SIAST Palliser Campus

Mathematics
MAT 226

Lecture Notes and Examples

Unit 4
Applications of the Derivative

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History

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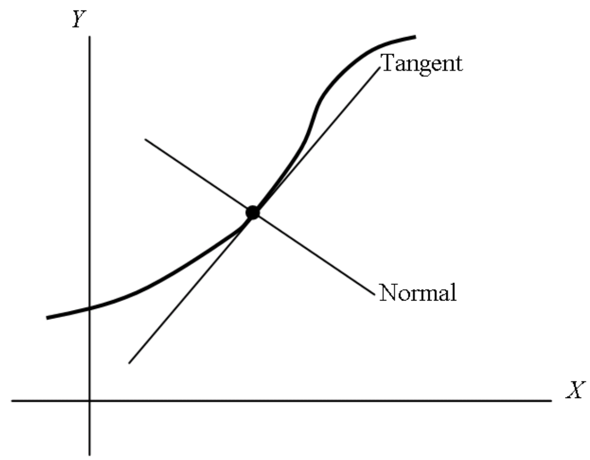
1 Tangents and Normals



The slope of the tangent to a curve at a given point is equal to the value of the derivative at that point.

The normal is perpendicular to the tangent, so its slope is the negative reciprocal of the slope of the tangent.

To find the equation of the tangent at a given point, we can evaluate the derivative at the point, and then use the point-slope form of the equation of a straight line. The equation of the normal can then be found in a similar way, using the negative reciprocal of the slope of the tangent.

Example

1. Find the equation of the tangent and normal of the curve $y = x^2 - 4x + 6$ at $(1, 3)$.

2. Find the equation of the tangent and normal of the curve $16x^2 + 9y^2 = 144$ at $(2, -2.98)$.

3. Find the angle of intersection of the curves:

$$y_1 = x^2 + x - 1$$

$$y_2 = x^2 - 5x + 5$$

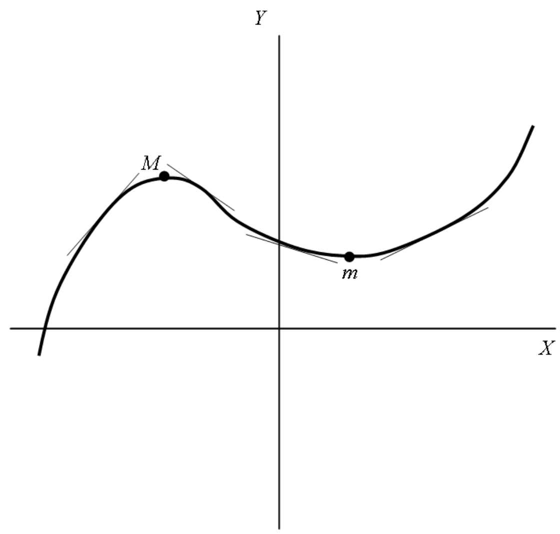
   at the point $(1, 1)$.

Reading:
Sec. 28-1
Problems:
Ex. 1 (P. 809) # 1-13 (odd)

2 Maximum, Minimum and Inflection Points

2.1 Increasing and decreasing functions

Consider the function $y = f(x)$ whose graph is shown on the right.



Any tangent line drawn to the left of point M will have a positive slope, as will any tangent line drawn to the right of point m. Since the slope of the tangent line at any point is equal to the value of the first derivative at that point, this means that the first derivative is positive to the left of M and to the right of m. We also see that, as x increases, the function increases in the region to the left of point M and in the region to the right of point m.

For points between M and m, any tangent line will have a negative slope, and the function decreases from left to right in this region.

We can summarize these results by stating the following rules:

- If the first derivative is **positive**, the function is **increasing**.
- If the first derivative is **negative**, the function is **decreasing**.

(It is assumed here that x is always increasing, i.e. we are observing the behavior of the function as we move from left to right along the x-axis.)

We can also state the above rules in symbols as follows:

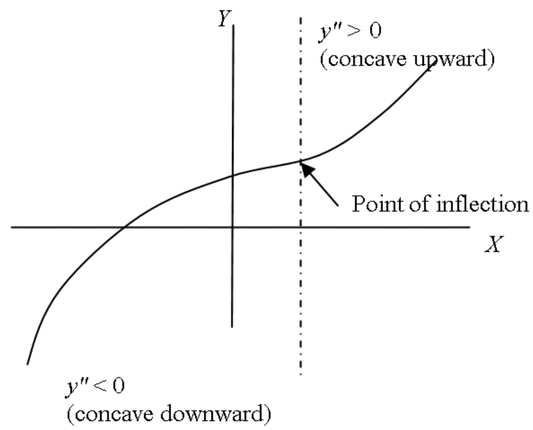
- If $f'(x) > 0$, $f(x)$ is **increasing**.
- If $f'(x) < 0$, $f(x)$ is **decreasing**.

Examples

1. Is the function $y = x^3 - 4x^2 + 5$ increasing or decreasing at $x = 3$?
2. For what values of x is the function $y = x^2 - 4x + 7$ increasing, and for what values of x is it decreasing?

2.2 Concavity

Concavity refers to the curvature of the graph of a function. As we have already seen in the previous unit, when the second derivative is positive, the slope of the curve is increasing, which means that the curve is concave upward. When the second derivative is negative, the curve is concave downward. At a point where the curve changes from being concave upward to concave downward (or vice-versa), the second derivative is zero. This is illustrated in the diagram on the right.

