SIAST Palliser Campus

Mathematics
MAT235

Lecture Notes and Examples

Unit 4

Linear Algebra

UNIT 1

Linear Algebra

Lecture #1	Determinants and Their Source.	P 279 - 300
Lecture #2	Cramer's Rule for 2 x 2 Systems.	P 259 – 260
Lecture #3	N x N Linear Systems and Determinants.	P 285 – 293
Lecture #4	Matrices and Matrix Operations.	P 319 - 334
Lecture #5	Matrices and Linear System solutions, introduction	Notes
Lecture #6	Matrices and Linear System solutions by inverse matrix operations.	P 322 – 324
Lecture #7	Determining the parameters of conic section curves based on known points using matrices.	Notes
Lecture #8	Translations and rotations by matrix operators.	Notes
Lecture #9	Applied problems using matrix operators	Notes

Unit Learning Objectives

A proficient student will be able to:

1. Evaluate second order determinants.
2. Evaluate third order determinants using calculator or PC.
3. Write a linear system of any dimension in standard form.
5. Solve a linear system of third order using determinants.
6. Identify various types of arrays, matrices, and vectors, and give their dimensions.
7. Add, subtract and multiply matrices both manually and using the matrix feature of the calculator.
8. Write a linear system in matrix format.
9. Compute the inverse of a matrix using calculator or PC and by the adjoint/transpose method for a 2 by 2 matrix.
10. Solve linear systems by and matrix operations.
11. Evaluate the parameters of conic section curves using matrix operations.
12. Apply matrix algebra to perform Euclidean transformations.
13. Apply matrix algebra to solving applied problems.

Lecture #1 **Determinants And Their Source**

Definition

A determinant is a **square** array of numbers that represents a single value. The array is denoted by a capital letter and an absolute value sign. For example | **B** | denotes a determinant named B.

Examples:

$$|A| = \begin{vmatrix} 3 & 0 & 1 \\ 7 & 2 & 4 \\ 9 & -1 & 6 \end{vmatrix} \qquad |V| = \begin{vmatrix} 3 & 0 \\ 7 & 2 \end{vmatrix}$$

The example on the left is called a 3 by 3 determinant; the one on the right is a 2 by 2 determinant. In a determinant, the number of rows must match the number of columns in order for it to be a square array. The # of rows x # of columns is called the **dimension** of the determinant. The values in the array are called **elements**.

Evaluating a Determinant

Evaluating a determinant means finding the value which the determinant represents. We can use a PC, a calculator or manual methods to evaluate.

PC Process for Evaluating a Determinant

1. Using Excel, or its equivalent (Openoffice Calc), enter the determinant array into the cells required. One element per cell.
2. In a different cell, find the result by entering the formula =mdeterm(alpha, beta), where alpha and beta are the ends of two diagonally opposite corners.
3. Pressing enter gives the value of the determinant.

** This process will be important for more complex determinants.

Calculator Process for Evaluating a Determinant

For the HP 50 Calculator:

1. Enter the array on the Matrix Writer of the HP calculator
2. **Enter** the array on the stack.
3. Choose the **MTH** menu.
4. In the Math menu, select the **DET** function to evaluate the determinant.

For the Sharp EL-546W Calculator:

See the appended two pages taken from the manual. Note the second page (the example) is likely more useful.

** This process will be important for more complex determinants.

Manual methods

See the Calter text Page 280 for the shortest method to evaluate a 2x2 determinant.

Evaluation of the previous examples returns: | **A** | = 23 and | **V** | = 6. Try these on your own with each of the two mechanical processes. Try the manual approach for | **V** |.

Evaluating determinants without the help of a tool like a calculator or PC gets intricate and time-consuming very quickly as the determinant's dimension grows. We will examine the manual methods for smaller determinants only.

The Mathematical Source of Determinants: Cramer's Rule

Recall:

1. In a linear equation all variable terms are of degree 1, i.e. all exponents are 1.
2. To solve a linear system with N variables requires N independent equations.
3. A linear system can be solved for a variable by eliminating all other variables.

Example:

Solve the following linear system for X:

		Eq #	
	$2X - 3Y = 1$	(1)	Eliminate Y
	$X + 4Y = -5$	(2)	
New Eq#		Process	
(3) ---->	$8X - 12Y = 4$ ---->	(1) x 4	
(4) ---->	$3X + 12Y = -15$ ---->	(2) x 3	
(5) ---->	$11X = -11$ ---->	(3) + (4)	
Solution:	$X = -11/11 = -1$ ---->	(5)/11	

A similar process yields the value for Y or we can substitute the X value into any equation and find the Y value. Soon we will generalize this process to find a faster solution.

Exercises

Solve these linear systems using the addition/subtraction elimination method.

$x + y = 3$
$x - y = 1$ ans(x, y) = (2, 1)

$2x + 3y = 13$
$4x + 5y = 25$ ans(x, y) = (5, 1)

Use addition/subtraction method on the text's exercises on two by two linear equation systems until you reach a comfort zone. You can challenge yourself with a few three by three systems.

