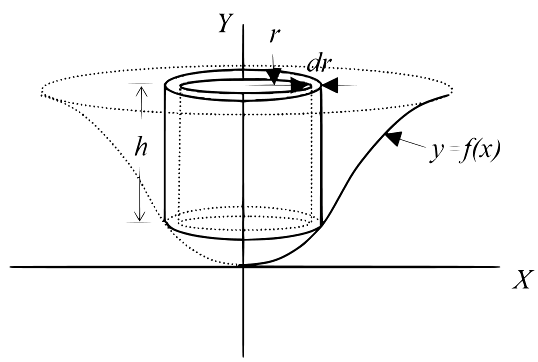
Examples

- Find the volume generated by rotating the first-quadrant area bounded by $y = 4 - x^2$, the x-axis and the y-axis, about the x-axis.
- Show that the volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$.

2.2 Shell Method

$$dV = 2\pi r h \, dr \quad (4)$$

$$V = \int_a^b 2\pi r h \, dr \quad (5)$$



Again, r and h must be expressed in terms of x and y for a particular problem.

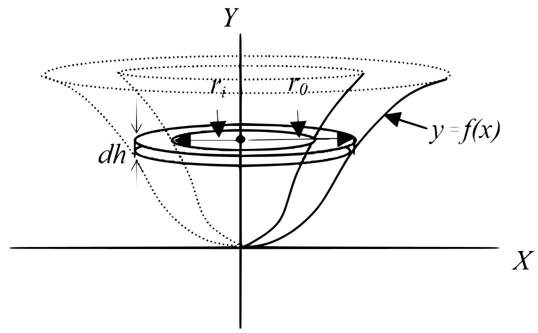
Example

- Find the volume generated by rotating the first area in the previous examples about the y-axis (use the shell method).

For a solid of revolution with a hole, the shell method can often be used. Another option is the ring method, which is similar to the disk method, with the central part of the disk missing.

2.3 Ring Method

$$
\begin{aligned}
dV &= (\pi r_0^2 - \pi r_i^2)\, dh && (6)\\
&= \pi(r_0^2 - r_i^2)\, dh\\
V &= \pi \int_a^b (r_0^2 - r_i^2)\, dh && (7)
\end{aligned}
$$



Example

- Find the volume generated by rotating the area bounded by $y = \sqrt{x}$ and $y = \frac{x}{2}$ about the line $y = 4$

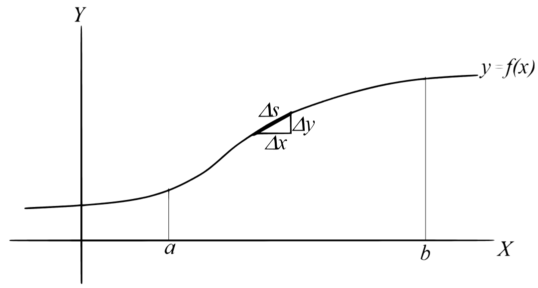
Additional Examples (Text, p. 908)

- Problem 16
- Problem 18
- Problem 20

Reading:
Sec. 31-4, pp. 902-907
Problems:
Ex. 4, p. 908 # 1, 3, 7-21 (odd)

3 Arc Length

In the graph shown, consider a small segment of the curve of length Δs, short enough to be considered a straight line.



By Pythagoras,

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

Dividing by $(\Delta x)^2$ gives

$$\frac{(\Delta s)^2}{(\Delta x)^2} = \frac{(\Delta x)^2}{(\Delta x)^2} + \frac{(\Delta y)^2}{(\Delta x)^2}$$

Taking the square root gives

$$\frac{\Delta s}{\Delta x} = \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}}$$

Taking the limit as Δx approaches zero, we get

$$\lim_{\Delta x \to 0} \frac{\Delta s}{\Delta x} = \lim_{\Delta x \to 0} \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}}$$

