SIAST Palliser Campus

Mathematics
MAT 226

Lecture Notes and Examples

Unit 5
Derivatives of Transcendental Functions

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published by the Department of Mathematics, SIAST Palliser Campus

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- Original document produced in 2011 entitled "Derivatives of Transcendental Functions" written by Robert G. Petry. Published by the Department of Mathematics, SIAST Palliser Campus. A transparent copy of this document is available via *http://www.campioncollege.ca/about-us/faculty-listing/dr-robert-petry*

1 Derivatives of Trigonometric Functions

In this course we have sufficient rules to differentiate any function we can imagine except for special functions like $\sin x$ or e^x. In order to differentiate such functions one has to appeal to the fundamental definition of the derivative:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \, ,$$

and carefully consider the properties of the function in question. For instance, one may use geometric considerations (see Text p. 943) to determine that the derivative of $y = \sin x$ is:

$$\frac{dy}{dx} = \frac{d(\sin x)}{dx} = \cos x$$

Here and in the other formulas, the angular variable x **must be in radians**. The derivative of $y = \cos x$ is:

$$\frac{dy}{dx} = \frac{d(\cos x)}{dx} = -\sin x$$

Note the minus sign in the last equation. Using these two relations, trig identities, and our basic rules of differentiation one can find other trigonometric derivatives as shown in the following example.

Example

Find the derivative of $y = \tan x$. (Hint: Use that $\tan x = \frac{\sin x}{\cos x}$.)

The following table summarizes the trig function derivatives.

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \cos x && (1)\\
\frac{d}{dx}(\cos x) &= -\sin x && (2)\\
\frac{d}{dx}(\tan x) &= \sec^2 x && (3)\\
\frac{d}{dx}(\cot x) &= -\csc^2 x && (4)\\
\frac{d}{dx}(\sec x) &= \sec x \tan x && (5)\\
\frac{d}{dx}(\csc x) &= -\csc x \cot x && (6)
\end{aligned}$$

Often these formulas will be written more generally as (for instance):

$$\frac{d}{dx}(\sin u) = \cos u \cdot \frac{du}{dx}$$

to indicate that the chain rule must be applied if the argument of the trig function differs from the variable which is being differentiated. Such notation is shown in the chain rule versions of our derivative formulas on the formula sheet. One must remember that the chain rule ($\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$) is required if the argument does differ from the variable differentiated. For instance, if $y = \sin(x^2)$ then

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{\left(\sin x^2\right)}_{y=\sin u} = \underbrace{\left(\cos x^2\right)}_{\frac{dy}{du}=\cos u} \cdot \underbrace{\frac{d}{dx}(x^2)}_{\frac{du}{dx}} = \left(\cos x^2\right) \cdot (2x) = 2x\cos x^2$$

Note that here $\sin x^2 = \sin(x^2)$ not $(\sin x)^2$. The latter we would write $\sin^2 x$.

These trigonometric derivatives are still just derivatives. All the rules of derivatives, higher order derivatives, implicit differentiation, tangents, extreme values, rates of change, related rates, and max-min problems can be applied to such functions. Examples of each of these can be found in the assignment and the problems at the end of the unit.

Reading:
Sec. 33-2
Problems:
Ex. 2 (P. 951) # 1-41 (odd)

2 Derivatives of Inverse Trigonometric Functions

As with trigonometric functions, it is desirable to find derivatives of **inverse** trigonometric functions like $\arcsin x$ (also written $\sin^{-1} x$). Since if $y = \arcsin x$ one has that

$$\sin y = \sin(\arcsin x) = x$$

one can use implicit differentiation of the latter to find the derivative of arcsin.

Example

Find the derivative $\frac{d}{dx}(\arcsin x)$. (Hint: Also use that if $\sin y = x = \frac{x}{1}$, then by the Pythagorean theorem $\cos y = \sqrt{1 - x^2}$.)

The following table summarizes the inverse trig function derivatives.

$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}} \tag{7}$$
$$\frac{d}{dx}\left(\cos^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}} \tag{8}$$
$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2} \tag{9}$$
$$\frac{d}{dx}\left(\cot^{-1}x\right) = \frac{-1}{1+x^2} \tag{10}$$
$$\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{|x|\sqrt{x^2-1}} \tag{11}$$
$$\frac{d}{dx}\left(\csc^{-1}x\right) = \frac{-1}{|x|\sqrt{x^2-1}} \tag{12}$$

Note that for $\sin^{-1}$ and $\cos^{-1}$ the derivative is only meaningfully defined where the function is, namely $|x| < 1$. For $\sec^{-1}$ and $\csc^{-1}$ one requires $|x| > 1$. Finally $\tan^{-1}$ and $\cot^{-1}$ have derivatives defined for all x.

As with the trigonometric derivatives, one simply uses these rules to differentiate any function that involves an inverse trig function. Examples may be found in the assignment and at the end of the unit.

Reading:
Sec. 33-3
Problems:
Ex. 3 (P. 956) # 1-21 (odd)

3 Derivatives of Logarithmic Functions

The derivative of the natural logarithm $\ln x$ gives the following surprisingly simple result:

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}} \tag{13}$$

(See Text p. 956 for proof.)

Logarithms to bases other than e (the base of $\ln = \log_e$), introduce a constant factor of $\ln b$ in the denominator:

$$\boxed{\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}} \tag{14}$$

These rules can be used to differentiate logarithmic functions as they arise in any differentiation problem as shown in the assignment below and at the end of the unit.

