SIAST Palliser Campus

Mathematics
MAT 201

Lecture Notes and Examples

Unit 2
The Definite Integral

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History

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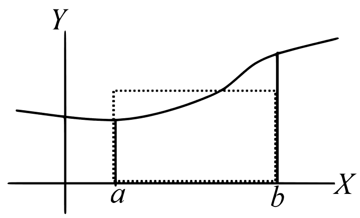
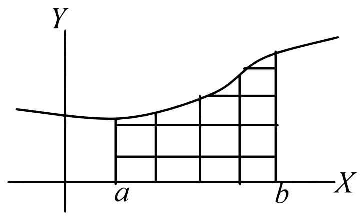
1 Area Under a Curve

There are a number of methods we can use for approximating the area under a curve between two given points. These include:

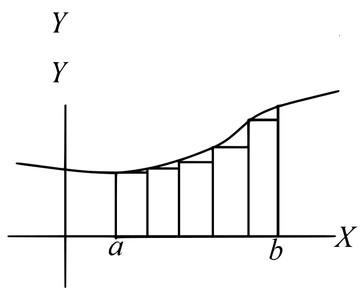
- sketching a triangle or rectangle of approximately the same area as the area under the curve and calculating the area



- subdividing the area under the curve into squares, counting the squares and multiplying by the area of one square (this is most easily done using squared paper)



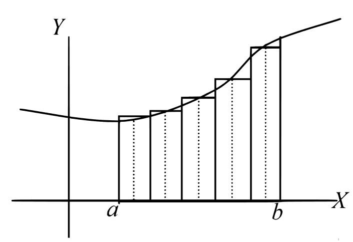
- subdividing the area under the curve into rectangular panels of known width, whose height is equal to the height of the curve at some point between the sides of the rectangle, and summing the areas of the rectangles



The first of these methods is the least accurate, but is good for getting a quick estimate. The accuracy of the second and third methods increases as the number of squares or rectangular panels increases.

1.1 The Midpoint Method

When using the third of the above methods, we need to decide at which point in the panels to take the height of the curve as the height of the panel. It turns out that, in most cases, we will get the best approximation of the area under the curve by defining the height of each panel as the height of the curve (i.e. the value of the function) at the *midpoint* of the panel. By doing this we will, to a large extent, balance out the extra area of the parts of the panels extending above the curve with the missing area of the parts of the panels that fall below the curve.



We call this approach the *midpoint method*. It should be fairly obvious that, as we decrease the width and increase the number of rectangular panels, the accuracy of the estimate also increases.

In symbols, we can write the sum of the areas found using the midpoint method as follows:

$$\boxed{A_{ab} \approx \sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x \qquad (1)}$$

where $f(x_i^*)$ is the value of the function at the midpoint x_i^* of each panel, and Δx is the width of each panel, $\Delta x = \frac{b-a}{n}$. A_{ab} is the actual area under the curve $f(x)$ between $x = a$ and $x = b$.

Examples

1. Estimate each of the following areas by sketching a triangle or rectangle:
   (a) the area under the curve $y = x^2$ from $x = 1$ to $x = 4$
   (b) the area given in Problem 6, p. 879 of the text
2. Estimate the area under the curve $y = \sqrt{2x + 1}$ from $x = 1$ to $x = 7$ by the midpoint method.
   (a) using panels of width 2
   (b) using panels of width 1

Reading:
Sec. 30-5, pp. 875-879
Problems:
Ex. 5, p. 879 # 1-9 (odd), 17, 18, 19, 20

2 The Definite Integral

If we are asked to approximate the area under the curve of a given mathematical function between two given x-values, we can inscribe rectangles between the curve and the x-axis and add the areas of these rectangles. If only a few such rectangles were used, we would only get a rough approximation of the area. If, however, we were to use a larger number of rectangles, the amount of "empty space" above the rectangles would be decreased, and we would get a better approximation of the area. (See the two diagrams in Figure 1, which illustrate this idea.)