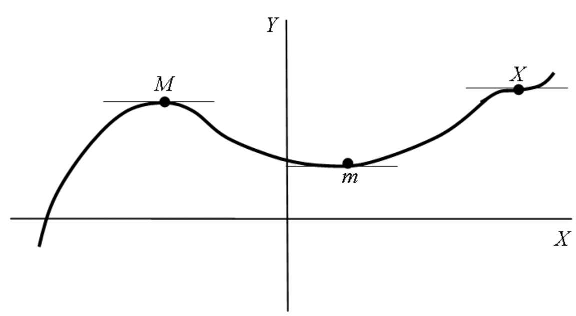


Examples

1. Is the function $y = x^3 - 4x^2 + 5$ concave up or down at $x = 3$?
2. Find the regions of concavity of $y = x^2 - 4x + 7$.

2.3 Stationary points

A *stationary point* is a point at which the first derivative is zero (i.e. the tangent is horizontal at the point).



Points M, m and X in the diagram are all stationary points.

Point M is called a *relative maximum point*. This means that the value of y is greater at M than at any other point near it.

Point m is called a *relative minimum point*. This means that the value of y is less at m than at any other point near it.

Point X is a stationary point, but it is *neither* a maximum nor a minimum point.

As can be seen in the diagram, functions change from increasing to decreasing and vice-versa at stationary points that are relative maxima and minima.

To find the stationary point(s) of a function, set the first derivative equal to zero and solve for x. Then substitute in the original function to find the $y-$coordinates of the stationary points.

Example

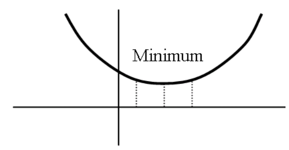
Find the stationary point(s) of the function $y = x^3 - 3x + 1$.

2.4 Testing for maximum or minimum points

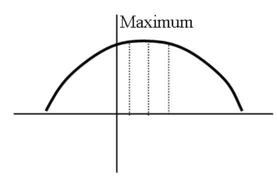
We have seen that the coordinates of the stationary points are found by setting the first derivative to zero, but this does not tell us whether each point is a maximum, a minimum or neither. To make this determination, we use the ordinate test, the first derivative test, or the second derivative test which are described below:

2.4.1 Ordinate test

The ordinate test determines whether a point is a relative maximum or minimum by evaluating the function $y = f(x)$ at points neighbouring the stationary point. It is clear from the diagrams at right that:



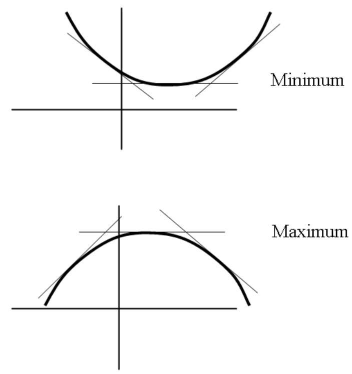
- The height of the curve on either side of a **minimum** point is **greater** than it is at the minimum point



- The height of the curve on either side of a **maximum** point is **less** than it is at the maximum point

To use the ordinate test, find the y-value at a point just to the left of the stationary point, and at a point just to the right of the stationary point. If both calculated y-values are greater than the y-value at the stationary point, it is a relative minimum. If both calculated y-values are less than the y-value at the stationary point, it is a relative maximum. If one of the y-values is greater and one is less than the y-value at the stationary point, it is neither a minimum nor a maximum.

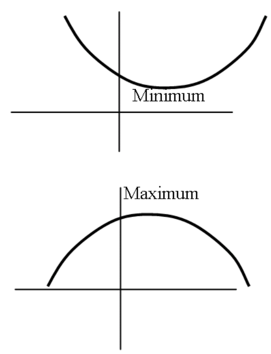
2.4.2 First derivative test



Referring to the diagrams on the right, and based on what we have already said about increasing and decreasing functions, we can state the following rules:

- The first derivative is **negative** to the **left** and **positive** to the **right** of a relative **minimum** point.
- The first derivative is **positive** to the **left** and **negative** to the **right** of a relative **maximum** point.

2.4.3 Second derivative test



We have already seen that the second derivative is a measure of the rate of change of the slope of a curve. Thus, if the second derivative is positive in an interval, the curve is concave upward in that interval. If the second derivative is negative in an interval, the curve is concave downward in that interval.

Since a curve is concave upward at a minimum point and concave downward at a maximum point (see diagram), we have the following rules for determining whether a given stationary point is a maximum or minimum:

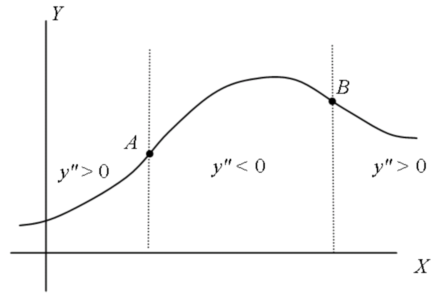
- If the second derivative is **positive**, the stationary point is a relative **minimum**.
- If the second derivative is **negative**, the stationary point is a relative **maximum**.
- If the second derivative is **zero**, the test is **inconclusive**.

Note that, if the second derivative is zero, the first derivative test or the ordinate test must be applied to determine whether the point is a maximum, a minimum, or neither.

Examples

1. Apply all three of the above tests to the stationary points of the function $y = x^3 - 3x + 1$ found in the previous example.
2. Find the maximum and minimum points for the implicit relation $x^2 + y^2 - 2x + 4y = 4$.

2.5 Inflection Points



An *inflection point* is a point where the curvature changes from concave upward to concave downward, or vice-versa. This means that the second derivative changes sign at an inflection point.

Points A and B in the graph on the right are inflection points.

To find inflection points, set the second derivative equal to zero and solve for x. Then check to see whether the second derivative changes sign on either side of the point(s).