Reading:
Sec. 33-4
Problems:
Ex. 4 (P. 960) # 1-51 (odd)

4 Derivatives of Exponential Functions

The derivative of the exponential function of base e is very special indeed as it is the only functions whose derivative is itself!

$$\frac{d}{dx}\left(e^x\right) = e^x \tag{15}$$

Example

Prove that the derivative of e^x is itself. (Hint: Use that if $y = e^x$ then $x = \ln y$ and implicitly differentiate.)

For exponentials involving bases other than e one has to introduce the constant $\ln b$ as with logarithms:

$$\frac{d}{dx}\left(b^x\right) = b^x \ln b \tag{16}$$

Examples using these derivatives may be found in the assignment and at the end of the unit.

Reading:
Sec. 33-5
Problems:
Ex. 5 (P. 964) # 1-61 (odd)

5 General Questions

Use calculus techniques and the new derivative formulas from this unit to solve the following problems.

1. Find the derivatives of the following:
   (a) $y = x^2 + \sin x + \cos^{-1} x - \ln x + 5^x$
   (b) $y = \sin x \cos x$
   (c) $y = \dfrac{\ln x}{x}$
   (d) $y = \tan^2 x$
   (e) $y = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
   (f) $y = 2\sqrt[3]{\sec^{-1} x}$
2. If $f(x) = \sin^{-1}(x)$, find $f''(x)$ and $f''(0)$.
3. Find the derivative $\frac{dy}{dx}$ of the implicitly defined function:
$$\ln y + x^3 = x^2 \cos y$$
4. Find the minima, maxima and inflection points of the function $y = \sin^2 \theta$ for $0 \leq \theta < 2\pi$.
5. The elevation above sea level of a hill (in km) is described by the function $y = xe^{-3x}$. Find the horizontal distance x at which the hill is a maximum as well the maximum elevation of the hill.
6. Find the horizontal distance x at which the hill is maximum from the last problem using *Newton's method*. (Hint: Since you will be solving $y' = 0$, you will want to call $f(x)$ in Newton's method your expression for y'. Then $f'(x)$ in the method will be your expression for y''.)
7. An airplane flies horizontally away from you at a constant altitude of 10.0 km. When the plane is at a ground distance (horizontal) from you of 5.00 km find the rate of change of the angle of inclination of the plane from your point of observation if the plane is travelling at a speed of $25\bar{0}$ m/s at that moment.

Solutions

1. (a) $\dfrac{dy}{dx} = 2x + \cos x - \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{x} + 5^x \ln 5$

   (b) $\dfrac{dy}{dx} = -\sin^2 x + \cos^2 x$

   (c) $\dfrac{dy}{dx} = \dfrac{1 - \ln x}{x^2}$

   (d) $\dfrac{dy}{dx} = 2\tan x \sec^2 x$

   (e) $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

   (f) $\dfrac{dy}{dx} = \dfrac{2}{3}\left(\sec^{-1} x\right)^{-\frac{2}{3}} \dfrac{1}{|x|\sqrt{x^2-1}}$

2. $f''(x) = x(1-x^2)^{-\frac{3}{2}}$, $f''(0) = 0$

3. $y' = \dfrac{2x\cos y - 3x^2}{\frac{1}{y} + x^2 \sin y}$

4. **relative minima:** 0, π

   relative maxima: $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

   inflection points: $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

5. **horizontal distance:** $\dfrac{1}{3}$ km

   maximum height: $\dfrac{1}{3}e^{-1} = .123$ km.

6. $x = .333333333 = \dfrac{1}{3}$ km

7. $\left.\dfrac{d\theta}{dt}\right|_{x=5 \text{ km}} = -.0200\dfrac{\text{rad}}{\text{s}}$

6 Further Review Problems

Chapter 33, page 966, # 1-25 (7 & 16 require implicit differentiation.) The following answers are for the even numbered problems:

2. $2\ln 5\left(5^{2x+3}\right)$

4. $2\sec^2 x\tan x$

6. $-(\arcsin 2x)^{-\frac{3}{2}}\left(1-4x^2\right)^{-\frac{1}{2}}$

8. $e^{2x}(2x+1)$

10. $x^2\cos x+2x\sin x$

12. $(2x+3)\cot\left(x^2+3x\right)$ Recall $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$

14. $2\log x\dfrac{1}{x\ln 10}$

16. $2\csc(x-y)+1$

18. $\dfrac{1}{x+10}$

20. $\dfrac{2x+3}{x^2+3x}$

22. $2\sec^2 x\tan x$

24. $\arcsin 2x+\dfrac{2x}{\sqrt{1-4x^2}}$

Basic Formulas

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \tag{1}$$
$$\frac{d}{dx}(\cos x) = -\sin x \tag{2}$$
$$\frac{d}{dx}(\tan x) = \sec^2 x \tag{3}$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x \tag{4}$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x \tag{5}$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x \tag{6}$$

Derivatives of Inverse Trigonometric Functions

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$$\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}} \tag{12}$$

Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \tag{13}$$
$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b} \tag{14}$$

Derivatives of Exponential Functions

$$\frac{d}{dx}(e^x) = e^x \tag{15}$$
$$\frac{d}{dx}(b^x) = b^x \ln b \tag{16}$$

Chain Rule Formulas

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx} \tag{1}$$
$$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx} \tag{2}$$
$$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx} \tag{3}$$
$$\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx} \tag{4}$$
$$\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx} \tag{5}$$
$$\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx} \tag{6}$$

Derivatives of Inverse Trigonometric Functions

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Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \frac{du}{dx} \tag{13}$$
$$\frac{d}{dx}(\log_b u) = \frac{1}{u \ln b} \frac{du}{dx} \tag{14}$$

Derivatives of Exponential Functions

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx} \tag{15}$$
$$\frac{d}{dx}(b^u) = b^u \ln b \, \frac{du}{dx} \tag{16}$$

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