SIAST Palliser Campus

Mathematics
MAT 201

Lecture Notes and Examples

Unit 1
Introduction to Integration

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1 A Simple Painting Problem

In this section I will try to motivate what the main component of this course is all about, integration. This is intended as a sketchy overview - the details will be fleshed out in the subsequent material. You may want to read this again when you have finished the course to see that it makes sense!

Suppose I was deciding to make a "feature wall" in a room in my house by painting the wall a unique colour. Since I would only be buying paint for the one wall it would make sense to buy only exactly the amount of paint I needed. This amount would be determined by the area of my wall. For a rectangular wall this area is

$$A = HB$$

where H is the height of the wall and B is its base length. See Figure 1. So if the wall has height $H = 4$ m and base $B = 3$ m we'd have an area $A = 12\text{ m}^2$ to cover. If a litre of paint covers 15 m^2, I would need $\frac{12\text{ m}^2}{15\text{ m}^2/\text{litre}} = .8$ litres of paint, so a single 1-litre can would be sufficient.

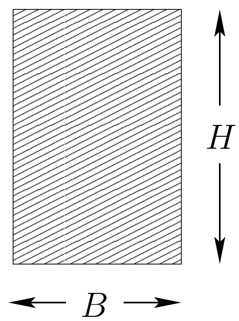

Figure 1: A rectangular wall

Now suppose my feature wall was not a rectangle but looked like Figure 2 on the next page. Here the height H is not fixed, but varies across the wall. How would we calculate the area[1] now? One way would be to try to estimate the area by covering it with shapes for which we know how to calculate the area such as rectangles and triangles. In Figure 3 on the following page we see one way to estimate the area by adding a rectangle as before and two triangles for which we could also calculate the areas ($A_\Delta = \frac{(\text{base})\cdot(\text{altitude})}{2}$). A second way to estimate the area is shown where we add several thin rectangles whose height varies with the wall height.

These approaches are only approximate however and if we were using expensive "paint" such as gilding a wall of a temple with gold, or if we were constructing a residential subdivision for which we had 20 identical feature walls to paint, the exact area of this wall would be desirable. If we order too much paint we are wasting money but if we don't order enough we would lose money on our painters as they wait for more paint to be delivered! This course will allow us to solve such problems like calculating such an irregular area (and many more things) exactly or at least to excellent approximation.

The first thing we need to do is represent the problem mathematically. For instance, in the case of our irregular wall let's draw an x-axis along the base of the wall and represent the varying height H with a *function* $f(x)$ as shown in Figure 4 on page 3 so that we know the height at every point x along the wall. If we let the wall start at $x = a = 1$ m and end at $x = b = 4$ m (so the base $B = b - a = 3$ m), then the function needs to be properly defined over the domain of values $[1, 4]$ (outside of this it can be whatever we want).

[1]One might question that such an irregular shape even has a clearly defined area. An intuitive way to define the area would be to paint the wall and see how much paint we used. If it took 1.2 litres to paint it, then its area would be $(1.2\text{ litres}) \cdot (15\text{ m}^2/\text{litre}) = 18\text{ m}^2$.

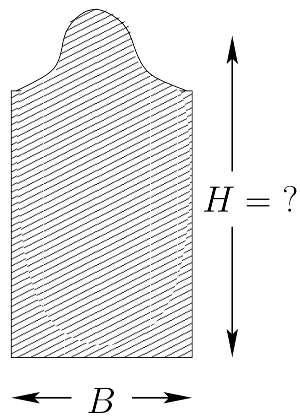

Figure 2: An irregular-shaped wall

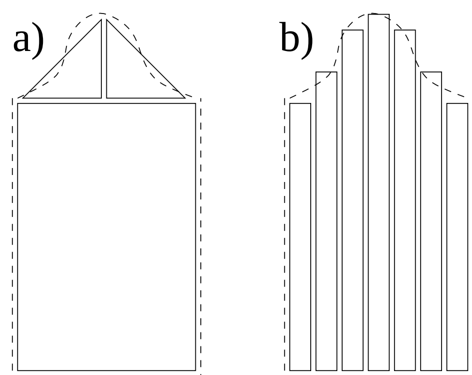

Figure 3: Two ways to estimate the irregular area with a) one rectangle and two triangles and b) seven thin rectangles.