In **general** a 2 x 2 linear system can be written in **standard form** as below where the a, b, c values are constants and x, y values are variables:

$$a_1X + b_1Y = c_1 \qquad (1)$$
$$a_2X + b_2Y = c_2 \qquad (2)$$

Solve this system for X by eliminating Y.

(3)	$b_2a_1X + b_2b_1Y = b_2c_1$	---->	(1) * b_2
(4)	$b_1a_2X + b_1b_2Y = b_1c_2$	---->	(2) * b_1
(5)	$b_2a_1X - b_1a_2X = b_2c_1 - b_1c_2$	---->	(3) - (4)
	$(b_2a_1 - b_1a_2)X = b_2c_1 - b_1c_2$	---->	factor LHS of (5)

Solve for X $\quad X = \dfrac{b_2c_1 - b_1c_2}{b_2a_1 - b_1a_2}$

Notice that this solution for X can be written in terms of determinants where:

$$b_2\,c_1 - b_1\,c_2 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \text{and} \quad b_2\,a_1 - b_1\,a_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Solve the same system for Y by eliminating X. This will produce the following result:

$$Y = \frac{a_2c_1 - a_1c_2}{b_1a_2 - b_2a_1}$$

Notice that the denominator for Y is identical to that for X with the exception of a factor of -1. The denominators for X and Y can be made identical by multiplying the denominator and the numerator for Y by a factor of -1. This produces the following result:

$$Y = \frac{a_1c_2 - a_2c_1}{b_2a_1 - b_1a_2}$$

These solutions always work for a 2 by 2 system. It’s just hard to remember the formulas. Remembering the formulas is easier when they are written in determinant form where:

$$a_1c_2 - a_2c_1 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \quad \text{and} \quad b_2a_1 - b_1a_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Note the position and source of the products. The first, a1c2, is called the principal diagonal. The second, a2c1, is called the minor or secondary diagonal.

The following result is called **Cramer's Rule** for a 2 x 2 linear system.

Given a 2 x 2 linear system of equations in **standard form**:

$$a_1X + b_1Y = c_1$$
$$a_2X + b_2Y = c_2$$

The solution is:

$$X = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad Y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Notice how the pattern of each determinant relates to the format of the system in standard form. The determinant in the denominator is called the **determinant of the system.** It is designated by the Greek letter **Δ,** (delta).

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

If **Δ** = 0, the values of X and Y cannot be found since division by 0 is not permitted. The explanation for the occurrence of a **Δ** = 0 relates to the graph of the system of linear equations.

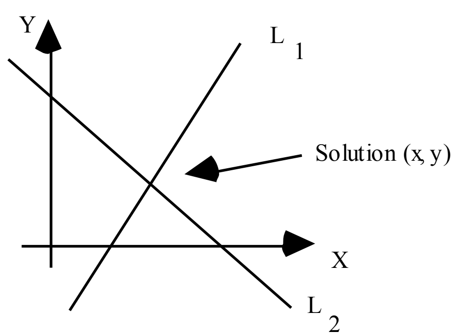
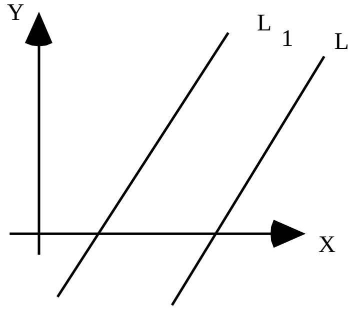


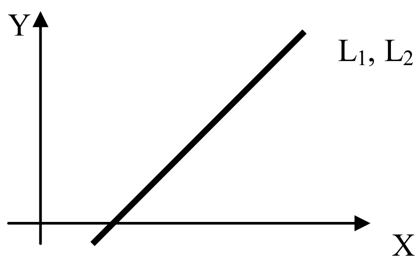
Figure #1: Consistent System

$\mathbf{\Delta \neq 0}$



$\Delta = 0$

Figure #2: Inconsistent System



Numerator & Denominator in Cramer's Rule are both 0.

Figure #3 Dependent System

When solving a linear system, always compute Δ first. The system may not have a unique solution, which means that you don't have to do more.

Example:

Solve the system $\quad 2X - 3Y = 1 \quad$ By Cramer's Rule.

$\quad X + 4Y = -5$

For this system $\Delta = \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = (2)(4) - (1)(-3) = 11$

This is a consistent system; it has a unique solution.

The solutions are:

$$X = \frac{\begin{vmatrix} 1 & -3 \\ -5 & 4 \end{vmatrix}}{11} = \frac{-11}{11} \quad \& \quad Y = \frac{\begin{vmatrix} 2 & 1 \\ 1 & -5 \end{vmatrix}}{11} = \frac{-11}{11} = -1$$

Check the solution by substituting back into both equations. This point must lie on both lines.

Exercises

Solve these linear systems using the determinant method.

x + y = 3	2x + 3y = 13	x + y = 3	x + y = 2
x - y = 1	4x + 5y = 25	2x + 2y = 6	x + y = 4
ans(x, y) = (2, 1)	ans(x, y) = (5, 1)	Dependent	Inconsistent

Use the text's exercises on two by two linear equation systems until you reach a comfort zone. You can challenge yourself with a few three by three systems. This time it's far less challenging.