Examples

Find the inflection point(s) of the following functions:

1. $y = 3x^4 - 4x^3$
2. $y = x^4$

2.6 Systematic Procedure for Finding Maximum, Minimum and Inflection Points of a Function

The formula sheets at the end of the unit summarize the steps for finding *critical points* (stationary or inflection points). Use that procedure to find any maximum, minimum and inflection points for the following function:

$$y = x^4 - 18x^2$$

Reading:
Sec. 28-2
Problems:
Ex. 2 (P. 819) # 1-23 (odd), 27, 29

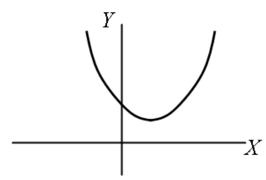
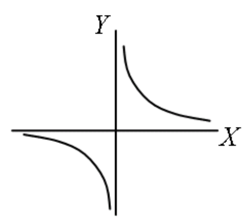
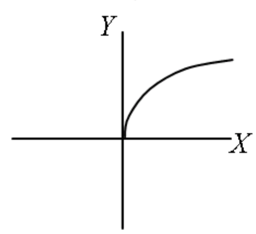
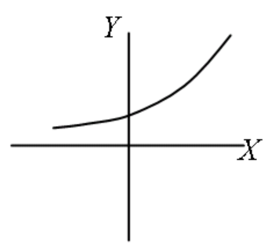
3 Curve Sketching

While it is usually possible to graph a function simply by plotting pairs of points, this is not always the best way to get an overall picture of how the function behaves. By looking at certain characteristics of the function, we can often get a good idea of what the graph of the function will look like, and make a sketch without using a table of x- and y-values. Going through these steps will enable us to analyze and interpret the characteristics of the function as well as allowing us to sketch the function.

Depending on the particular function, some or all of the following characteristics may be used in sketching the graph of a function.

3.1 Type of function

Certain types of functions have characteristic shapes, a few of which are illustrated below:

Function Type	Example	Graph
Quadratic	$y = x^2 - 3x + 5$	
Reciprocal	$y = \frac{5}{x}$	
Square Root	$y = 2\sqrt{x}$	
Exponential	$y = e^x$	

3.2 Intercepts

Intercepts are points where a curve crosses a coordinate axis. To find them do the following:

***y*-intercepts** : Set x equal to zero and solve for y.

***x*-intercepts** : Set y equal to zero and solve for x.

Example: Find the x and y-intercepts of the function $y = x^2 - 5x + 6$.

y-intercepts: $(x = 0)$

$$y = 0^2 - 5(0) + 6 = 6$$

$$y\text{-intercept: } (0, 6)$$

x-intercepts: $(y = 0)$

$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x - 2 = 0 \text{ or } x - 3 = 0$$
$$x = 2 \text{ or } x = 3$$

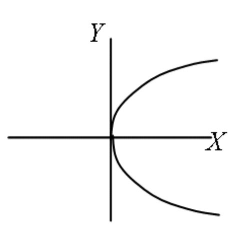
$$x\text{-intercepts: } (2, 0),\ (3, 0)$$

3.3 Symmetry

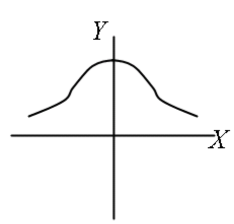
If the equation remains unchanged when $-y$ is substituted for y, the curve is **symmetric about the *x*-axis**.

If the equation remains unchanged when $-x$ is substituted for x, the curve is **symmetric about the *y*-axis**.

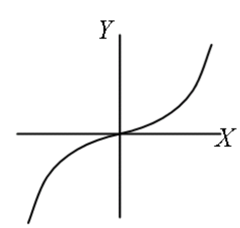
If the equation does not change when $-x$ is substituted for x and $-y$ is substituted for y, the curve is **symmetric about the origin**.



Symmetry about x-axis



Symmetry about y-axis



Symmetry about origin

Examples

1. $y^2 - 3x = 0$

   If $-y$ is substituted for y we get

$$\begin{aligned} (-y)^2 - 3x &= 0 \\ y^2 - 3x &= 0 \end{aligned}$$

   Thus the curve is symmetric about the x-axis.

2. $y = x^4 + 2x^2$

   If $-x$ is substituted for x we get

$$\begin{aligned} y &= (-x)^4 + 2(-x)^2 \\ &= x^4 + 2x^2 \end{aligned}$$

   Thus the curve is symmetric about the y-axis.

3. $y = x^3 - x$

   If $-x$ is substituted for x and $-y$ is substituted for y we get

$$-y = (-x)^3 - (-x) = -x^3 + x$$

   Dividing both sides by -1 gives

$$y = x^3 - x$$

   Thus the curve is symmetric about the origin.

3.4 Extent (Domain and Range)

For some functions, there may be certain x or y values that are not allowed. For example, some x values may lead to division by zero, or may result in a negative number under a square root sign. The set of allowed x values is the *domain* of the function. The set of y values that correspond to the allowed x values is the *range* of the function.

Example

$$y = \sqrt{5 - x}$$

For this function, only x values less than or equal to 5 are allowed, so the domain of the function is $x \leq 5$. Since the square root function by definition gives only non-negative values, and since the x values in the domain make possible any such value, only y values greater than or equal to zero are possible. (The range of the function is $y \geq 0$.)

3.5 Asymptotes

If y approaches infinity (positive or negative) as x approaches some value (from below or from above), there will be a *vertical asymptote* at that x value.

If y approaches some value as x approaches positive or negative infinity, there will be a *horizontal asymptote* at that y value.

Example

$$y = \frac{3}{x-1}$$

As x approaches 1 from below, y becomes infinite in the negative direction. As x approaches 1 from above, y becomes infinite in the positive direction. In symbols:

$$\lim_{x\to 1^-} \frac{3}{x-1} = -\infty \qquad \lim_{x\to 1^+} \frac{3}{x-1} = +\infty$$

Thus $x = 1$ is a vertical asymptote.