Now what's the area? We take our direction from case b) in Figure 3 and add up thin rectangles of *varying* height $H = f(x)$ and *equal* base width $B = dx$ (see Figure 4 on the next page), and we write the area as[2]

$$\int_a^b f(x)\,dx = \underbrace{\int_a^b}_{\substack{\text{Sum over rectangles}\\ \text{from } a \text{ to } b}} \underbrace{\underbrace{f(x)}_{H} \cdot \underbrace{dx}_{B}}_{\substack{\text{Area of rectangle}\\ \text{at } x}}$$

Here the $\int$ sign is an S which reminds us we want to *sum* (or add) all the rectangles. We need to put the limits a and b on the sum so that we know the domain over which we want to add the rectangles. This expression is called a *definite integral* and it becomes mathematically precise if we make the base width dx "infinitely thin". We'll see how to do that in this course.

Now taking limits as things get infinitely small is what calculus is about. You've seen how to take a *derivative* of a function like our $f(x)$. Geometrically the derivative represents the slope of a tangent to the curve $y = f(x)$ at a point x. We wrote that derivative as $f'(x)$ or $\frac{dy}{dx}$. Now it turns out that evaluating the definite integral above (finding our area) can be done exactly if we calculate something

[2] Here dx is taken to be a single entity. The d is like the Δ in an expression like Δx. We write a d to distinguish the infinitessimal nature of the width.

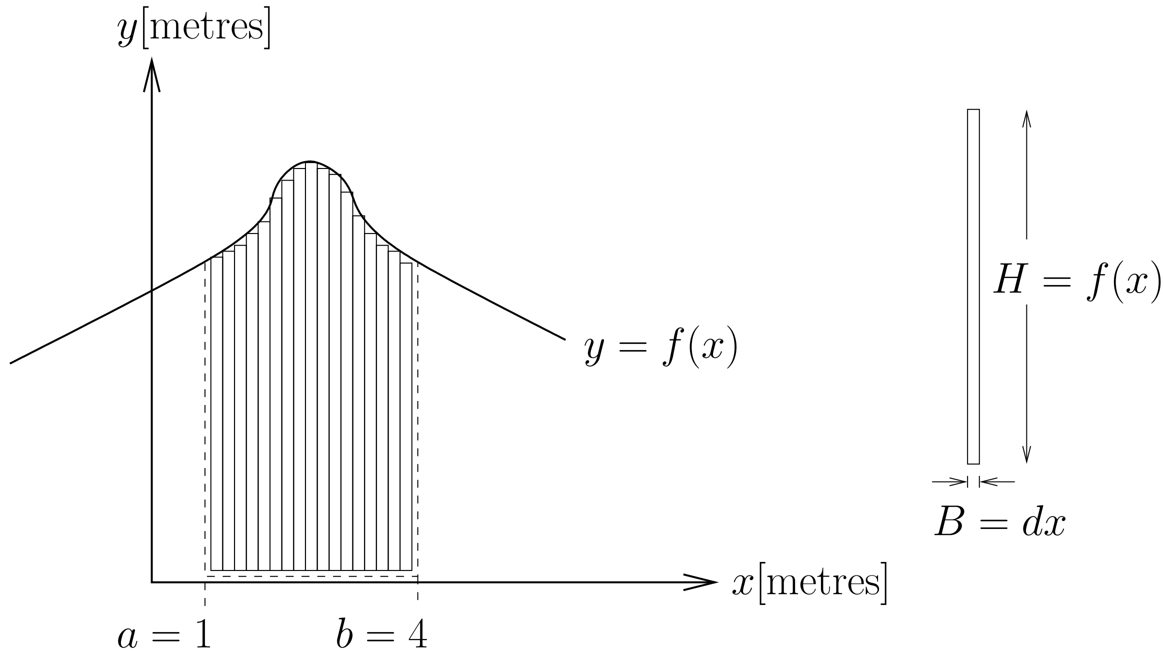

Figure 4: We can represent the varying wall height H with a function $f(x)$. The exact wall area will be the sum of rectangles of height $f(x)$ and infinitessimal width dx between $x = a$ and $x = b$.