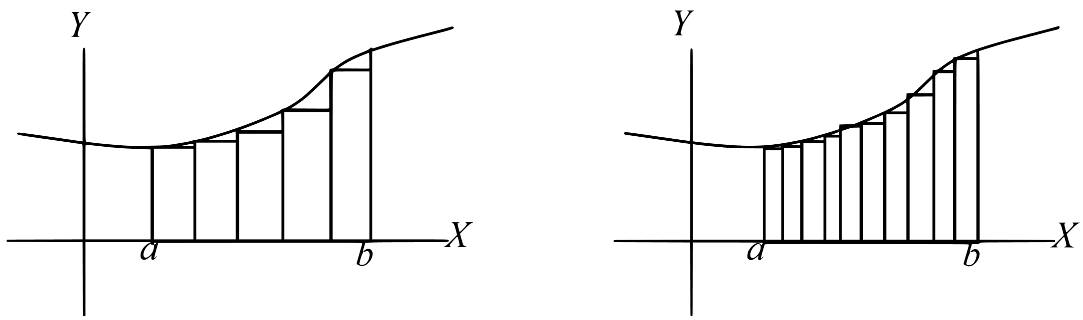
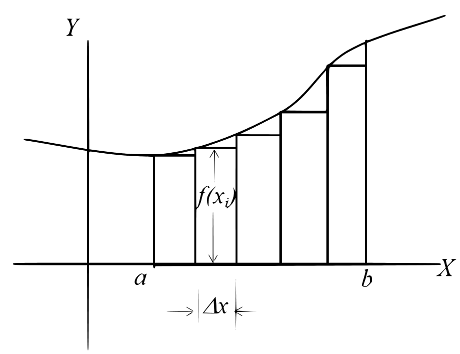


Figure 1: Approximating the area under a curve.

If you imagine that the number of rectangles is increased without bound, the sum of the areas of the rectangles will approach a limiting value that is equal to the exact area under the curve.

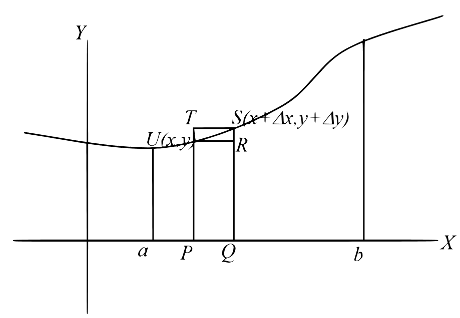


In general, the area under a curve $y = f(x)$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n}$$



Let ΔA represent the area $PQSU$ under the curve.

$$\begin{array}{rcl} A_{PQRU} \le & \Delta A & \le A_{PQST} \\ y\Delta x \le & \Delta A & \le (y+\Delta y)\Delta x \end{array}$$

Dividing by Δx gives

$$y \le \frac{\Delta A}{\Delta x} \le y + \Delta y$$

Taking the limit as Δx approaches zero, we get

$$y \le \lim_{\Delta x\to 0} \frac{\Delta A}{\Delta x} \le y$$

since, as $\Delta x \to 0$, $\Delta y \to 0$.

The middle term, which is "sandwiched" between two terms that both become equal to y in the limit, is, in fact, the derivative of the area with respect to x, i.e. dA/dx. Thus, we can write

$$\frac{dA}{dx} = y$$

In differential form this becomes what one would intuitively expect for the infinitessimal area dA of the rectangle of height y and infinitessimal width dx, namely

$$dA = y\ dx$$

Taking the integral of both sides, we get

$$\int dA = \int y\ dx$$

But the integral of dA is, by definition, A. Using the notation A_{ax} to represent the area between $x = a$ and any given value of x, we can write

$$A_{ax} = F(x) + C$$

where $F(x)$ is the antiderivative of the function y, and C is the constant of integration.

The area between $x = a$ and $x = a$ must, of course be zero. We can use this information to determine the constant C.

$$A_{aa} = F(a) + C = 0$$

Solving for C, we get

$$C = -F(a)$$

We can then write an expression for the area between $x = a$ and $x = b$:

$$A_{ab} = F(b) + C$$

Substituting $C = -F(a)$ we get

$$A_{ab} = F(b) - F(a)$$

This leads us to the following formal definition of the definite integral.

We define the *definite integral* of a function $y = f(x)$ as follows:

$$\boxed{\int_a^b f(x)\ dx = F(b) - F(a)} \tag{2}$$

where a and b are called the *limits of integration.* $F(b)$ and $F(a)$ are the values of the integral (or antiderivative) for $x = b$ and $x = a$ respectively.

As we have shown above, the definite integral is numerically equal to the area under the curve $y = f(x)$ between $x = a$ and $x = b$, i.e.

$$A_{ab} = \int_a^b f(x)\ dx = F(b) - F(a) \tag{3}$$

It should be noted that the area A_{ab} will be *negative* if the function lies below the x-axis (i.e. is negative) over the interval $[a, b]$.