As x approaches positive infinity, y approaches 0 from above. As x approaches negative infinity, y approaches 0 from below. In symbols:

$$\lim_{x\to \infty} \frac{3}{x-1} = 0 \qquad \lim_{x\to -\infty} \frac{3}{x-1} = 0$$

Thus $y = 0$ is a horizontal asymptote.

3.6 Large x values

What happens to y as x gets very large in the positive or negative direction? In addition to horizontal asymptotes one may find the function is approximately equal to a simpler function for large x which will determine its large x behaviour.

Example

$$y = x^4 - 3x^2 + 5$$

Since

$$\lim_{x\to\pm\infty} \left(x^4 - 3x^2 + 5\right) = \lim_{x\to\pm\infty} x^4 \left(1 - \frac{3}{x^2} + \frac{5}{x^4}\right) \approx x^4 \ ,$$

it follows that as $x \to \pm\infty$, $y \to \infty$ since the x^4 term predominates.

3.7 Stationary Points (Maxima and Minima)

Find stationary points by setting y' equal to zero and solve for x. Apply the second derivative test (or the first derivative or ordinate tests if the second derivative test fails) to determine if a stationary point is a maximum, a minimum or neither.

Example

$$y = x^2 + 3x$$

Take the derivatives:

$$\begin{aligned} y' &= 2x + 3 \\ y'' &= 2 \end{aligned}$$

Solving $0 = 2x + 3$ gives $x = -3/2$. The corresponding y value is

$$y = (-3/2)^2 + 3(-3/2) = -9/4 \ .$$

Since $y''(-3/2) = 2$ is greater than zero we have a relative minimum at $(-3/2, -9/4)$.

3.8 Increasing and Decreasing Functions

- When y' is positive the function is increasing.
- When y' is negative the function is decreasing.

Example

Continuing with the last example, we now solve $y' < 0$:

$$\begin{aligned} 2x + 3 &< 0 \\ 2x &< -3 \\ x &< -\frac{3}{2} \end{aligned}$$

Hence when $x < -3/2$, $y' < 0$, so the function is decreasing when $x < -3/2$. Similarly solving $2x + 3 > 0$ shows $y' > 0$ when $x > -3/2$ so the function is increasing when $x > -3/2$.

Note that regions of increasing or decreasing are often broken up by maxima and minima since y' often changes sign at such points. In this example there was a minimum with x-coordinate of $-3/2$.

3.9 Inflection Points

- Set y'' equal to zero and solve for x. Test to see whether the sign of y' changes on either side of the x value. If it does, it is an inflection point.

Example

$$y = -x^3 + 2x^2$$

Take the derivatives:

$$\begin{aligned} y' &= -3x^2 + 4x \\ y'' &= -6x + 4 \end{aligned}$$

Solving $y'' = 0$ results in $x = -4/-6 = 2/3$ with corresponding y coordinate of $y = -(2/3)^3 + 2(2/3)^2 = 16/27$. To the left and right of $x = 2/3 = .667$ we have $y''(.6) = .4 > 0$ and $y''(.7) = -.2 < 0$ so there is an inflection point at $(2/3, 16/27)$.

3.10 Curvature

- When y'' is positive the function is concave upward.
- When y'' is negative the function is concave downward.

Example

Contining the last example we solve the inequality $y'' < 0$:

$$\begin{aligned} -6x + 4 &< 0 \\ -6x &< -4 \\ x &> \frac{-4}{-6} \\ x &> \frac{2}{3} \end{aligned}$$

Hence when $x > 2/3$, $y'' < 0$, so the function is concave downward for $x > 2/3$. Similarly solving $-6x + 4 > 0$ shows $y'' > 0$ when $x < 2/3$ so the function is concave upward when $x < 2/3$. (Note in solving the above inequality we had to change the direction of the inequality because we divided (or multiplied) by a negative quantity.)

Note that regions of curvature may be separated by inflection points as can be seen here with an inflection point with x-coordinate of $x = 2/3$.

3.11 Summary of Steps in Curve Sketching

The above steps for curve sketching are summarized on the formula sheets at the end of the unit. Follow through them as you do systematic curve sketches of the following functions.

Examples

Sketch the graphs of the following functions:

1. $y = \dfrac{2}{x^2 + 1}$
2. $y = \dfrac{x^2}{x - 2}$

Reading:
Sec. 28-3
Problems:
Ex. 3 (P. 823) # 1-21 (odd)

4 Newton's Method

Newton's method is a numerical method for solving equations. Newton's method involves making an initial guess as to the solution of an equation of the form $f(x) = 0$, then refining this guess to get a better approximation of the solution. The process is repeated until the desired degree of accuracy is obtained.

The root, or solution, of the equation, is the value of x at which the graph of the function crosses the x-axis.

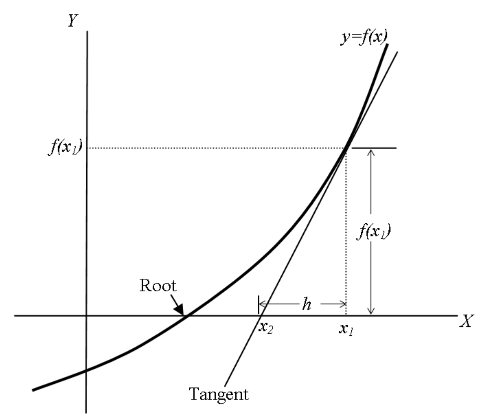
Let x_1 be the first guess of the root.