called the *antiderivative* of $f(x)$ which, like the derivative f' of f, is itself a function, now called $F(x)$. Given the name "antiderivative" you could guess correctly that finding this F is related to how you take derivatives. We'll see how to do this. Once you know how to find $F(x)$ given a function f, you can get the definite integral (our area) easily by evaluating the expression

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

We can find the area just by subtracting two numbers, F evaluated at $x = b$ minus F evaluated at $x = a$! Obviously figuring out how to get these antiderivatives will be pretty useful. The last equation is known as the *fundamental theorem of calculus.*

Now we need a symbol for this antiderivative F to make it clear which function f it is associated with. Just like we write the derivative of f by f', we similarly write our antiderivative with a new symbol involving f,

$$F = \int f(x)\,dx$$

The symbol on the right hand side is chosen because of the relation of F to the definite integral above. In fact we call F not only an antiderivative but also an *indefinite integral* for this reason. (Indefinite since there aren't any a's or b's.)

Sadly we can't always find an antiderivative function $F(x)$ for every function $f(x)$ and then we have to use some *numerical methods* (i.e. adding up small areas by hand) to estimate our definite integrals. We'll see how to do that too.

So in summary in our course we will learn to do the following:

- Figure out how to calculate the antiderivative/indefinite integral $\int f(x)\,dx$ for a given function $f(x)$.

- Calculate the definite integral $\int_a^b f(x)\,dx$ using antiderivatives when they exist and numerically otherwise.
- Apply these integrals to other problems besides area.

Finally we'll introduce a different mathematical concept, *statistics*, in a final unit.

If you feel overwhelmed, this is only a road map. Hopefully at least we're now motivated to take the journey.

2 The Indefinite Integral

If we are given a function, we can differentiate the function, i.e. find its derivative. The derivative of a function is itself a function.

For example, if the given function is x^4, its derivative is $4x^3$.

Just as we can reverse a process such as squaring (by taking the square root), we can also reverse the process of differentiation.

This means that if we are given a function, we can find another function that has the given function as its derivative.

We see from the above example that the derivative of x^4 is $4x^3$. Thus, if we are asked to find a function that has $4x^3$ as its derivative, we could give x^4 as an answer. In this case we say that x^4 is an *antiderivative* of $4x^3$.

It should be noted that x^4 is not the only possible antiderivative of $4x^3$. Since the derivative of a constant is zero, adding *any constant* to x^4 will give us another function whose derivative is $4x^3$. For example, $x^4 + 5$, $x^4 - 7$ and $x^4 + \pi$ are all antiderivatives of $4x^3$.

To show that x^4+*any constant* can be an antiderivative of $4x^3$, we write the general form of the antiderivative (also called the *indefinite integral*) of $4x^3$ as $x^4 + C$. Here C is called the *constant of integration*, and can have any value. We will see later on that, if additional information about the given function is available (such as its value for a given value of x), we can assign a specific value to the constant C.

We use a special symbol called an *integral sign* ($\int$) to denote the indefinite integral of a function. If $f(x)$ is a function whose indefinite integral is to be found, we write

$$\int f(x)\,dx = F(x) + C$$

where $f(x) = F'(x)$ and C is the constant of integration.[3]

What this means is, the function whose derivative is $f(x)$ (or equivalently, the function whose differential dF is $f(x)dx$) is $F(x) + C$.

The process of finding the indefinite integral of a function is called *integration*.

We can derive some basic rules for integration simply by reversing some of the basic rules for differentiation. These rules are given in Figure 5 on the following page.