The Mathematical Source of N x N Determinants
The standard form of an N x N linear system has this format:

$$
\begin{array}{ccccccccccc}
a_{11}X_1 & + & a_{12}X_2 & + & ---- & + & a_{1n}X_n & = & c_1 & \text{--->(1)} \\
a_{21}X_1 & + & a_{22}X_2 & + & ---- & + & a_{2n}X_n & = & c_2 & \text{--->(2)} \\
---- & --- & ---- & ---- & ---- & ---- & ---- & ---- & ---- & \\
---- & --- & ---- & ---- & ---- & ---- & ---- & ---- & ---- & \\
a_{n1}X_1 & + & a_{n2}X_2 & & ---- & + & a_{nn}X_n & = & c_n & \text{--->(n)}
\end{array}
$$

The doubly subscripted notation, a_{ij}, is used to identify the location within the array for the constant coefficients. The letter i denotes the row and j denotes the column.

Example:
a_{12} refers to the coefficient of the second variable in the first equation.
a_{ij} refers to the coefficient of the j^{th} variable in the i^{th} equation.

The determinant of this system, **Δ**, is found by the same process as was used for the simple 2 x 2 case.

$$
\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}
$$

To solve this N x N system, the value of **Δ** must first be found. **Δ** is evaluated differently than a 2 x 2 determinant because all elements must be considered.

The 3 x 3 system is often written without the doubly subscripted notation because only 3 symbols are required for the 3 variables involved. See the text.

Notice that the fractions in the solution to the 3 x 3 system all have the same denominator just as in the 2 x 2 case. This denominator, **Δ,** can be expressed as a 3 x 3 determinant.
Example:

Find **Δ** for the following linear system:

$$
\begin{aligned}
X + 2Y + 3Z &= 14 \\
2X + Y + 2Z &= 10 \\
3X + 4Y - 3Z &= 2
\end{aligned}
$$

$$
\boxed{D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 4 & -3 \end{vmatrix}}
$$

Each third order linear equation describes a two dimensional plane in space. Three two dimensional planes have exactly one point in common in a consistent system.

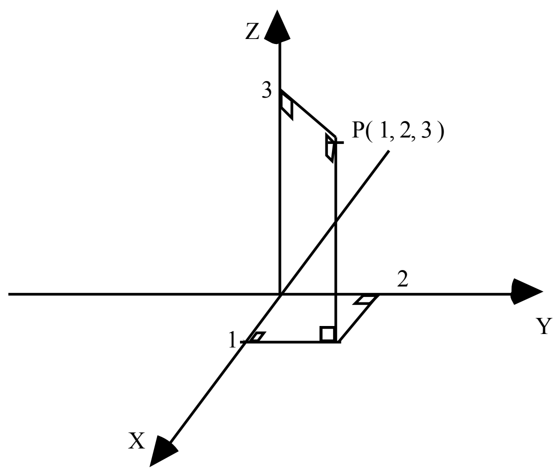
Evaluate Δ using the calculator or PC. Here $\Delta = 28$, so the system is consistent and has a unique solution. The solution to this system is an ordered triple, (X, Y, Z), a point in space. It's also a vector with one end at the origin and the other at the point (X, Y, Z),

To solve this system by Cramer's Rule replace the appropriate columns by the constants and divide by Δ.

$$X = \frac{\begin{vmatrix} 14 & 2 & 3 \\ 10 & 1 & 2 \\ 2 & 4 & -3 \end{vmatrix}}{28} = \frac{28}{28} \qquad Y = \frac{\begin{vmatrix} 1 & 14 & 3 \\ 2 & 10 & 2 \\ 3 & 2 & -3 \end{vmatrix}}{28} = \frac{56}{28}$$

$$Z = \frac{\begin{vmatrix} 1 & 2 & 14 \\ 2 & 1 & 10 \\ 3 & 4 & 2 \end{vmatrix}}{28} = \frac{84}{28}$$

Here is the solution: (X, Y, Z) = (1, 2, 3)



The Mathematics Required to Evaluate a Determinant of Order More Than 2.
All elements of a determinant are involved in its evaluation, therefore the method of finding products along the diagonal does not work for orders higher than 2. There is a special "trick" that can be used on order 3 determinants but it does not work on orders higher than 3.

Special Method for Order 3

1. Rewrite the first 2 columns to the right of the determinant.
2. Multiply elements starting in row 1 diagonally downwards and sum the products.
3. Multiply elements starting in row 3 diagonally upwards and sum the products.
4. Subtract the sum in 3 from the sum in 2.

Example:

9 8 - 12 = 5, **M**inor diagonal is **M**inus

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 4 & -3 \end{vmatrix} \quad \text{rewrite as} \quad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 4 & -3 \end{vmatrix}\begin{matrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{matrix} \qquad \text{Value} = 33 - 5 = 28$$

-3 12 24 = 33, **P**rincipal diagonal is **P**lus

Try several examples and verify with the calculator or PC.

The next section begins matrices, a different but closely related topic.