This is equivalent to the following equation written in terms of derivatives:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Multiplying by dx gives us the following expression for the differential ds:

$$\boxed{ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx} \qquad (8)$$

To find the total arc length between two x-values, a and b, we integrate this expression between the limits of a and b.

$$\boxed{s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx} \qquad (9)$$

If the equation of the curve is written as $x = f(y)$, the arc length between $y = c$ and $y = d$ can be found from the following formula:

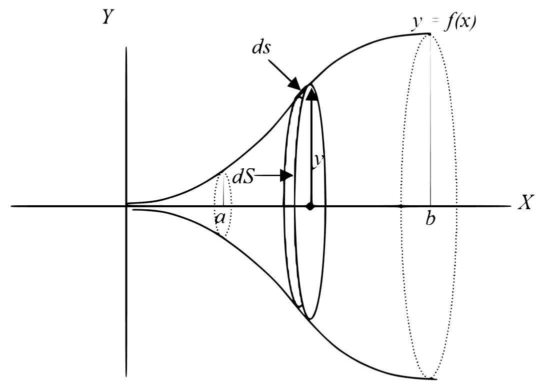
$$\boxed{s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy} \qquad (10)$$

Examples (Text, p. 914)

- Problem 2
- Problem 6
- Problem 12

Reading:
Sec. 32-1, pp. 912-914
Problems:
Ex. 1, p. 914 # 1-15 (odd)

4 Surface Area of Solids of Revolution



If a curve is rotated about an axis, it generates of surface of revolution.

For rotation about the x-axis, the circular element of surface area dS is given by

$$dS = 2\pi y \, ds$$

where ds is an element of arc length on the curve. But we have already calculated ds in Equation (8) in the previous section.

Substituting that expression for ds into the expression for dS, we get

$$\boxed{dS = 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \qquad (11)}$$

We get the total surface area between $x = a$ and $x = b$ by integrating dS between these limits.

$$\boxed{S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \qquad (12)}$$

For rotation about the y-axis, the total surface area is given by

$$\boxed{S = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \qquad (13)}$$

Examples (Text, p. 917)

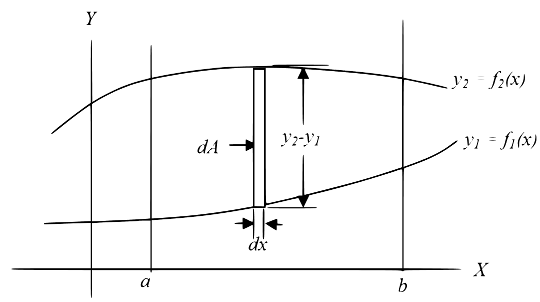
- Problem 2
- Problem 12
- Problem 14
- Problem 16

Reading:
Sec. 32-2, pp. 915-917
Problems:
Ex. 2, p. 917 # 1-15 (odd)

5 Centroids

5.1 Centroid of an Area by Integration

Suppose we want to find the *centroid* $(\bar{x}, \bar{y})$ of the area bounded by the curves $y_1 = f_1(x)$, $y_2 = f_2(x)$, and the lines $x = a$ and $x = b$, as shown.



The moment of the element of area dA about the y-axis is given by[1]

$$\begin{aligned} dM_y &= x\,dA \\ &= x(y_2 - y_1)dx \end{aligned}$$

The moment of the whole bounded area about the y-axis is found by integrating this expression between a and b.

$$M_y = \int_a^b x(y_2 - y_1)\,dx \tag{14}$$

But the moment about the y-axis is just equal to the product of the area and the distance to the centroid, i.e.