[3]The student will note that we neglected this constant in our casual discussion in the introductory section.

$$\int f(x)\,dx = F(x) + C \qquad (\ \textit{where}\ f(x) = F'(x)\ ) \tag{1}$$

$$\int du = u + C \tag{2}$$

$$\int a\ f(x)\,dx = a\int f(x)\,dx = a\ F(x) + C \tag{3}$$

$$\int [f(x) + g(x) + \ldots]\ dx = \int f(x)\,dx + \int g(x)\,dx + \ldots \tag{4}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1) \tag{5}$$

Figure 5: Rules for Integration

Examples

Perform the following integrations:

1. $\displaystyle\int 3x\ dx$
2. $\displaystyle\int x^4\ dx$
3. $\displaystyle\int 2x^{\frac{3}{2}}\ dx$
4. $\displaystyle\int 6\sqrt[3]{x}\ dx$
5. $\displaystyle\int \frac{4}{\sqrt{x}}\ dx$
6. $\displaystyle\int \left(\frac{1}{x^3} + \frac{1}{2}\right)\ dx$
7. $\displaystyle\int (x\sqrt{x} - 5x^2)\ dx$
8. $\displaystyle\int (1 + 2x)^2\ dx$

If we are given the derivative of a function in the form of an equation (this is a simple form of *differential equation*), we can find the original function (i.e. we can solve the differential equation) by integration, as in the example on the following page.

Additional Example (Text, p. 864)

- Problem 34

Reading:
Sec. 30-1, pp. 857-863
Problems:
Ex. 1, p. 863 # 1-38 (all)

3 More Rules of Integration

3.1 Integral of a Power of a Function

Suppose we are given a function $y = \frac{u^{n+1}}{n+1}$, where u is a function of x.

We can take the derivative of both sides using the rule for the derivative of a constant times a power of a function.

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{n+1}\,(n+1)\,u^n\,\frac{du}{dx} \\ \frac{dy}{dx} &= u^n\,\frac{du}{dx} \end{aligned}$$

We can put this into differential form by multiplying both sides by dx, then integrate both sides.

$$\begin{aligned} dy &= u^n\,du \\ \int dy &= \int u^n\,du \\ y + C &= \int u^n\,du \end{aligned}$$

But $y = \frac{u^{n+1}}{n+1}$. Substituting this expression for y gives us the following formula for the integral of a power of a function:

$$\boxed{\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1) \qquad (6)}$$

Note that the left hand side of this formula contains du, which is the differential of the function u. This differential is given by

$$du = \frac{du}{dx}dx$$

The quantity under the integral sign in Equation (6) **must** contain the differential du in addition to u^n. If it does not contain du, but contains a constant multiple of du, we can multiply by a constant to obtain the required expression for du, *provided we compensate by dividing by the same constant outside the integral sign.* The objective is to have both the function u and the differential du present following the integral sign. This must be done before the above integration formula can be used.

Examples

Integrate the following:

1. $\int (5x+1)^2 \, dx$
2. $\int (x^2+1)^3 x \, dx$
3. $\int (x^3-2)(5x^2) \, dx$
4. $\int \frac{dx}{\sqrt{3x+1}}$
5. $\int \sqrt[3]{4-3x} \, dx$
6. $\int \frac{x+1}{\sqrt{x^2+2x}} \, dx$

   Find the function whose derivative is given (i.e. solve each differential equation):

7. $\frac{dy}{dx} = \sqrt[4]{1-2x}$
8. $\frac{dy}{dx} = x(x^2-5)^2$

It should be noted that the solution of a differential equation, such as the last two examples, consists not of a number, but a function that makes the differential equation true.

3.2 Integral of a Reciprocal Function

You may have wondered why the rule (6) cannot be applied when $n = -1$. In that case the right hand side of the equation would then have zero in the denominator and we know dividing by zero is meaningless. However we can still integrate u^{-1}. If u is a function of x, the indefinite integral of $\frac{du}{u}$ is given by

$$\boxed{\int \frac{du}{u} = \ln|u| + C} \tag{7}$$

where $\ln|u|$ is the natural logarithm of the absolute value of u.

Note that, just as in the case of the power formula, the differential du *must be present* before the formula can be applied.

