SIAST Palliser Campus

Mathematics
MAT 226

Lecture Notes and Examples

Unit 1
Trigonometric Identities and Equations

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published by the Department of Mathematics, SIAST Palliser Campus

History

- Original document produced in 2011 entitled "Trigonometric Identities and Equations" written by Robert G. Petry. Published by the Department of Mathematics, SIAST Palliser Campus. A transparent copy of this document is available via *http://www.campioncollege.ca/about-us/faculty-listing/dr-robert-petry*

1 Fundamental Identities

We recall that an *identity* is an equation that is true for all meaningful values of the variable; for example, $x(x+2) = x^2 + 2x$ is true for any value of the variable x. This is to be distinguished from a *conditional equation* such as $x^2 - 5x + 6 = 0$ which is true for only certain values of the variable x, here $x = 2$ or $x = 3$.

1.1 Reciprocal Relations

The simplest *trigonometric* identities are the reciprocal relations involving definitions of the reciprocal functions:

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta\csc\theta = 1 \tag{1}$$

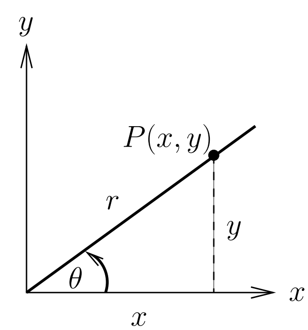
$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta\sec\theta = 1 \tag{2}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta\cot\theta = 1 \tag{3}$$

1.2 Quotient Relations

An angle θ in standard position can have its basic trig functions defined in terms of a point P at (x, y) on the terminal side as follows:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$



where r is the hypotenuse length. Dividing the left and right sides of the first two equations proves the first of the following quotient relations:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{4}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{5}$$

Reciprocating both sides of equation (4) and using the reciprocal relation (3) results in the second equation.

1.3 Pythagorean Relations

Returning to the coordinate trig functions of the previous section, one has, by the Pythagorean theorem,

$$x^2 + y^2 = r^2$$

Dividing both sides of this equation by r^2 and identifying the trig functions by their coordinates yields the first of the following Pythagorean relations.

$$\boxed{\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 && (6) \\ 1 + \tan^2\theta &= \sec^2\theta && (7) \\ 1 + \cot^2\theta &= \csc^2\theta && (8) \end{aligned}}$$

The two other Pythagorean relations follow by dividing both sides of the first equation by $\cos^2\theta$ and $\sin^2\theta$ respectively and using the reciprocal relations (1)-(3).

We can use our trigonometric identities to write a trigonometric expression in terms of only $\sin\theta$ and $\cos\theta$ as the following example shows:

Example

Express the following in terms of only sin and cos:

$-\tan x \sin x + \sec x$

Our trig identities can be used to *simplify trigonometric expressions* as the following examples show.

Examples

Simplify the following:

1. $$\frac{\tan\theta(\csc^2\theta - 1)}{\cos\theta\cot\theta + \sin\theta}$$

2. $$\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta}$$

Trig identities can be used to *prove* whether other trig identities are true. This is done by using known identities to transform one side of the equation until it is identical to the other side. Often converting expressions to $\sin\theta$ or $\cos\theta$ can help.

Example

Prove the following identity:

$$\frac{\sin\theta + \tan\theta}{1 + \cos\theta} = \tan\theta$$

Reading:
Sec. 18-1
Problems:
Ex. 1 (P. 466) # 1-49 (odd)

2 Sum or Difference of Angles

2.1 Sine and Cosine of $\alpha \pm \beta$

A geometric consideration of the combination of two angles (see Text p. 468) results in the following formulas for the sine and cosine of the *sum of two angles*:

$$\begin{aligned}
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta
\end{aligned}$$

For β in the first quadrant, $-\beta$ is in the fourth quadrant so sine, but not cosine will change sign:

$$\cos(-\beta) = \cos\beta \qquad \sin(-\beta) = -\sin\beta$$

Using these results and replacing $\beta \to -\beta$ in our sum formulas above gives the difference formulas:

$$\begin{aligned}
\sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta
\end{aligned}$$

The sum and difference formulas can be written simply as:

$$\boxed{\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta && (9) \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta && (10)
\end{aligned}}$$

2.2 Tangent of $\alpha \pm \beta$

Since, by (4),

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)}$$

one can use the results (9) and (10) to arrive at the sum and difference formulas for tangent:

$$\boxed{\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \qquad (11)}$$

(Here we divided both the numerator and denominator by $\cos\alpha\cos\beta$.)