The slope of the tangent at x_1 is given by

$$f'(x_1) = \frac{f(x_1)}{h}$$

Solving for h, we get

$$h = \frac{f(x_1)}{f'(x_1)}$$



Let x_2 be the point where the tangent crosses the x-axis.

$$\begin{aligned} x_2 &= x_1 - h \\ &= x_1 - \frac{f(x_1)}{f'(x_1)} \end{aligned}$$

In general, if x_n is the n^{th} approximation of the solution, the $(n+1)^{\text{st}}$ approximation is given by the following formula:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \tag{1}$$

Examples

1. Use Newton's method to find the positive root of the equation $x^4 + 8x - 12 = 0$ to two decimal places.

2. Use Newton's method to find the positive root of the equation $x^2 = \sqrt{2x+1}$ to three decimal places.

Reading:
Sec. 28-4
Problems:
Ex. 4 (P. 825) # 1-11 (odd)

5 Rates of Change

5.1 Introduction

Recall that the value of the first derivative gives the rate of change of a function with respect to the independent variable.

Example

Illumination (I) from a light source is inversely proportional to the distance squared (d^2) from the source. If the illumination from a light source is 702 lux at a distance of 3.25 m, what is the rate of change of illumination with respect to distance at 3.25 m?

5.2 Equations of Motion

An important case of a rate of change is that of position with time. Position can be measured in one spatial dimension or higher. As well we can speak of angular position changing in time.

5.2.1 Straight-line motion

If an object is moving along one dimension (perhaps straight up and down or along a straight horizontal line) we can give its position by stating its displacement s along that axis. If it is moving along the axis then s is a function of time, $s(t)$.

The *velocity* of a moving object is defined as the instantaneous rate of change of displacement with respect to time, i.e. the derivative of displacement with respect to time.

$$\boxed{v = \frac{ds}{dt}} \tag{2}$$

The *acceleration* of a moving object is defined as the instantaneous rate of change of the velocity with respect to time. This is the derivative of the velocity, or the second derivative of the displacement.

$$\boxed{a = \frac{dv}{dt} = \frac{d^2s}{dt^2}} \tag{3}$$

Note that the velocity and acceleration themselves will, in general, depend on time.

Example

A ball thrown straight up has a vertical displacement given by

$$s = 30t - 4.9t^2$$

where s is the vertical displacement in metres and t is the time in seconds. Find

a) Its velocity as a function of time

b) Its acceleration as a function of time

c) The greatest height reached by the ball

d) The velocity of the ball when it reaches the ground. (Answer to 2 significant digits.)

5.2.2 Curvilinear Motion

Curvilinear motion is motion which is not in a straight line. It may occur in two or three spatial dimensions. We will consider two spatial dimensions but our results are easily generalized to three. In two spatial dimensions the position is determined by the displacent vector $\boldsymbol{s} = (x, y)$ from some fixed origin. Since the position is changing in time it is given by the vector function $\boldsymbol{s}(t) = (x(t), y(t))$, where $x(t)$ and $y(t)$ are just real functions giving the x- and y-coordinates at some time t.

With two spatial dimensions the velocity $\boldsymbol{v} = (v_x, v_y)$ and acceleration $\boldsymbol{a} = (a_x, a_y)$ are also vectors. Their components are determined by differentiating $\boldsymbol{s} = (x, y)$ component-wise with respect to time as shown on the following table:

Vector	x-component	y-component
Velocity $\boldsymbol{v} = (v_x, v_y)$	$v_x = \dfrac{dx}{dt}$	$v_y = \dfrac{dy}{dt}$
Acceleration $\boldsymbol{a} = (a_x, a_y)$	$a_x = \dfrac{dv_x}{dt} = \dfrac{d^2x}{dt^2}$	$a_y = \dfrac{dv_y}{dt} = \dfrac{d^2y}{dt^2}$

To find the magnitude and direction of the velocity or acceleration, find the x and y components of the velocity or acceleration then find the magnitude and direction from the components as done when converting from rectangular to polar vectors in Math 120.

Example

A point moves so that its horizontal and vertical displacements are given by

$$\begin{aligned} x &= 4t^2 - 5 \\ y &= 2 - 3t^3 \end{aligned}$$

where x and y are in metres, t is in seconds.

1. Find the magnitude and direction of its velocity when $t = 1.25$ s. (Give the answers to 3 significant digits)
2. Find the acceleration (x and y components) at $t = 1.25$ s.

Projectiles Projectile motion is the special case of curvilinear motion which occurs when an object is shot or thrown with an initial velocity $\boldsymbol{v_0} = (v_{0x}, v_{0y})$ and subsequently only feels a downward acceleration (negative y direction) due to gravity.

The motion of a projectile moving in the earth's gravitational field is then described by the following equations for x and y:

$$\begin{aligned} x &= v_{0x}t \\ y &= v_{0y}t - \frac{gt^2}{2} \end{aligned}$$

Here v_{0x} is the initial velocity x component, v_{0y} is the initial velocity y component, g is the acceleration of gravity (9.8 m/s^2 or 32 ft/s^2). These are all constants while t is time. One can solve for the velocity and acceleration of a projectile once and for all by differentiating components. The veolocity $\boldsymbol{v} = (v_x, v_y) = (dx/dt, dy/dt)$ has components:

$$\begin{aligned} v_x &= v_{0x} \\ v_y &= v_{0y} - gt \ , \end{aligned}$$

while the acceleration $\boldsymbol{a} = (a_x, a_y) = (dv_x/dt, dv_y/dt)$ has components:

$$\begin{aligned} a_x &= 0 \\ a_y &= -g \end{aligned}$$

Example

An arrow is shot with initial speed of 220 ft/s at an angle of 33.0°. Find the position and velocity (components) at time $t = 4.00$ s

5.2.3 Rotational motion

An object that is rotating, such as a horse on a merry-go-round, can have its position specified by the angle θ it makes with a fixed set of coordinates with origin at the centre of rotation. If the merry-go-round is moving in time then one has the angle $\theta(t)$ at time t.