Examples

1. Find the exact area under the curve $y = \sqrt{2x+1}$ from $x = 1$ to $x = 7$. (This is the same area that we estimated earlier by the midpoint method.)
   Evaluate the following definite integrals. Give the answers in exact form, and in approximate form to 3 significant digits.
2. $\displaystyle\int_2^3 \frac{1}{x^2}\ dx$
3. $\displaystyle\int_{-1}^2 (x^2 - 7)x\ dx$
4. $\displaystyle\int_0^1 \frac{x\ dx}{\sqrt{2 - x^2}}$
5. $\displaystyle\int_{-1.6}^{0.7} \sqrt[3]{1 - x}\ dx$
6. $\displaystyle\int_1^e \frac{dx}{x}$
7. $\displaystyle\int_0^2 \frac{dx}{8 - 3x}$

Additional Examples (Text, p. 875)

- Problem 6
- Problem 8
- Problem 12

Reading:
Secs. 30-4, 30-6; pp. 873-875, 880-882
Problems:
Ex. 4, p. 875 # 1-14 (all); Ex. 6, p. 883 # 1-12 (all)

3 Numerical Integration

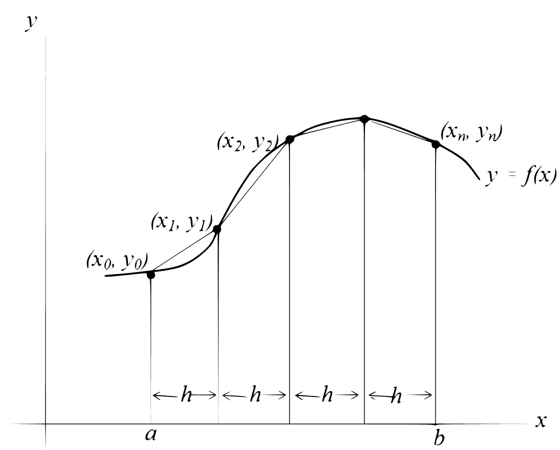
We have seen that we can find the exact area under a curve by finding the definite integral. This works well as long as the function is one that we can integrate using the rules for integration or the table of integrals. This may not always be the case. The function may be one that cannot be integrated, or it may be given in the form of a table of values rather than an algebraic expression. In cases like these, we can use numerical methods to find the value of the definite integral to whatever degree of accuracy we need, using a calculator or computer.

We have already seen one numerical method for approximating areas under a curve – the midpoint method. We will look at two more methods that give us more accuracy than the midpoint method for a given number of intervals – the trapezoidal rule and Simpson's rule. Of course, the accuracy of all three methods is improved by increasing the number of intervals. When the number of intervals is very large, a computer is absolutely essential.

3.1 The Trapezoidal Rule

In the diagram shown on the right, suppose we wish to estimate the area under the curve $y = f(x)$ between $x = a$ and $x = b$.

We have divided the interval from $x = a$ to $x = b$ into n equal intervals of width h. If we join the tops of the vertical lines by straight line segments, we form trapezoids each of whose areas is *approximately* equal to the area under the curve between successive x-values.



The height of each trapezoid (measured horizontally) is equal to h, and the lengths of the parallel sides are equal to the y-values on either side of the trapezoid. Making use of the fact that the area of a trapezoid is equal to half the height times the sum of the parallel sides, we get the following expression for the sum of the areas of the trapezoids between a and b:

$$A_T = \frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \frac{1}{2}h(y_2 + y_3) + \cdots + \frac{1}{2}h(y_{n-2} + y_{n-1}) + \frac{1}{2}h(y_{n-1} + y_n)$$

Factoring out the common factor $\frac{1}{2}h$ and collecting like terms, we get

$$\begin{aligned} A_T &= \frac{h}{2}\left[y_0 + y_1 + y_1 + y_2 + y_2 + \cdots + y_{n-1} + y_{n-1} + y_n\right] \\ &= \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right] \end{aligned}$$

Since the sum of the areas of the trapezoids is approximately equal to the area under the curve from a to b, which is in turn equal to the definite integral from a to b, we can state the following formula, called the *trapezoidal rule*:

$$\boxed{\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + \cdots + y_{n-1}\right) + y_n\right] \qquad (4)}$$

where $h = \frac{b-a}{n}$ and n is the number of intervals used.

It should be fairly obvious that, as the number of intervals is increased and the width of the intervals is decreased, the area of each trapezoid approaches the area under the curve for each interval more and more closely. Thus any desired degree of accuracy can be obtained simply by using a sufficiently large number of intervals.

It should be noted that the trapezoidal rule can be used whether the function is given as a mathematical equation or as a table of values. Also, it can be used to evaluate a definite integral even for a function that cannot be integrated analytically.