These new trigonometric identities allow the simplification of more expressions as well as the proof of further identities as the next examples show.

Examples

1. Expand: $\sin(\alpha + \beta + \gamma)$

2. Simplify: $\sin 5x \cos 2x - \cos 5x \sin 2x$

3. Prove: $\sin(\theta + \phi) + \sin(\theta - \phi) = 2 \sin\theta \cos\phi$

4. Prove: $0 = \tan\left(\theta - \frac{\pi}{4}\right) + \cot\left(\theta + \frac{\pi}{4}\right)$

2.3 Adding a Sine Wave and Cosine Wave of the Same Frequency

A function of the form

$$A \sin \omega t$$

is a *sine wave*. The constants A and ω are called the *amplitude* and *(angular) frequency* respectively. As time changes the wave oscillates between $+A$ and $-A$ with a frequency $\omega/2\pi$. A *cosine wave* may be similarly defined:

$$B \cos \omega t$$

This oscillates in the same way as a sine wave except the peaks of $\pm B$ occur at the times where the cosine wave vanishes and vice-versa.

A common problem is to combine a sine wave and a cosine wave of the *same* frequency ω. If one writes A and B in terms of two new constants R and ϕ as follows:

$$\boxed{A = R\cos\phi \qquad B = R\sin\phi} \tag{12}$$

then

$$\begin{aligned} &A \sin \omega t + B \cos \omega t \\ &\quad = R \sin \omega t \cos \phi + R \cos \omega t \sin \phi \\ &\quad = R\left(\sin \omega t \cos \phi + \cos \omega t \sin \phi\right) \end{aligned}$$

Using the sine angle addition formula (9) on the last line gives:

$$\boxed{A \sin \omega t + B \cos \omega t = R \sin\left(\omega t + \phi\right)} \tag{13}$$

Thus the new wave has the same frequency ω as the original waves but now peaks with an amplitude R at times shifted by a phase angle ϕ. These values are found by solving (12) for R and ϕ to get:

$$\boxed{R = \sqrt{A^2 + B^2} \qquad \phi = \arctan \frac{B}{A}} \tag{14}$$

as done for converting between polar and Cartesian forms of a vector.
(Or see diagram in text page 472.)

Example

Write as a single sine function:

$y = 1.91 \sin \omega t + 2.87 \cos \omega t$

Reading:
Sec. 18-2
Problems:
Ex. 2 (P. 473) # 1-35 (odd)

3 Double Angle Formulas

Trigonometric functions of 2α can be reduced to functions of α. Setting $\beta = \alpha$ in the trig addition formulas (9)-(11) the left-hand side of these equations become functions of 2α. For sine one has

$$\boxed{\sin 2\alpha = 2\sin\alpha\cos\alpha} \tag{15}$$

For cosine one arrives at the first of the following identities:

$$\begin{aligned} \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ \cos 2\alpha &= 1 - 2\sin^2\alpha \\ \cos 2\alpha &= 2\cos^2\alpha - 1 \end{aligned} \tag{16}$$

The latter two identities for $\cos 2\alpha$ are found by using the Pythagorean identity (6) on the first one to simplify it further to be in terms of either just sine or cosine.

Tangent of 2α may similarly be found by setting $\beta = \alpha$ in (11):

$$\boxed{\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}} \tag{17}$$

Examples

1. Simplify: $\quad 2\sin 3x\cos 3x$

2. Simplify: $\dfrac{2-\sec^2\theta}{\sec^2\theta}$

3. Prove: $\tan x = \dfrac{\sin x + \sin 2x}{\cos 2x + \cos x + 1}$

4. If θ and ϕ are two acute angles in a right triangle prove:

$$-\cot 2\theta = \tan(\theta - \phi)$$

Reading:
Sec. 18-3
Problems:
Ex. 3 (P. 476) # 1-19 (odd)

4 Half-Angle Formulas

Formulas for trig functions of $\alpha/2$ in terms of α can be found by manipulating the double-angle formulas of the previous section. Replacing $\alpha \to \alpha/2$ in the second identity of (16) and solving for $\sin(\alpha/2)$ gives:

$$\boxed{\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}} \tag{18}$$

Here (and below) the choice of $+$ or $-$ must be determined by the quadrant in which $\alpha/2$ is found. (Recall the CAST prescription.)