Lecture #4 **Matrices and Matrix Operations**

Definition

A **matrix** is a rectangular array of values. It does not represent a value as a determinant does. Instead, it represents a mathematical operation or a process applied to a system.
Matrices are found in many types of calculations. We will study them from the point of view of the solution to an N x N linear system as we did in the case of determinants.

The dimension of a matrix specifies the number of rows and columns. The number of rows does not need to equal number of columns in a matrix.

Matrices are identified by a capital letter. The matrix can be included in square brackets or within round parentheses, e.g. {}, (). It cannot be included between absolute value signs because this represents a determinant. It is unusual to see a matrix with missing row or column elements.
Example

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 9 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 9 & 6 \end{bmatrix}$$

2 rows by 3 columns
Dimensions (2 x 3)

1 row by 3 columns
Dimensions (1 x 3)

Unlike determinants, a matrix need not have square dimensions. A matrix which consists of a single row or a single column is called a **vector**. Matrix **B**, above, is a **row vector.**

An element in a vector needs only a single subscript to identify its location but an element in a table requires a double subscript as in a determinant.
Example

$$\begin{bmatrix} 1 & 9 & 6 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 1 \\ 9 & 0 & 6 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Note: $a_3 = 6 \quad a_{21} = 9$

Equality of Matrices

Two matrices **A** and **B** are equal if $a_{ij} = b_{ij}$ for all i, j.

Example:

$$\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} X & Y \\ 4 & 7 \end{bmatrix} \quad \text{if } X = 2 \text{ and } Y = 3$$

Special Matrices

There are some matrices which have special properties that are useful in matrix applications.

Null Matrix

O is the null matrix if $a_{ij} = 0$ for all i, j. That is all elements are 0.

Example

$$\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Unit Matrix

This matrix is square and has the property that $a_{ij} = 1$ if i = j else $a_{ij} = 0$. This means that all elements along the diagonal are ones and all other elements are zeros.

The unit matrix is always identified by the symbol **I.** Example

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The Transpose of a Matrix

$\mathbf{A}'$ is the transpose of **A** if $a_{ij}' = a_{ji}$ for all i, j. This means that rows are interchanged with the corresponding columns. That is row one with column one and etc.

Example

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \\ 4 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{A}' = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 7 \end{bmatrix}$$

Operations with Matrices

Arithmetic-like operations can be performed with matrices.

Addition and Subtraction

$$\mathbf{A} \pm \mathbf{B} = \mathbf{C} \quad \textbf{where} \quad c_{ij} = a_{ij} \pm b_{ij}$$

Note: The dimensions of A and B must be identical for addition & subtraction to work.

Example

$$\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 7 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 4 & 8 \end{bmatrix}$$

Addition and subtraction of matrices obey the same properties that addition and subtraction of scalars obey with respect to order and grouping. It doesn't matter what order matrices are added, the sum is the same. It doesn't matter how the matrices are grouped when adding, the sum is the same.

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$
$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

The Product of a Scalar, k, and a Matrix, A

k**A = B** where $b_{ij} = k\, a_{ij}$ for all i,j. This means that all elements of the matrix are multiplied by the scalar.

Example

$$3 \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 6 & 3 \end{bmatrix}$$

Multiplication of Matrices

Only **conformal** matrices can be multiplied. Matrices are said to conform when the number of rows in one equals the number of columns in the other.

Example

$$\mathbf{A} = \begin{bmatrix} 2 & 7 \\ 4 & 3 \\ 1 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

A, B can be multiplied

A is a (3 x 2) matrix. B is a (2 x 3) matrix. Thus they can be multiplied, because of the column count on **A** and the row count on **B** are equal. The result is a (3 x 3) matrix because the column count of A and the row count on B are both two. If the column count in **B** were 17, the result of multiplication would be a (3 x 17) matrix.

In the above example the product **A** x **B** can be found; but the product **B** x **A** can not be found. Think of multiplication as the application of one matrix to the other. **B** can be applied to **A** but **A** cannot be applied to **B**. Here is a demonstration of the process:

Here is how to find **A** x **B**.

$$\mathbf{A} \xrightarrow{\text{input}} \boxed{\begin{matrix}\text{Apply}\\ \text{Operation}\\ \text{B}\end{matrix}} \xrightarrow{\text{output}} \mathbf{C}$$

Here is how the process works:

apply the first column to
each row in turn then apply the second
column to each row in turn.

$$\begin{bmatrix} 2 & 7 \\ 4 & 3 \\ 1 & 2 \end{bmatrix} \text{ x } \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2(1)+7(2) & 2(3)+7(1) \\ 4(1)+3(2) & 4(3)+3(1) \\ 1(1)+2(2) & 1(3)+2(1) \end{bmatrix} = \begin{bmatrix} 16 & 13 \\ 10 & 15 \\ 5 & 5 \end{bmatrix}$$

Notice that the product has the same number of rows as **A** and the same number of columns as **B**. This is true in general. If **A** is of dimension **n x m** and **B** is of dimension **r x s** and **m = r**, then **A x B** will be of dimension **n x s**.