$$M_y = A\bar{x}$$

Thus we get

$$A\bar{x} = \int_a^b x(y_2 - y_1)\,dx$$

Solving for $\bar{x}$,we get the following formula for the x-coordinate of the centroid:

$$\bar{x} = \frac{1}{A}\int_a^b x(y_2 - y_1)\,dx \tag{15}$$

The moment of the element of area dA about the x-axis is

$$dM_x = \frac{y_1 + y_2}{2}(y_2 - y_1)\,dx$$

so the moment about the x-axis, M_x, is the integral of dM_x

$$M_x = \int_a^b \frac{y_1 + y_2}{2}(y_2 - y_1)\,dx \tag{16}$$

The y-coordinate of the centroid, $\bar{y}$, is similarly defined through $M_x = A\bar{y}$, so solving for $\bar{y}$ and inserting M_x we have the result

$$\bar{y} = \frac{1}{2A}\int_a^b (y_1 + y_2)(y_2 - y_1)\,dx \tag{17}$$

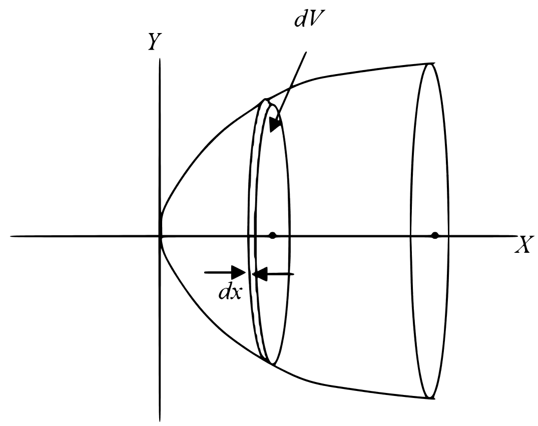
Note: If horizontal elements are used, interchange x and y in formulas (15) and (17).

[1]If one wanted to find the *centre of mass* rather than the centroid, the moment would also be multipled by a density m and rather than using the area A we would use the total mass which would be $M = \int dM = \int m\,dA$. The centroid is effectively just the centre of mass in the special case when the density m is constant.

Examples (Text, p. 925)

- Problem 6
- Problem 12
- Find the centroid of a right triangle with base b and height h.

5.2 Centroid of a Solid of Revolution



For a solid of revolution about the x-axis, the element of volume dV is

$$dV = \pi y^2 \, dx$$

The moment of this element of volume about the y-axis is

$$dM_y = x(\pi y^2 \, dx)$$

The moment of the entire volume about the y-axis is found by integrating from $x = a$ to $x = b$.

$$\boxed{M_y = \pi \int_a^b xy^2 \, dx \qquad (18)}$$

Since $M_y = V\bar{x}$, we get the following for the x-coordinate of the centroid:

$$\boxed{\bar{x} = \frac{\pi}{V} \int_a^b xy^2 \, dx \qquad (19)}$$

Because the x-axis is the axis of symmetry, the y-coordinate of the centroid is zero.

For a solid of revolution about the y-axis, the x-coordinate of the centroid is zero, and the y-coordinate is given by

$$\boxed{\bar{y} = \frac{\pi}{V} \int_c^d yx^2 \, dy \qquad (20)}$$

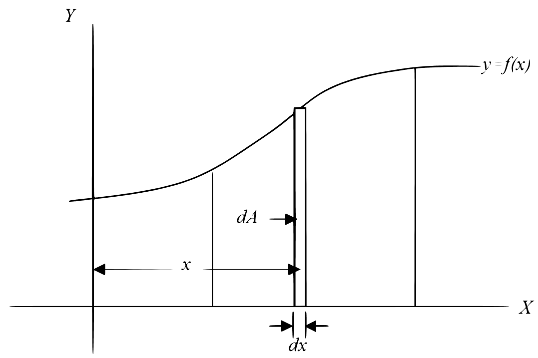
Example (Text, p. 926)

- Problem 16

Reading:
Sec. 32-3, pp. 918-925
Problems:
Ex. 3, p. 925 # 5-21 (odd)

6 Moment of Inertia

6.1 Moment of Inertia of an Area



The *moment of inertia* (or *second moment*), dI, of an element of area dA about a line at a distance r from the element of area is defined by