The angular velocity, ω, of a rotating body is defined as the instantaneous rate of change of the angular displacement, θ.

$$\boxed{\omega = \frac{d\theta}{dt}} \tag{4}$$

The angular acceleration, α, of a rotating body is defined as the instantaneous rate of change of the angular velocity, ω.

$$\boxed{\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}} \tag{5}$$

The SI units of angular velocity are rad/s, and the SI units of angular acceleration are rad/s^2.

Example

The angular displacement of a rotating body is given by

$$\theta = 184 + 271t^3$$

Find the angular velocity and angular acceleration at t = 1.25 s.

Reading:
Sec. 29-1; 29-2
Problems:
Exercise 1, p. 830 # 1-7 (odd); Exercise 2, p. 837 # 1-19, 22, 24

6 Related Rate Problems

If a problem involves two variables, both of which vary with respect to time, and the relationship between the two variables is known, it is possible to express the time rate of change of one in terms of the time rate of change of the other. This is done by differentiating the equation relating the variables with respect to time. The steps to be followed are outlined on page 838 of the text and are summarized on page 33 at the end of this unit.

Examples

1. A leaking pipe drips water on a floor creating a circular pool of water whose area is increasing a constant rate of 2.00 cm^2/s. Find the rate at which the radius of the pool is changing when the radius is 5.00 cm.
2. A spherical balloon is being blown up at a constant rate of 2.00 m^3/min. Find the rate at which the radius is increasing when it is 3.00 m.
3. A tank in the shape of an inverted cone has a radius of 8.00 m at the top and a height of 15.0 m. At the instant when the water in the tank is 5.00 m deep, the surface level is rising at a rate of 0.500 m/min. Find the rate at which water is being added.

Reading:
Sec. 29-3
Problems:
Exercise 3, p. 841 # 1-29 (odd) – except # 13

7 Optimization Problems

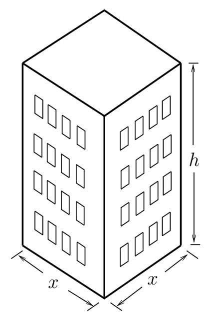
Optimization problems require us to find the value of the independent variable that results in either a maximum or minimum value of the dependent variable. The steps for solving optimization problems are given on page 845 of the text and summarized at the end of these notes on page 33. Effectively one is just finding the maxima and minima of functions derived from a word problem.

Examples

1. What are the dimensions of the rectangle with the largest area that can be enclosed with a perimeter of 16.0 m?

2. An architect is designing a rectangular building in which the front wall costs twice as much per linear metre as the other 3 walls. The building is to cover 1350 m^2. What dimensions must it have to minimize the cost of the walls?

3. A construction company desires to build an apartment building in the shape of a rectangular parallelpiped (shown) with fixed volume of $32\bar{0}00$ m^3. The building is to have a square base. In order to minimize heat loss, the total above ground surface area (the area of the four sides and the roof) is to be minimimized. Find the optimal dimensions (base length x and height h) of the building.



4. A can is to be made in the shape of a right circular cylinder. Find the optimal dimensions (radius r and height h) which maximize its volume if it is to have a fixed total surface area a. (Your expressions for r and h should be in terms of the constant a.) Find the ratio $\frac{h}{r}$ of this optimal can. Why would a manufacturer want to have a maximal container volume for a fixed surface area?

Reading:
Sec. 29-4
Problems:
Ex. 4 (p. 850) # 7, 9, 11, 17, 27, 29

8 The Differential of a Function

8.1 Definition

The *differentials* dx and dy for a function $f(x)$ are defined as follows:

dx is an arbitrary change in x (analagous to an increment Δx).

dy is defined by the equation

$$\boxed{dy = f'(x)dx} \tag{6}$$

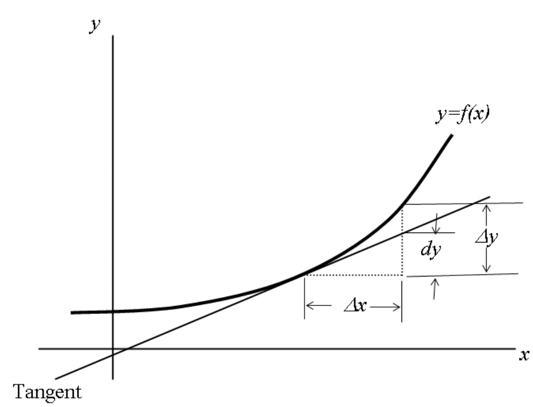
Note that the above equation can be rearranged to write the ratio of dy to dx as follows:

$$\frac{dy}{dx} = f'(x)$$

Thus the derivative, $\frac{dy}{dx}$, can now be viewed as the quotient of the two differentials, dy and dx.

Geometric meaning of the increment, Δy and the differential, dy:

- Δy is the change in y on the curve for a given change in x, Δx.
- dy is the change in y along the tangent for a given change in x, dx (or Δx).



Examples

Find the differential dy for each of the following:

1. $y = 4x^3 - x^2$
2. $y = \dfrac{x}{2x+1}$
3. $x^2 - 2xy + y^3 = 7$

8.2 Estimating Small Changes in a Function Using Differentials

If Δx is small, then the increment Δy is approximately equal to the differential dy, i.e., setting $dx = \Delta x$ in our differential formula

$$dy = \frac{dy}{dx}dx$$

gives

$$\Delta y \approx \frac{dy}{dx}\Delta x \ .$$

The calculation of the differential is often simpler than calculation of the increment, as illustrated in the first example below.

Examples

1. Estimate the change in the function $y = 3x^2$ when x changes from 3.00 to 3.01. Compare the estimate with the exact value of Δy.
2. A metal ball when heated increases in radius from 1.250 cm to 1.252 cm. Estimate the resulting change in volume.