If you try to multiple **B** x **A,** you'll see that the rows and columns don't align.

Verify **A x B** on the calculator.
1. Enter **A** first.
2. Enter **B** second.
3. Then multiply to find the product C, the solution above.

Now try multiplying **B** x **A** on the calculator. Notice that the calculator cannot do this because of what it lists as a dimension error. To see why this is so, try the above procedure in the reverse order.

$$\mathbf{B} \xrightarrow{\text{input}} \boxed{\begin{matrix}\text{Apply}\\ \text{Operation}\\ \text{A}\end{matrix}} \xrightarrow{\text{output}} \mathbf{C}$$

apply the first column to
each row in turn then apply the second
column to each row in turn.

$$\underset{\mathbf{B}}{\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}} \text{ x } \underset{\mathbf{A}}{\begin{bmatrix} 2 & 7 \\ 4 & 3 \\ 1 & 2 \end{bmatrix}}$$

The rows in **B** have 2 elements but the columns in **A** have 3 elements so there is no element available in a row of **B** to multiply the last element of a column of **A.** The product is not defined because of this mismatch in dimensions.

For a matrix product **A x B** may be defined but **B x A** is not necessarily defined. The product is defined in both directions if all matrices are square in dimensions. The product will be defined in both directions but it will not necessarily be the same product in both directions. This is a different property than that for scalars. Order is not important in scalar multiplication.

Matrix multiplication on a PC with Excel

Try finding **A** x **B** and **B** x **A with** Excel.

1. Enter the matrix **A** from above with each number in a separate cell.
2. Repeat for matrix **B.**
3. Determine the size of the product. Here it's 3x2, three rows and two columns.
4. Highlight an area of that size and shape in the area where you want the resultant matrix.
5. Enter the matrix multiplication formula, =MMULT(ARRAY_A,ARRAY_B) in the upper left of this resultant region. ARRAY_A, for example, is entered by providing the two diagonally opposite boundaries of the region where you entered matrix **A.**
6. After you've entered the two ranges, close the parentheses.
7. Press and hold CTRL, the SHIFT, then ENTER. Release
8. The resultant region now contains the product of the two constituent matrices.

Next we'll investigate how to solve system of linear equations with matrices.

Exercises in the text. Try enough to get comfortable with each of the three methods. We can use either matrices or determinants to solve complex systems of linear equations. Both processes benefit with the application of good mechanical or paper-based tools.

Note that the free office suite OpenOffice contains a spreadsheet equivalent to Excel which can perform these calculations. To download it visit:
http://www.openoffice.org

Using Wikipedia, your textbook, these notes or other resources, determine the definitions of the following terms; square matrix, row matrix, column matrix, diagonal matrix, triangular matrix, identity matrix, zero matrix, conformable matrix, transpose of a matrix, symmetric matrix, and inverse of a matrix.

Examples of internet sources are
http://en.wikipedia.org/wiki/Conformable_matrix
http://en.wikipedia.org/wiki/Matrix_(mathematics)
http://en.wikipedia.org/wiki/Gauss_Jordan_elimination
http://www.ping.be/~ping1339/stels2.htm

Lecture #5 **Matrices and Linear Systems**

An N x N linear system can be represented in matrix form using the operation of a matrix product. We'll use this process to solve an N x N system of linear equations

Example

Here is an example of a 2 x 2 linear system represented in matrix form.

$$2x - 3y = 5$$
$$4x - 1y = 7$$

Define the following matrices:

$$A = \begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad C = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Matrix **A,** similar to the delta from the determinant section**,** is called the coefficient matrix, matrix **X** is called the variable vector and vector **C** is called the constant vector. Multiplying matrix A by matrix X gives the left side above, and Matrix C is the right side.

The form below is called the **compact form** of the above linear system:

$$\boxed{\mathbf{AX = C}}$$

Notice that by substitution for **A, X** and **C,** and then by matrix multiplication:

$$\mathbf{AX} = \begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 2\mathbf{x} - 3\mathbf{y} \\ 4\mathbf{x} - 1\mathbf{y} \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \mathbf{C}$$

There are two main methods for solving a system of linear equations using matrices; Gauss and multiplication by the inverse. The Gauss method is not part of the evaluation structure of this course, but is useful to view to link the matrix methods with the linear equation and determinant methods. We'll look quickly at the Gaussian method first.

Gauss Elimination Method of Solving Linear Systems

This method is equivalent to the addition/subtraction method which eliminates all variables but one and then solves for the other variables by back substitution. Recall that when equations are solved by this method, it doesn't matter which equation is first or if an equation is multiplied through by a constant. The meaning of a matrix does not change if rows are interchanged or if a row is multiplied by a constant. This is another instance in which matrices and determinants differ.