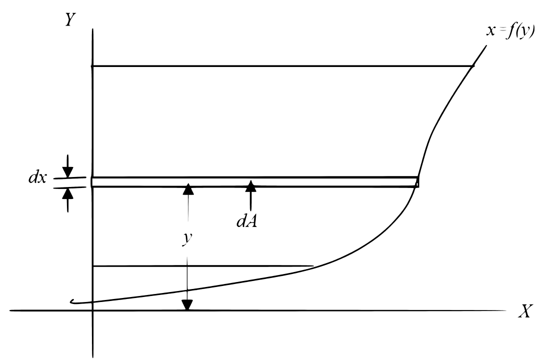
$$dI = r^2\, dA$$

For the element of area shown on the right, the moment of inertia about the y-axis is given by

$$dI_y = x^2\, dA = x^2 y\, dx$$

The moment of inertia of the whole area between $x = a$ and $x = b$ about the y-axis is obtained by integration[2]

$$\boxed{I_y = \int_a^b x^2 y\; dx \qquad (21)}$$

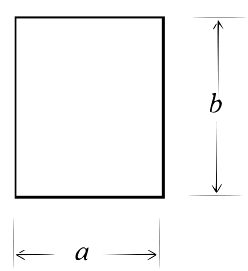


For the area shown on the right, if we take horizontal elements, the moment of inertia of the element dA about the x-axis is given by

$$dI_x = y^2\, dA = y^2 x\, dy$$

The moment of inertia of the whole area between $y = c$ and $y = d$ is then

$$\boxed{I_x = \int_c^d y^2 x\; dy \qquad (22)}$$



Example

- Show that the moment of inertia of the rectangle shown about an x-axis through its base is $\frac{b^3 a}{3}$.

[2]Technically the moment of inertia requires a factor of m, the area mass density. We will consider only areas of constant density so any such factor will factor out of the integrals. When we consider the moment of inertia of a volume we will include the volume mass density explicitly. This arbitrariness is for consistency with the textbook.

Using the result of the previous example, we can find a formula for the moment of inertia of the area at the top of the page about the x-axis (by summing/integrating vertical elements of height y and width dx).

$$dI_x = \frac{y^3\,dx}{3}$$

$$\boxed{I_x = \frac{1}{3}\int_a^b y^3\;dx} \tag{23}$$

If the area whose moment of inertia has been found were stretched into a thin strip and placed at a distance r from the axis such that the moment of inertia of the strip is the same as that of the original area, the distance r is called the *radius of gyration*.[3] The radius of gyration is found from the equation

$$I = Ar^2$$

Solving for r, we get

$$\boxed{r = \sqrt{\frac{I}{A}}} \tag{24}$$

Examples (Text, p. 939)

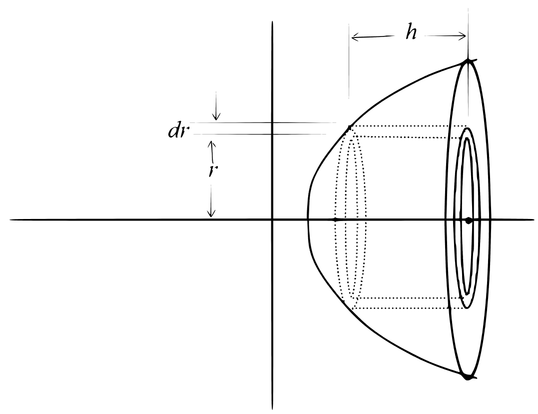
- Problem 2
- Problem 4
- Problem 6

[3] The radius of gyration may be thought of as the rotational equivalent of the centroid (centre of mass). It gives the point at which all the mass of the system could be placed to have an equivalent mechanical system.

6.2 Polar Moment of Inertia

The *polar moment of inertia* (also called the *mass moment of inertia*) is a quantity that is required when studying the rotation of solid rigid bodies, such as flywheels.