8.3 Approximate Volumes of Shells and Rings

The volume of a thin shell or ring is estimated as the change in volume when the radius (or other linear dimension) increases by an amount equal to the wall thickness. In symbols,

$$\Delta V \approx \frac{dV}{dr}\Delta r$$

Examples

1. (a) Derive a formula for the volume of material in a closed thin-walled cubical box of edge length L and wall thickness t.
   (b) Use the formula to estimate the volume of material in a cubical box 75.0 cm on a side with walls 0.240 cm thick.
2. (a) Derive a formula for the volume of material in a metal ring of radius r, height h and thickness t.
   (b) Use the formula to find the volume of material in a ring 25.0 cm in diameter, 3.00 cm high and 0.120 cm thick.

8.4 Errors in Calculated Quantities

If a measured quantity has an *absolute error* Δx, a quantity y calculated from it will have an absolute error given, to a good approximation, by

$$\boxed{\Delta y \approx \frac{dy}{dx}\Delta x} \tag{7}$$

The *relative error*, $\Delta y/y$, can be approximated by dividing the right hand side of the last equation by y.

Examples

1. What error in the area of a square room results from an error of 0.02 m in the length, which is measured as 15.25 m?
2. To what percent accuracy must the radius of a ball bearing be measured so the calculated volume will be correct to 0.3%?

Exercise

1. Find the differential of each of the following functions:
   (a) $y = 3x^2 + 6$
   (b) $y = (x^2 - 1)^4$
2. Find Δy and dy for each of the following:
   (a) $y = 7x^2 + 4x$; $x = 4$, $\Delta x = 0.2$
   (b) $y = (1 - 3x)^5$; $x = 1$, $\Delta x = 0.01$
3. Estimate the volume of rubber in a rubber-covered ball 25.0 cm in diameter if the rubber is 0.125 cm thick. (The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.)
4. If the measurement of the radius of a circular patio has an error of 0.50 cm, what is the error in the calculated area if the radius is measured as 1.35 m? (Area of a circle is given by $A = \pi r^2$.)

Reading:
Sec. 27-6
Problems:
Ex. 6 (p. 801) # 29-39 (odd)

Answers

1. (a) $dy = 6x\,dx$
   (b) $dy = 8x(x^2 - 1)^3\,dx$
2. (a) $\Delta y = 12.28$, $dy = 12$
   (b) $\Delta y = -2.473$, $dy = -2.4$
3. 245 cm^3
4. 0.042 m^2

Review Problems: Applications of the Derivative

1. Write the equations, in general form, of the tangent and normal to the curve $y^2 = 4x$ at the point $(1, 2)$.

2. Use Newton's method to find the positive root of the equation

$$x^3 + 3x - 8 = 0$$

   to 2 decimal places.

3. Find the maximum, minimum and inflection points of the curve

$$y = x^3 - 3x + 4$$

4. Using curve sketching techniques demonstrated in class, sketch the graph of the function $y = \dfrac{3}{x} + x^2$ (give critical points in decimal form to 3 significant digits).

5. The height h (in kilometres) to which a balloon rises in t minutes is given by the formula

$$h = \frac{10t}{\sqrt{4000 + t^2}}$$

   At what rate is the balloon rising at the end of 30 minutes?

6. The horizontal and vertical displacements of a projectile moving in the earth's gravitational field are given by

$$x = v_{0x}t \quad y = v_{0y}t - \frac{gt^2}{2}$$

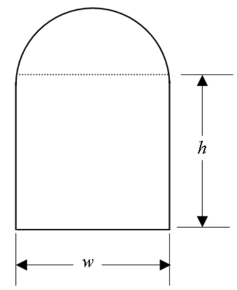
   where v_{0x} and v_{0y} are the initial horizontal and vertical velocities respectively, and $g = 9.80$ m/s^2. A cannonball is fired with an initial velocity of 175 m/s at an angle of 45.0° to the horizontal. Find its horizontal and vertical velocities after 10.0 s.

7. A ladder 10.0 m long is leaning against a vertical wall. If the foot of the ladder is being pulled away from the wall at 0.500 m/s, how fast is the top of the ladder moving down the wall when the foot of the ladder is 2.00 m from the wall?

8. A rope attached to a boat is being pulled in by a winch at a rate of 2.50 m/s. If the boat is 5.00 m below the level of the winch, how fast is the boat approaching the wharf when 13.0 m of rope are out?

9. Sand is emptied down a chute at the rate of 3.00 m^3/s, forming a conical pile whose height is always twice the radius. At what rate is the radius changing when the height is 2.00 m? (Volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$.)

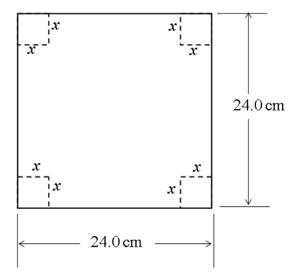
10. A window at the front of a house with a cathedral ceiling is to be designed in the shape of a rectangle with a semicircle on top, as shown. If the perimeter of the window is to be 8.00 m, find the dimensions, w and h, of the window that will maximize the area of the window.



11. An open rectangular box is to be formed by cutting a square from each corner of a square piece of cardboard 24.0 cm on each side, as shown. After the squares are removed the sides are folded up to form the box.

(a) How large a square must be removed from each corner in order for the box to have the maximum volume?

(b) What will the maximum volume of the box be?



12. A coat of paint 0.050 mm thick is to be applied to the outside surface of a spherical storage tank with a radius of 20.0 m. Use differentials to estimate the number of litres of paint that will be needed.