Similarly one may replace $\alpha \to \alpha/2$ in the third identity of (16) and solve for $\cos(\alpha/2)$ to get the following cosine identity:

$$\boxed{\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}} \tag{19}$$

Three different identities for $\tan\alpha/2$ can be obtained:

$$\boxed{\begin{aligned} \tan\frac{\alpha}{2} &= \frac{1-\cos\alpha}{\sin\alpha} \\ \tan\frac{\alpha}{2} &= \frac{\sin\alpha}{1+\cos\alpha} \\ \tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \end{aligned}} \tag{20}$$

The last of these is obtained by noting that by quotient relation (4) that

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \qquad (*)$$

into which one may substitute (18) and (19). The first identity of (20) may be found by multiplying the numerator and denominator of $(*)$ by $2\sin(\alpha/2)$ and simplifying the numerator using the second identity of (16) and the denominator using (15). Finally the second identity of (20) may be shown by multiplying the numerator and denominator of the first identity of (20) by $1+\cos\alpha$ and simplifying the new numerator with the Pythagorean relation (6).

Examples

1. Prove: $\dfrac{\cos x - \cos^2 \frac{x}{2}}{\sin^2 \frac{x}{2}} = -1$

2. In right triangle ABC show that:

$$\frac{b+c}{a} = \cot \frac{A}{2}$$

Reading:
Sec. 18-4
Problems:
Ex. 4 (P. 480) # 1-13 (odd)

5 Trigonometric Equations

5.1 Single Trigonometric Function with a Single Angle

Unlike an identity, a conditional trigonometric equation is true for only certain values of the angle. The simplest trigonometric equations involve a single trig function with a single angle.

In general there are an infinite number of solutions due to the periodicity of the trig function (e.g. $\sin(20°) = \sin(20° + 360°)$) so we will restrict ourselves to finding *all non-negative solutions less than* 360°.

Examples

Solve for all θ with $0 \leq \theta < 360°$:

1. $2\cos\theta + \sqrt{3} = 0$

2. $2\cos 3\theta = \dfrac{1}{2}$

Often trigonometric identities can be used to convert a more complicated equation into one involving a single trigonometric function of a single angle.

Example Solve for all x with $0 \leq x < 360°$:

$$\sqrt{3}\cos x = \sin x$$

If the equation involves a single trig function that is squared we have an equation in *quadratic form*. In this case we can use our general techniques for solving a quadratic equation followed by solving the trigonometric equations generated by equating the given trig function with the quadratic solutions. As above, trig identities may be first used to put the equation into a quadratic form as shown in the following example.

Example Solve for all θ with $0 \leq \theta < 360°$:

$$3 + \cos\theta = -2\sec\theta$$

5.2 Equations with One Angle and More Than One Function

If trig identities cannot reduce an equation to a single trig function, one can try transposing all terms to one side of the equation and then factor it. Set each factor to zero and solve as before.

Example Solve for all x with $0 \le x < 360°$:

$$0 = 2\tan x + 3\sin x \tan x$$

Reading:
Sec. 18-5
Problems:
Ex. 5 (P. 486) # 1-25 (odd)

6 Inverse Trigonometric Functions

We recall from MAT120 that for a given function $f(x)$ an inverse function $f^{-1}(x)$ may be defined. For the trigonometric functions *sine*, *cosine*, and *tangent* the inverses are given the special names arcsin, arccos, and arctan respectively. Because they are inverses they are also often written $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$, especially on calculators.