6.2.1 Polar Moment of Inertia by the Shell Method



Using the shell method, the volume of the cylindrical shell shown is given by

$$dV = 2\pi rh\,dr$$

If m is the density (mass per unit volume), the mass of the cylindrical shell is then

$$dM = 2\pi mrh\,dr$$

The polar moment of inertia of the cylindrical shell about its axis of rotation is defined as the product of its mass and the square of its distance r from the axis.

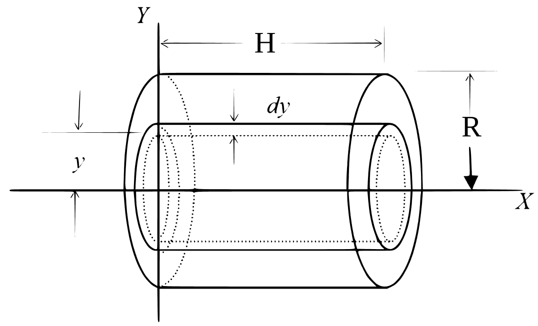
$$dI = r^2\,dM = 2\pi mr^3h\,dr$$

The polar moment of inertia of the entire solid is then the integral of this quantity over the whole range of r-values.

$$\boxed{I = 2\pi m \int r^3 h\; dr \qquad (25)}$$

It should be noted that, in a particular problem, r and h must be expressed in terms of x and y.

6.2.2 Polar Moment of Inertia of a Solid Cylinder



As an example of applying the shell method, suppose we have a solid cylinder of radius R and length H.

If we orient the cylinder along the x-axis as shown, then the shell radius of Equation (25) is $r = y$ (so $dr = dy$), and we integrate from shells of radius 0 to R. The shell length h is constant in this example, $h = H$. Making these substitutions in Equation (25) gives

$$I = 2\pi m \int_0^R y^3 H\, dy = 2\pi m H \int_0^R y^3\, dy = 2\pi m H \left[\frac{y^4}{4}\right]_0^R = \frac{\pi m H R^4}{2}$$

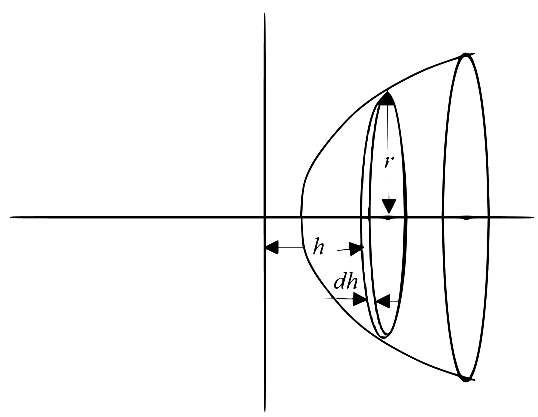
We can also write this in terms of the volume of the cylinder, given by

$$V = \pi R^2 H$$

Using this gives the following formula for polar moment of inertia of a cylinder:

$$I = \frac{\pi m H R^4}{2} = \frac{1}{2} m V R^2$$

6.2.3 Polar Moment of Inertia Using the Disk Method



We can use the result of the previous equation to derive an alternative formula for finding polar moment of inertia called the disk method.

In the diagram shown, we have a disk-shaped element of volume which is effectively a thin cylinder of radius $R = r$ and height $H = dh$. Inserting these into the equation for the polar moment of inertia of a cylinder gives the disk's moment of inertia dI

$$dI = \frac{m\pi r^4\, dh}{2}$$

Integrating gives the total moment of inertia of the volume, $I = \int dI$, to be