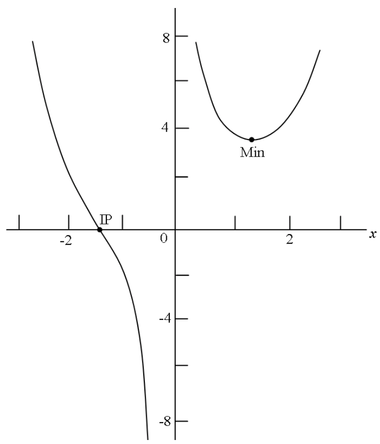
    (Volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.)

13. A disk-shaped solar cell has a measured radius of 120.0 mm. The error in measurement of the radius of the cell is 0.050 mm. Use differentials to estimate:

    (a) The absolute error in the area of the top surface of the solar cell

    (b) The relative (percent) error in the area of the top surface of the solar cell

Answers:

1. Tangent: $x - y + 1 = 0$; Normal:$x + y - 3 = 0$
2. $x = 1.51$
3. Maximum: $(-1, 6)$; Minimum: $(1, 2)$; Inflection point: $(0, 4)$
4. Minimum: $(1.14, 3.93)$; Inflection Point: $(-1.44, 0)$
   Plot:
   

5. 0.117 km/min
6. $v_x = 124$ m/s, $v_y = 25.7$ m/s
7. 0.102 m/s
8. 2.71 m/s
9. 0.477 m/s
10. $w = 2.24$ m, $h = 1.12$ m
11. (a) 4.00 cm
    (b) 1020 cm^3
12. 251 L
13. (a) 38 mm^2
    (b) 0.083%

Additional Review Problems

- Chapter 28 Review (P. 826) # 1-16 (all except # 12)
- Chapter 29 Review (p. 854) # 3, 5, 6, 7, 11, 13, 19, 21

Even-Numbered Answers

- Chapter 28:

  2. (0,0)

  4. Maximum: (-1, 5.33); Minimum: (3, -5.33); Inflection Point: (1,0)

  6. (2,3)

  8. Graph

  10. Decreasing

  14. x-intercept of tangent: (1.29, 0), x-intercept of normal: (818, 0)

  16. Graph

- Chapter 29:

  6. $v = 10$ units/s, $a = 8$ units/s^2

Formulas

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

Equations of Motion

$$v = \frac{ds}{dt} \tag{2}$$

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2} \tag{3}$$

$$\omega = \frac{d\theta}{dt} \tag{4}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{5}$$

Differentials

$$dy = f'(x)dx \tag{6}$$

$$\Delta y \approx \frac{dy}{dx}\Delta x \tag{7}$$

Systematic Procedure for Critical Points

Stationary Points (Maxima, Minima):

1. Write down the original function y, the first derivative y', and the second derivative y''.
2. Set the first derivative, y', to zero and solve for x. This gives the x-coordinates of any stationary points.
3. Substitute these x-values into the original function, y, to find the y-coordinates of the stationary points.
4. Perform the **second derivative test** on each stationary point by substituting its x-value into the second derivative, y''.
   - If $y'' > 0$, the point is a minimum point.
   - If $y'' < 0$, the point is a maximum point.
   - If $y'' = 0$, the test is inconclusive.
5. If y'' turns out to be zero, use one of the following tests to determine if the stationary point is a maximum, a minimum or neither:

   First derivative test: The first derivative y' is:
   - negative to the left and positive to the right of a minimum point.
   - positive to the left and negative to the right of a maximum point.
   - the same sign on both sides of a stationary point that is neither a maximum nor a minimum.

   Ordinate test: the y-value is:
   - greater on both sides of a minimum point.
   - less on both sides of a maximum point.
   - greater on one side and less on the other side of a stationary point that is neither a maximum nor a minimum.
6. Write down the coordinates of each stationary point, and state whether each one is a maximum, a minimum or neither.

Inflection Points:

1. To find inflection points, set the second derivative equal to zero and solve for x.
2. Check to see whether the second derivative changes sign on either side of the point(s).
3. Substitute the x-coordinates of any inflection points into the original function to find the y-coordinates of the inflection points.
4. Write down the x and y-coordinates of the inflection points.

Steps for Curve Sketching

1. Check to see if the equation represents a standard type of curve (e.g. power function, exponential function, etc.).
2. Determine the x and y-intercepts (if possible).
3. Determine whether the curve is symmetric about the x-axis, y-axis or origin.
4. Determine the domain and (if possible) the range of the function.
5. Determine whether there are any vertical or horizontal asymptotes.
6. Determine what happens to the function for large (positive and negative) x-values.
7. Locate any stationary points, and determine if they are maximum or minimum points, or neither.
8. Determine for what x-values the function is increasing or decreasing.
9. Locate any inflection points.
10. Determine for what x values the curve is concave upward or concave downward.

Steps for Related Rate Problems

1. Locate the given rate and express it as a derivative with respect to time.
2. Determine the unknown rate and express it as a derivative with respect to time.
3. Find an equation linking the variables in the known and unknown rates. If there are other variables in the equation, eliminate them using other known relationships.
4. Take the derivative of the equation with respect to time.
5. Subsitute known values and solve for the unknown rate.

Steps for Optimization Problems

1. Locate the quantity (for purpose of these steps call it y) to be maximized (or minimized).
2. Locate the quantity (call it x) which is to be varied to maximize (or minimize) y.
3. Write an equation linking y and x. If the equation contains any other variables, eliminate them using a second equation. A graph of y versus x would show maximum (or minimum) points.
4. Take the derivative $\frac{dy}{dx}$.
5. Set the derivative equal to zero and solve for x. Find the corresponding y values for these stationary points.
6. Check the stationary points found to see if they are relative maxima (or minima) using either the second derivative test, the first derivative test or the ordinate test.
7. Find the absolute maximum (or absolute minimum) by comparing y values of the relative maxima (or relative minima) found. Also check the y values at the endpoints of the physically allowed values of x since the largest (or smallest) y value may also lie there.

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