Example

Solve the previous example by the Gauss elimination method. Augment the coefficient matrix by adding on the constant vector. The **augmented matrix** of A is:

$$\left\langle\begin{array}{cc|c} 2 & -3 & 5 \\ 4 & -1 & 7 \end{array}\right\rangle$$

Use row operation on this augmented matrix to transform the coefficient matrix into a triangular matrix with ones along the diagonal and zeroes below the diagonal.

$$\left\langle\begin{array}{cc|c} 1 & -1.5 & 2.5 \\ 4 & -1 & 7 \end{array}\right\rangle \quad \mathbf{R}1/2$$

$$\left\langle\begin{array}{cc|c} 1 & -1.5 & 2.5 \\ 0 & 5 & -3 \end{array}\right\rangle \quad \mathbf{R}1(-4)+\mathbf{R}2$$

$$\left\langle\begin{array}{cc|c} 1 & -1.5 & 2.5 \\ 0 & 1 & -0.6 \end{array}\right\rangle \quad \mathbf{R}2/5$$

Once it is in the proper form, rewrite the system in compact form:

$$\begin{bmatrix} 1 & -1.5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 2.5 \\ -0.6 \end{bmatrix}$$

Apply the variable vector to the last row of the coefficient matrix: 0(x) + 1(y) = -0.6

y = -0.6

Back substitute by replacing y by -0.6 in the variable matrix and applying the variable matrix to the first row of the coefficient matrix: 1(x) + (-1.5)(-0.6) = 2.5.

x = 1.6

The solution is the ordered pair (x, y) = (1.6, -0.6). Check this solution in both original equations.

The Gauss Elimination Method Extended, also called Gauss-Jordan Method
Transform the coefficient matrix into the identity matrix, **I**, in the augmented matrix.
Example
In the previous example instead of stopping at

$$\left\langle\begin{array}{cc|c} 1 & -1.5 & 2.5 \\ 0 & 1 & -0.6 \end{array}\right\rangle \quad \mathbf{R}2/5,$$

continue with row operations to produce

$$\left\langle\begin{array}{cc|c} 1 & 0 & 1.6 \\ 0 & 1 & -0.6 \end{array}\right\rangle \quad \mathbf{R}2(1.5)+\mathbf{R}1$$

Once it is in the proper form, rewrite the system in compact form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 1.6 \\ -0.6 \end{bmatrix}$$

The answer upon multiplication is x = 1.6 and y = -0.6 without back substitution.
Use the Gauss and Extended Gauss for exercises in text and the following.
2x + 3y = 13
3x + 4y = 25

Lecture #6 **Solution of Linear Systems By Inverse Matrix Operations**

The inverse of a matrix **A** is represented by the symbol $\mathbf{A}^{-1}$. $\mathbf{A}^{-1}$ has the special property that:

$$\mathbf{A}^{-1} \text{ x } \mathbf{A} = \mathbf{A} \text{ x } \mathbf{A}^{-1} = \mathbf{I}$$

Recall that **I** is the identity matrix with ones along the diagonal and zeroes elsewhere. A necessary condition for any matrix **A** to have an inverse $\mathbf{A}^{-1}$ is that **A** be of square dimensions.

There are four ways to find the inverse of a square matrix

1. Calculator
2. PC with Excel
3. Row operations (not used in this class)
4. Adjoint-transpose method

Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix}$

There is a shortcut version of the Adjoint-transpose method that is very quick for a 2x2 matrix. See Calter page 321.

1. Exchange the items in the main diagonal.
2. Reverse the signs of items in the minor diagonal.
3. Divide the result by the determinant of either of the two matrices.
4. Use this method to confirm the result below.

This produces an inverse: $\mathbf{A}^{-1} = \begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix}$

1. Use your calculator and do the following:

1. Create the matrix **A**.
2. Enter **A** on the stack (HP) or select matrix A (Sharp).
3. Use the $\boxed{\frac{1}{x}}$ key (HP) or x^{-1} (Sharp) to find the inverse, $\mathbf{A}^{-1}$.

This produces an inverse: $\mathbf{A}^{-1} = \begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix}$

2. Using a PC to find matrix inverses

1. Enter the matrix **A** from above with each number in a separate cell.
2. Highlight an area of the same size in the area where you want the inverse matrix.
3. Enter the matrix multiplication formula, =MINVERSE(ARRAY_A) in the upper left of this resultant region. ARRAY_A, for example, is entered by providing the two diagonally opposite boundaries of the region where you entered matrix **A.**
4. After you've entered the range, close the parentheses.
5. Simultaneously Press CTRL+SHIFT+ENTER.
6. The resultant region now contains the inverse of the matrix.

Notice: $$\mathbf{A} \text{ x } \mathbf{A}^{-1} = \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix} \text{ x } \begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{A}^{-1} \text{ x } \mathbf{A} = \begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix} \text{ x } \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Solving an N x N linear system by the inverse matrix method requires writing the system in compact form. Recall that the compact form is:

$$\mathbf{AX} = \mathbf{C}$$

Multiply both sides of this equation by A^{-1}.

$$A^{-1}\mathbf{AX} = A^{-1}\mathbf{C}$$

This produces:

$\mathbf{IX} = A^{-1}\mathbf{C}$ and $\mathbf{IX} = \mathbf{X}$ since $\mathbf{I}$ is the identity matrix.

So the solution to the system is $\mathbf{X} = A^{-1}\mathbf{C}$.