$$\boxed{I = \frac{m\pi}{2} \int_a^b r^4\, dh \qquad (26)}$$

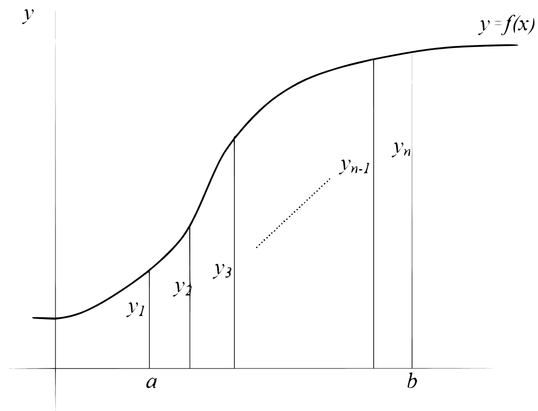
Note that a radius of gyration for solids may be defined as for areas, namely $r = \sqrt{\frac{I}{M}}$, where now the total mass $M = mV$, where V is the volume of constant density m.

Example (Text, p. 940)

- Problem 10

Reading:
Sec. 32-6, pp. 933-939
Problems:
Ex. 6, p. 939 # 1-11 (odd)

7 Average Value of a Function



Suppose we want to find the average value of a function $y = f(x)$ between two x-values, a and b. We can start by dividing the interval between a and b into n equal intervals of width $\Delta x = \frac{b-a}{n}$, then summing the y-values and dividing by n. The average y-value would then by approximated by

$$y_{avg} \approx \frac{1}{n}\sum_{i=1}^{n} y_i$$

Solving the equation $\Delta x = \frac{b-a}{n}$ for $\frac{1}{n}$, we get

$$\frac{1}{n} = \frac{\Delta x}{b-a}$$

We can substitute this for $\frac{1}{n}$ in the previous equation to get

$$y_{avg} \approx \frac{\Delta x}{b-a}\sum_{i=1}^{n} y_i$$

Since Δx is a constant, we can move it inside the summation sign and get

$$y_{avg} \approx \frac{1}{b-a}\sum_{i=1}^{n} y_i \Delta x$$

Of course, the larger the number of y-values, the more accurate the average value of y will be. To get an exact value for the average, we take the limit as n approaches infinity.

$$\begin{aligned} y_{avg} &= \lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} y_i \, \Delta x \\ &= \frac{1}{b-a}\left(\lim_{n\to\infty}\sum_{i=1}^{n} y_i \, \Delta x\right) \\ &= \frac{1}{b-a}\left(\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i) \, \Delta x\right) \end{aligned}$$

The quantity in parentheses is just the definite integral of $f(x)$ from $x = a$ to $x = b$, so we can write the following formula for the average value of $y = f(x)$ from a to b:

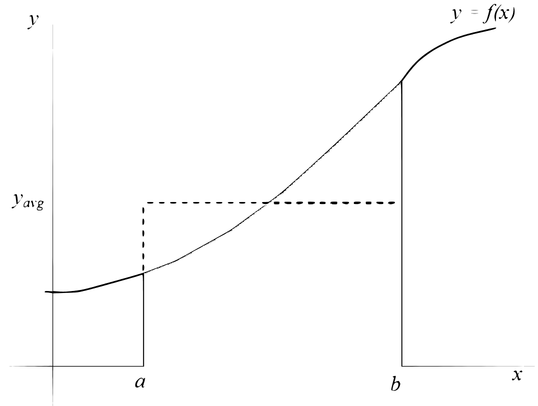
$$\boxed{y_{avg} = \frac{1}{b-a}\int_a^b f(x)\, dx \qquad (27)}$$

We can give a geometric interpretation of y_{avg}. Since the definite integral is equal to the area A_{ab} under the curve between a and b, we can write

$$y_{avg} = \frac{1}{b-a} A_{ab}$$

Solving for A_{ab}, we get

$$A_{ab} = y_{avg}(b-a)$$



Since $b-a$ is the width of the rectangle shown, whose height is y_{avg}, we see that y_{avg} is just the height of a rectangle that is equal in area to A_{ab} under the curve from $x = a$ to b.