Examples

1. $\displaystyle\int \frac{dx}{1-x}$
2. $\displaystyle\int \frac{x\,dx}{3x^2+1}$
3. $\displaystyle\int \frac{(x-1)\,dx}{x^2-2x}$
4. $\displaystyle\int \frac{x^2\,dx}{1-x^3}$
5. $\displaystyle\int \frac{x^2+1}{x^3+3x}\,dx$

3.3 Using a Table of Integrals or a Computer

The Table of Integrals (Textbook Appendix C, p. A-44) can be used to find many more integrals.[4] The trick is to choose a given function u of x which makes your indefinite integral match an expression in the table. (The differential du must appear correctly as well up to a multiplicative constant which can be factored out of the integral as usual using Formula (3).) In this course students are not expected to solve these other possible indefinite integrals, but if they would like may try some by looking at Examples 29-43, on page 869 of the text.[5]

In practice one will often use computer programs to *symbolically* evaluate an integral in which you are interested. (i.e. to find an indefinite integral by searching for patterns as being done by hand here.) Figure 6 on the next page shows the computer program *wxMaxima* computing an indefinite integral. wxMaxima is a free (see `wxmaxima.sourceforge.net`) cross-platform (i.e. runs on Linux, Windows, or Mac) mathematics software package. Proprietary packages that perform similar calculations also exist (*Maple*, *Mathematica*, etc.). Note, however, that computers can only calculate what is possible. Unlike taking derivatives of common functions, not all such functions have an antiderivative which can be written in a closed form in terms of familiar functions.

[4] Note that there is nothing magic about a table of integrals, they arise simply by considering the forms that derivatives of known functions can take.

[5] Being able to only integrate x^n is still quite powerful. Formulas (3) and (4) then allow us to calculate the indefinite integral of any *polynomial* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. The *Weierstrass approximation theorem* states that every continuous function defined on an interval $[a,b]$ can be approximated as closely as desired by such a polynomial function. As such our minimal integration tools can, in theory, find an approximate indefinite integral of arbitrary accuracy to any function by first approximating it with a polynomial.

Reading:
Sec. 30-2, pp. 864-868
Problems:
Ex. 2, p. 869 # 1-28 (all)