Examples

- Evaluate the definite integral $\int_0^{10} x^{\frac{3}{2}}\, dx$
  1. using the trapezoidal rule
     (a) with 5 panels of equal width
     (b) with 10 panels of equal width
  2. by integration

  Carry 6 significant digits and round your answers to 4 significant digits. Compare each of your results for part (a) with the answer found in part (b).

Additional Examples (Text, p. 1000)

- Problem 2
- Problem 6
- Problem 10
- Problem 14

Reading:
Sec. 34-9, pp. 996 to 998
Problems:
Ex. 9, p. 1000 # 1-17 (odd) – use trapezoidal rule only

3.2 Simpson's Rule

In the previous section, we approximated the area under a curve by summing the areas of trapezoids formed by connecting successive points of the curve with straight-line segments. Suppose that, instead of straight-line segments, we connect each successive set of three points on the curve with a parabolic arc. This leads to a formula called *Simpson's Rule*, which is derived in the text (pp. 998-1000). According to Simpson's Rule, the definite integral of a function between two points is given by the following formula:

$$\boxed{\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right] \qquad (5)}$$

where $h = \frac{b-a}{n}$ is the width of each panel and n is the number of panels.

It is intuitively clear that, by using parabolic arcs instead of straight line segments to join the points on a curve, we should get a better approximation of the area under a curve for the same number of intervals.

It should be noted that, in order to apply Simpson's Rule, we must have an *even* number of intervals.

As was the case with the trapezoidal rule, using more intervals leads to higher accuracy.

Example

- Evaluate the definite integral $\int_0^{10} x^{\frac{3}{2}}\,dx$ using Simpson's rule with 10 intervals. Carry 6 significant digits and round your answer to 4 significant digits. Compare your answer to that found on the previous page using the trapezoidal rule, and to the answer found by integration.

Additional Examples (Text, p. 1000)

- Problem 12
- Problem 16

Reading:
Sec 34-9, pp. 998-1000
Problems:
Ex. 9, p. 1000 # 1-17 (odd) – use Simpson's rule only

4 Review Problems: The Definite Integral

Questions:

1. Evaluate the following definite integrals. Give the answers in exact form and in approximate decimal form to three significant digits.

   (a) $\displaystyle\int_1^3 x^2(3x^3-1)^2\,dx$

   (b) $\displaystyle\int_0^1 \frac{t\,dt}{4+t^2}$

2. Given the function $y = 2x^3 + 5$,

   (a) Use *Simpson's rule* with 4 intervals to find the approximate area under the curve between $x = 0$ and $x = 4$.

   (b) Find the *exact* area under the above curve from $x = 0$ to $x = 4$ by integration.

   (Note: If your answers to (a) and (b) are not reasonably close to each other, you'd better recheck your work!)

3. Use the *trapezoidal rule* to find the approximate area under the curve defined by the following set of experimental data:

x	5.0	6.0	7.0	8.0	9.0	10.0
y	4.2	3.9	3.8	4.0	3.5	3.4

4. Additional Review Problems:

   - Chapter 30 Review Problems (p. 883) # 10-14 (all) , 28, 29
   - Chapter 34 Review Problems (p. 1002) # 21 (use the trapezoidal rule with 8 intervals)

Answers:

1. (a) $\frac{56{,}888}{3} \approx 1.90 \times 10^4$
   (b) $\frac{1}{2}(\ln 5 - \ln 4) \approx 0.112$
2. (a) 148 square units
   (b) 148 square units
3. 19 square units
4. (Answers to Even-Numbered Review Problems):
   - Chapter 30:
     10. $\ln 2 \approx 0.693$
     12. $\frac{\ln 2}{2} \approx 0.347$
     14. $-\frac{14}{9} \approx -1.56$
     28. 280

5 Formulas

The Definite Integral

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{1}$$

where F is the antiderivative (indefinite integral) of f.

$$A_{ab} = \int_a^b f(x)\,dx \tag{2}$$

where A_{ab} is the area under the curve $f(x)$ between $x = a$ and $x = b$.

Numerical Approximations of the Definite Integral

Midpoint Method:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x \tag{3}$$

where $f(x_i^*)$ is the value of the function at the midpoint x_i^* of each panel, n is the number of panels, and Δx is the width of each panel, $\Delta x = \frac{b-a}{n}$.

Trapezoidal rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + \cdots + y_{n-1}\right) + y_n\right] \tag{4}$$

where $h = \frac{b-a}{n}$, n is the number of intervals used, and $y_i = f(x_i)$.

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right] \tag{5}$$

where $h = \frac{b-a}{n}$ is the width of each panel, n is the (even) number of panels, and $y_i = f(x_i)$.

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