Examples (Give answers to 3 significant digits)

1. Find the average value of the function $y = \sqrt{2x+5}$ between $x = 0$ and $x = 2$.
2. If an object is falling freely in the earth's gravitational field, its velocity as a function of time is $v = -9.8t$. Find its average velocity over the first 10.0 seconds.

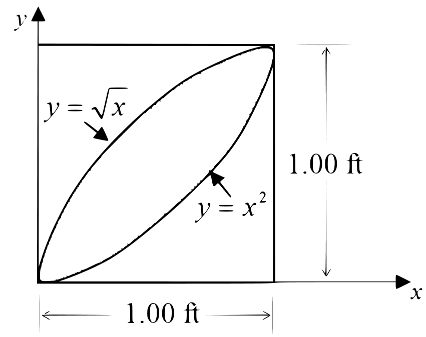
Reading:
Sec. 34-3, pp. 975-976
Problems:
Ex. 3, p. 976 # 1, 3

8 Review Problems: Applications of the Definite Integral

Questions

Round all numerical answers to 3 significant digits where applicable.

1. A tile manufacturing company plans to produce enamel tiles with the design shown. Find the area of the inner lenticular-shaped tile and the area of the surrounding tile that will be required.



2. A hemispherical water tank has a radius of 4.00 m. The water is 1.50 m deep at the centre. How much water is in the tank?
3. Find the length of the curve $y = \sqrt{8x^3}$ from $x = 0$ to $x = 2$. Do a rough estimate, and then find the length by integration. Give the answer in exact form and in approximate decimal form.
4. Find the area of the surface generated by rotating the curve $y = 2\sqrt{x}$ from $x = 0$ to 8 about the x-axis. Give the answer in exact form and in approximate decimal form.
5. Locate the centroid of the plane region bounded by the curves $y = 2 - x$ and $y = x^2$.
6. Find the moment of inertia, about the y-axis, of the plane region bounded by the curve $y = x^2$, the x-axis, and the line $x = 2$.
7. Find the average value of the function $f(t) = 4.9t^2 - 4.2t + 9.0$ over the interval $t = 0$ to $t = 1.6$.

Answers

1. inner: 0.333 ft^2, outer: 0.667 ft^2
2. 24.7 m^3
3. $\dfrac{1}{27}(37^{\frac{3}{2}} - 1) \approx 8.30$ units
4. $\dfrac{208\pi}{3} \approx 218\ \text{u}^2$
5. $\left(-\frac{1}{2}, \frac{8}{5}\right)$
6. 6.4
7. 9.82

9 Additional Review Problems: Applications of the Definite Integral

Questions

- Chapter 31 Review Problems (p. 909) # 1, 2, 5, 7, 13, 15 (Note question 7 requires use of integration Rule 52 of Appendix C or numerical integration.)
- Chapter 32 Review Problems (p. 940) # 1, 3, 6, 8, 10, 12, 16, 17, 18

Answers to Even-Numbered Review Problems

- Chapter 31

  2. $\frac{8}{3}$

- Chapter 32

  6. $\frac{3Mr^2}{10}$

  8. $\frac{8\pi m r^5}{15}$

  10. $(0.800, 1.00)$

  12. 13.75 (Requires Rule 66 of Appendix C or numerical integration.)

  16. 102.6 m (Requires Rule 66 of Appendix C or numerical integration.)

  18. 2p/3

10 Formulas

Area between Curves

Area bounded by the vertical lines $x = a$ and $x = b$ and the curves $y_1 = f_1(x)$ on the bottom and $y_2 = f_2(x)$ on top:

$$A = \int_a^b (y_2 - y_1)\, dx \tag{1}$$

Volumes of Solids of Revolution

Disk Method:

$$\begin{aligned} dV &= \pi r^2\, dh && (2)\\ V &= \int_a^b \pi r^2\, dh && (3) \end{aligned}$$

Shell Method:

$$\begin{aligned} dV &= 2\pi r h\, dr && (4)\\ V &= \int_a^b 2\pi r h\, dr && (5) \end{aligned}$$