Example

Solve the following system by the inverse matrix method:

$$1\mathbf{x} - 4\mathbf{y} = -19$$

$$-2\mathbf{x} + 9\mathbf{y} = 43$$

In matrix form this is:

$$\begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} -19 \\ 43 \end{bmatrix}$$

Find the value A^{-1} of on the calculator: $A^{-1} = \begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix}$. If A^{-1} is applied on the left of both sides of the equation:

$$\begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -19 \\ 43 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 9(-19) + 4(43) \\ 2(-19) + 1(43) \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Multiplying the two matrices on the left side gives $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$

The solution $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. Check this solution in the original equations.

3. By Row Operations.(omitted)

4. By the Adjoint Matrix Method

The steps in finding $\mathbf{A}^{-1}$ by the adjoint matrix method are:
1. Establish the cofactor matrix of **A.**
2. Establish the adjoint matrix of **A** as the transpose of its cofactor matrix
3. $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}$ [adj **A],** where $|A|$ is the determinant of A

A **cofactor matrix** is a matrix whose elements are the cofactors of the corresponding elements of the original matrix. Recall that a cofactor is a determinant expressed as a signed minor.
Example:

Construct the cofactor matrix of $\mathbf{A} = \begin{bmatrix} -1 & 3 & 7 \\ 4 & -2 & 0 \\ 7 & 2 & -9 \end{bmatrix}$.

The cofactor of element a_{11} is $+\begin{vmatrix} -2 & 0 \\ 2 & -9 \end{vmatrix} = +18$ so a_{11} is replaced with +18. Now replace each element with its cofactor.

The cofactor matrix of **A** is $\begin{bmatrix} 18 & 36 & 22 \\ 41 & -40 & 23 \\ 14 & 28 & -10 \end{bmatrix}$

Recall that the transpose of a matrix is one in which corresponding rows and columns have been interchanged.

By definition, the adjoint matrix of matrix **A,** adj **A**, is the transpose of its cofactor matrix.

In our case adj **A** = $\begin{bmatrix} 18 & 41 & 14 \\ 36 & -40 & 28 \\ 22 & 23 & -10 \end{bmatrix}$

In the case above, the determinant of **A,** $|\mathbf{A}|$ is **:**

First simplify by row operations: $|\mathbf{A}| = \begin{vmatrix} -1 & 3 & 7 \\ 4 & -2 & 0 \\ 7 & 2 & -9 \end{vmatrix} = -\begin{vmatrix} 1 & -3 & -7 \\ 0 & 10 & 28 \\ 0 & 23 & 40 \end{vmatrix} = -(400 - 644) = 244$

So $\mathbf{A}^{-1} = \frac{1}{244}\begin{bmatrix} 18 & 41 & 14 \\ 36 & -40 & 28 \\ 22 & 23 & -10 \end{bmatrix}$

This method has advantages over the PC and row operation methods in that decimal values and fractional values do not have to be used.

Also, this method works very fast with 2 x 2 matrices because the cofactor of each element is a single element rather than a value calculated by the determinant of a reduced array. See the text, page 321, for the shortcut method.

Example

If $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$ find $\mathbf{A}^{-1}$. The cofactor matrix is $\begin{bmatrix} -4 & -2 \\ -1 & 3 \end{bmatrix}$ and its transpose is $\begin{bmatrix} -4 & -1 \\ -2 & 3 \end{bmatrix}$.

Since $|\mathbf{A}| = -14$, $\mathbf{A}^{-1} = \frac{1}{-14}\begin{bmatrix} -4 & -1 \\ -2 & 3 \end{bmatrix}$. Check this value on your calculator or on a PC.

Exercises in Textbook

(Use your calculator or PC to evaluate any determinants).

Try enough to get comfortable with each of the three methods.

This process has two stages. First we determine how to document three points to determine the equation of a conic curve that has the three points on the desired equation. Second, we determine the required values and simplify the result.

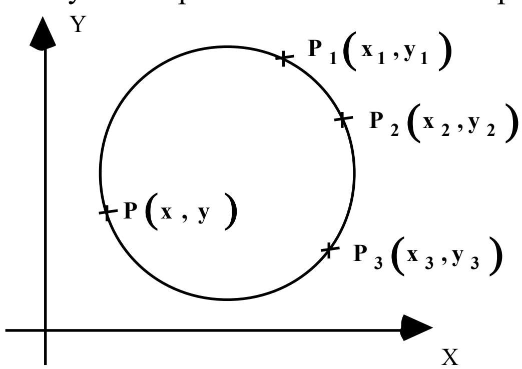
Recall that the general equation in standard form for a conic section curve is:

$$Ax^2 + By^2 + Cx + Dy + E = 0$$

This equation represents one of the circle, the ellipse, the nonrotated hyperbola, the parabola or the straight line depending on the values of the constants A, B, C, D, E.

If A= B and the equation is divided through by this constant the equation of a circle results.