If for some value of x and y one has

$$y = \sin x$$

then one can, by definition of the inverse, define the arcsin of the value y to be x:

$$x = \arcsin y$$

The problem however is that there are (infinitely) many angles x which have the same value of sin. For instance if

$$1 = \sin x$$

then the following are all solutions:

$$x = 90^\circ, (90 + 360)^\circ, (90 - 360)^\circ, (90 + 720)^\circ, (90 - 720)^\circ, \ldots$$

A function can have only a *single* value. To make the inverse a *function* we restrict the range of angles the inverse is allowed to take. For instance for arcsin we force the angle to be between -90° and $+90^\circ$, so, in the above example, $\arcsin 1 = 90^\circ$. In radians the range is $-\pi/2$ to $\pi/2$. The numbers in this range are called the *principal values* of arcsin. These are the values which are given by a calculator.

For arctan the principal values lie in the same range as arcsin. For arccos, however, the principal values are chosen to lie in the range 0° to 180° (0 to π radians).

The range of principal values must be chosen to include a quadrant for which the trig function is positive and one for which it is negative. Remembering the CAST mnemonic, this explains why the range of principal values for arccos must differ from the other two functions, since cosine is positive in both quadrant I and IV.

Further graphs and analysis of the inverse trig functions may be found in the textbook pages 487-9.

The inverse trig functions of the reciprocal functions csc, sec, and cot are called arccsc, arcsec, and arccot respectively (or $\csc^{-1}$, $\sec^{-1}$, and $\cot^{-1}$). They have the same principal value ranges as arcsin, arccos, and arctan respectively. The inverse reciprocal functions are evaluated on a calculator by first reciprocating the argument and then taking the corresponding inverse trig function.

Examples Evaluate the following in degrees:

1. $\sin^{-1}(-0.819)$

2. $\sec^{-1}(2.91)$

Reading:
Sec. 18-6
Problems:
Ex. 6 (P. 489) # 1-9 (odd)

Review Problems: Trigonometric Identities & Equations

1. Simplify: $\dfrac{(\sec\theta+1)\tan\frac{\theta}{2}}{\sec\theta}$
2. Prove the following identities:
   (a) $\cos\theta = \sec\theta - \sec\theta\sin^2\theta$
   (b) $\dfrac{1+\cos\theta}{1-\cos\theta} = \csc^2\dfrac{\theta}{2} - 1$
3. Solve for all θ, $0° \le \theta < 360°$:
   (a) $\tan\theta + 5.00 = 0$
   (b) $\sec 2\theta = 2$
   (c) $2\sin^2\theta + \sin\theta = 0$
4. In right triangle ABC show that acute angle A satisfies:

   $$\sin(2A) = \frac{2ab}{c^2}$$

   where c is the hypoteneuse length.
5. Evaluate in degrees: $\csc^{-1}(1.495)$

Solutions

1. $\sin\theta$

3.(a) $\theta = 101^\circ, 281^\circ$

(b) $\theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

(c) $\theta = 0^\circ, 180^\circ, 210^\circ, 330^\circ$

5. 41.98°

Formulas

Reciprocal Relations

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta\csc\theta = 1 \tag{1}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta\sec\theta = 1 \tag{2}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta\cot\theta = 1 \tag{3}$$

Quotient Relations

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{4}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{5}$$

Pythagorean Relations

$$\sin^2\theta + \cos^2\theta = 1 \tag{6}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{7}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{8}$$

Sum or Difference of Angles

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \tag{9}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{10}$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \tag{11}$$

Adding a Sine Wave and Cosine Wave of the Same Frequency

$$A\sin\omega t + B\cos\omega t = R\sin(\omega t + \phi) \tag{12}$$

$$A = R\cos\phi \qquad B = R\sin\phi \tag{13}$$

$$R = \sqrt{A^2 + B^2} \qquad \phi = \arctan\frac{B}{A} \tag{14}$$

Double Angle Formulas

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \tag{15}$$

$$\begin{aligned} \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ \cos 2\alpha &= 1 - 2\sin^2\alpha \\ \cos 2\alpha &= 2\cos^2\alpha - 1 \end{aligned} \tag{16}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} \tag{17}$$

Half-Angle Formulas

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \tag{18}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} \tag{19}$$

$$\begin{aligned} \tan\frac{\alpha}{2} &= \frac{1-\cos\alpha}{\sin\alpha} \\ \tan\frac{\alpha}{2} &= \frac{\sin\alpha}{1+\cos\alpha} \\ \tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \end{aligned} \tag{20}$$

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