Ring Method:

$$\begin{aligned} dV &= (\pi r_0^2 - \pi r_i^2)\, dh && (6)\\ V &= \pi \int_a^b (r_0^2 - r_i^2)\, dh && (7) \end{aligned}$$

Arc Length

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{8}$$

Arc length between $x = a$ and $x = b$ of $y = f(x)$:

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{9}$$

Arc length between $y = c$ and $y = d$ of $x = f(y)$:

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \tag{10}$$

Surface Area of Solids of Revolution

$$dS = 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{11}$$

Surface area between $x = a$ and $x = b$ of $y = f(x)$ rotated about the $x - axis$:

$$S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{12}$$

Surface area between $x = a$ and $x = b$ of $y = f(x)$ rotated about the $y - axis$:

$$S = 2\pi \int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{13}$$

Centroid of an Area

In the following the area is bounded by the vertical lines $x = a$ and $x = b$ and the curves $y_1 = f_1(x)$ on the bottom and $y_2 = f_2(x)$ on top. The area of the region is A. $(\bar{x}, \bar{y})$ is the centroid and M_x and M_y are the moments about the $x-$ and $y-$axes respectively.

$$M_y = \int_a^b x(y_2 - y_1)\,dx \tag{14}$$

$$\bar{x} = \frac{1}{A}\int_a^b x(y_2 - y_1)\,dx \tag{15}$$

$$M_x = \int_a^b \frac{y_1 + y_2}{2}(y_2 - y_1)\,dx \tag{16}$$

$$\bar{y} = \frac{1}{2A}\int_a^b (y_1 + y_2)(y_2 - y_1)\,dx \tag{17}$$

Centroid of a Solid of Revolution

The following two equations apply for a solid of revolution about the $x-$axis. Here M_y is the moment about the $x-$axis and $\bar{x}$ is the the $x-$coordinate of the centroid.

$$M_y = \pi\int_a^b xy^2\,dx \tag{18}$$

$$\bar{x} = \frac{\pi}{V}\int_a^b xy^2\,dx \tag{19}$$

The equation for the $y-$coordinate of the centroid for a solid of revolution about the $y-$axis is:

$$\bar{y} = \frac{\pi}{V}\int_c^d yx^2\,dy \tag{20}$$

Moment of Inertia of an Area

The following gives the moment of inertia of the area bounded by $y = f(x)$ and the $x-$axis, and the vertical lines $x = a$ and $x = b$ about the $y-$axis:

$$\boxed{I_y = \int_a^b x^2 y \, dx} \tag{21}$$

The following gives the moment of inertia of the area bounded by $x = f(y)$ and the $y-$axis, and the horizontal lines $y = c$ and $y = d$ about the $x-$axis:

$$\boxed{I_x = \int_c^d y^2 x \, dy} \tag{22}$$

The following gives the moment of inertia of the area bounded by $y = f(x)$ and the $x-$axis, and the vertical lines $x = a$ and $x = b$ about the $x-$axis:

$$\boxed{I_x = \frac{1}{3}\int_a^b y^3 \, dx} \tag{23}$$

The **Radius of gyration** of an area about an axis is given by the following where I is the moment of inertia and A is the area.

$$\boxed{r = \sqrt{\frac{I}{A}}} \tag{24}$$

Moment of Inertia of a Solid

Using the **Shell Method** where r is the radius and h is the height of the cylindrical shell about the axis of rotation, the moment of inertia of the solid of density m is:

$$\boxed{I = 2\pi m \int r^3 h \, dr} \tag{25}$$

Using the **Disk Method** where the disk is of radius r the moment of inertia is:

$$\boxed{I = \frac{m\pi}{2}\int_a^b r^4 \, dh} \tag{26}$$

Average Value of a Function

$$\boxed{y_{avg} = \frac{1}{b-a}\int_a^b f(x) \, dx} \tag{27}$$

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