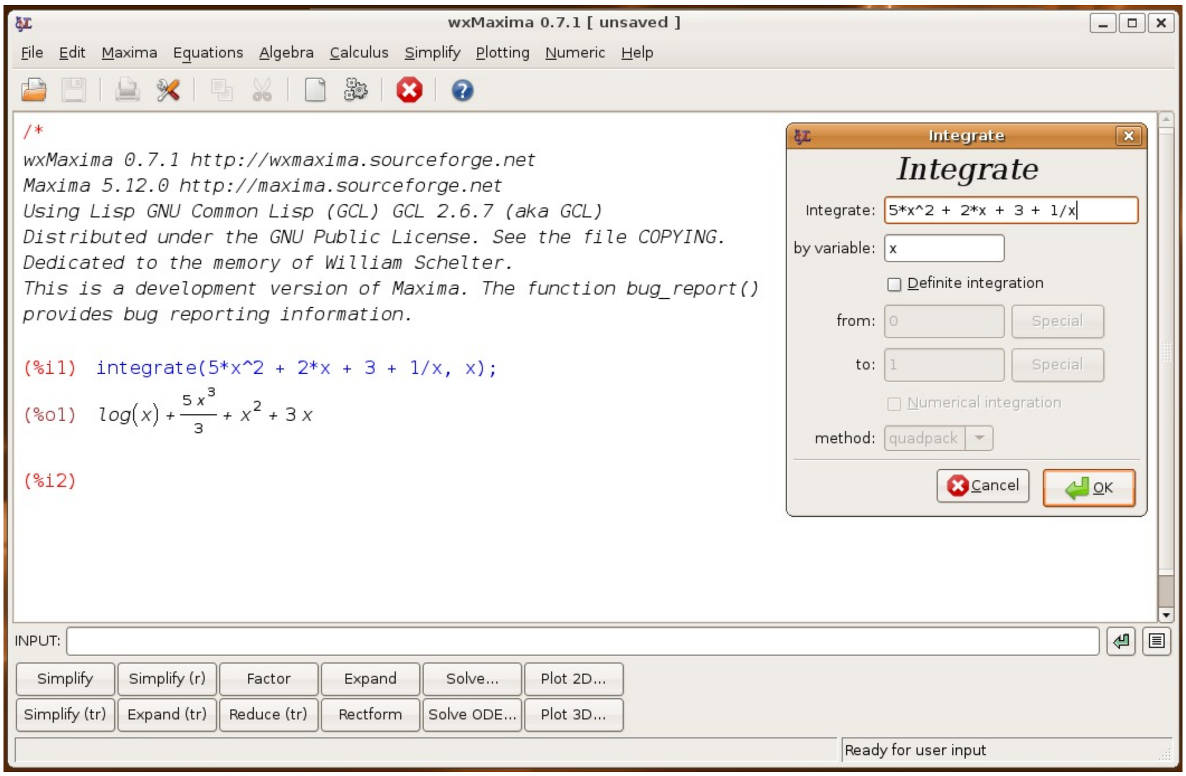

Figure 6: Screenshot of the computer program *wxMaxima* computing the indefinite integral $\int (5x^2 + 2x + 3 + \frac{1}{x})\, dx$.

4 The Constant of Integration

When we integrate, we are finding a function whose derivative (or differential) is given. But since there is always a constant of integration, C, the resulting function is not unique, but actually represents a whole family of functions, each of which has a different value of C, but is otherwise the same (see Fig. 30-2 on page 871 of the text).

In order to "pin down" the value of C and get a unique function, we must be given some additional information, such as the coordinates of a point through which the function passes. We can use this information to evaluate the constant of integration.

If we are given the second derivative of a function, we can get the original function by integrating twice in succession. For each step, we get a constant of integration. To evaluate both constants, we need two additional pieces of information (for example, the coordinates of a point and the slope of the tangent line at that point).

Examples

1. Find the equation of the curve whose tangent slope at x is $\sqrt{6x-3}$ and which passes through $(2, -1)$.
2. The second derivative of a function is $12x^2$. The curve passes through $(1, -1)$ and the tangent slope at that point is 9. Find the function.

Additional Examples (Text, p. 873)

- Problem 8
- Problem 10
- Problem 12

Reading:
Sec. 30-3, pp. 870-872
Problems:
Ex. 3, p. 873 # 5-11 (odd)

5 Review Problems: Introduction to Integration

Questions:

1. Integrate the following:
   (a) $\displaystyle\int (2 - 4x^2)\, dx$
   (b) $\displaystyle\int x^2(x^3 - 3)^4\, dx$
   (c) $\displaystyle\int \frac{t+2}{t}\, dt$
2. Chapter 30 Review Problems, p. 883 # 1-9 (all), 24

Answers:

1. (a) $2x - \frac{4}{3}x^3 + C$
   (b) $\frac{1}{15}(x^3 - 3)^5 + C$
   (c) $t + 2\ln|t| + C$

2. Chapter 30 (Answers to Even-Numbered Review Problems):

   2. $\dfrac{x^2}{2} + x - \dfrac{1}{2x^2} + C$

   4. $-\dfrac{2}{t} + C$

   6. $\dfrac{1}{9}(x^3 - 4)^3 + C$

   8. $\dfrac{1}{2}\ln|x^2 + 3| + C$

   24. $y = \frac{x^3}{6} + \frac{9}{2}x - 18$

6 Formulas

Indefinite Integrals

$$\int f(x)\,dx = F(x) + C \qquad (\textit{ where } f(x) = F'(x) \text{ }) \tag{1}$$

$$\int du = u + C \tag{2}$$

$$\int a\,f(x)\,dx = a\int f(x)\,dx = a\,F(x) + C \tag{3}$$

$$\int [f(x) + g(x) + \ldots]\,dx = \int f(x)\,dx + \int g(x)\,dx + \ldots \tag{4}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1) \tag{5}$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1) \tag{6}$$

$$\int \frac{du}{u} = \ln|u| + C \tag{7}$$

Note that a is a *constant* in (3). In formulas (6) and (7) u is a function of x (for example $u = 2x^2 + 9$) and $du = \frac{du}{dx}dx$, where $\frac{du}{dx}$ is the derivative of u with respect to x.

Derivatives

$$\frac{d}{dx}(a) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(au) = a\frac{du}{dx}$$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

In these formulas a is a *constant*, u and v are *functions* of x.

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