The equation of a circle has the form $\mathbf{x^2 + y^2 + A\,x + B\,y + C = 0}$, the **A, B, C** here are not the ones above. In this equation, there are three unknown constants, A, B, C all of degree one. From linear algebra, we know that three equations are required to determine three unknowns. As a result, 3 known points on a circle will completely determine the path of a circle. If only two points are known, an infinite number of circles will pass through them. If four points are known, it may not be possible to draw a circle passes through all of them.



Substitute the 3 points into the general form:

$\mathbf{x_1^2 + y_1^2 + A\,x_1 + B\,y_1 + C = 0}$ --->P_1
$\mathbf{x_2^2 + y_2^2 + A\,x_2 + B\,y_2 + C = 0}$ --->P_2
$\mathbf{x_3^2 + y_3^2 + A\,x_3 + B\,y_3 + C = 0}$ --->P_3

Take the squared terms to the right hand side to produce a linear 3 x 3 system in A, B and C.

Example: Use the general equation to get an equation for each (x,y) pair.
Determine the equation of the circle passing through: $P_1(4, 6)$, $P_2(-2, -2)$,$P_3(-4, 2)$.
By substituting point 1: $\mathbf{16 + 36 + 4\,A + 6\,B + C = 0 \text{ or } 4\,A + 6\,B + C = -52}$
By substituting point 2: $\mathbf{-2A - 2B + C = -8}$
By substituting point 3: $\mathbf{-4A + 2B + C = -20}$

Now we have three equations and three unknowns, we can write this system in compact form and use matrix algebra to solve this system:

$$\begin{pmatrix} 4 & 6 & 1 \\ -2 & -2 & 1 \\ -4 & 2 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} -52 \\ -8 \\ -20 \end{pmatrix}$$

Using the calculator and inverse matrix operations:

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -20 \end{pmatrix}$$

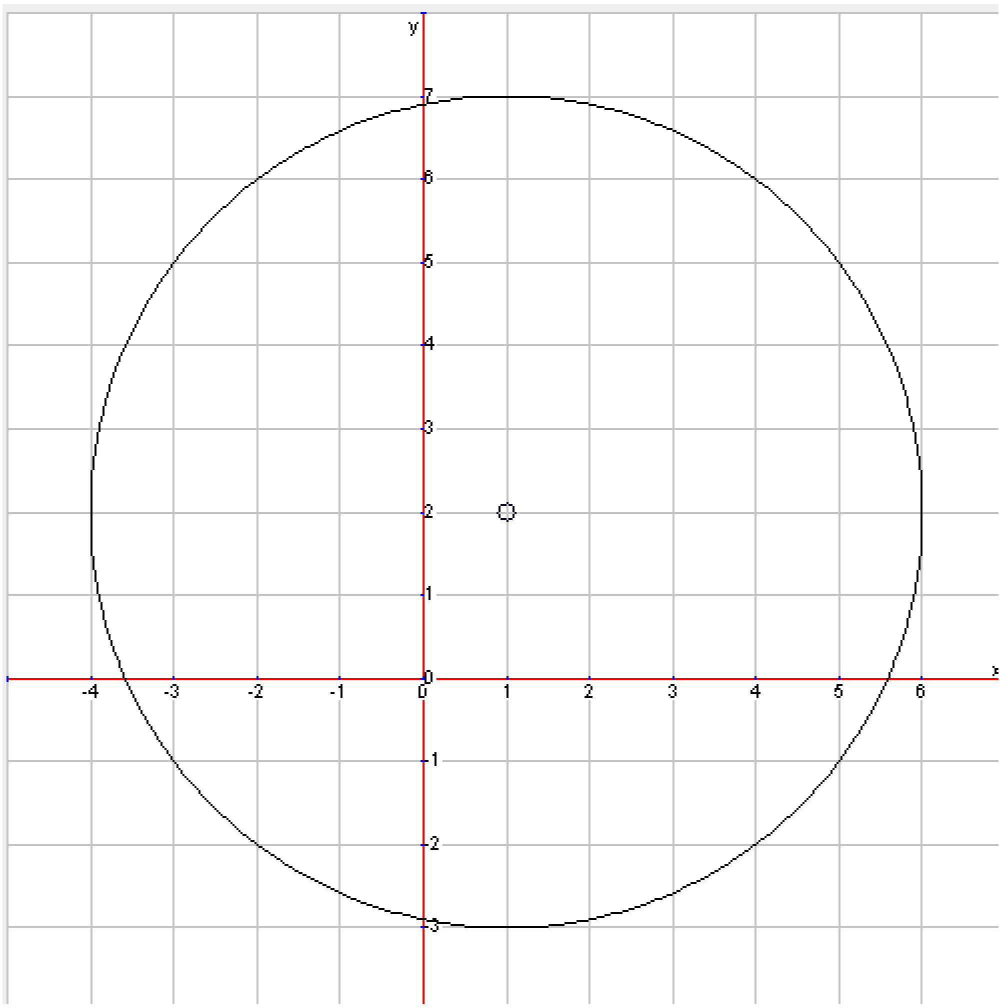
Substituting A, B and C back into the equation for a circle gives $\mathbf{x^2 + y^2 - 2\,x - 4\,y - 20 = 0}$
With this equation, we can compute the coordinates of all points in the circle's path.

By completing the square on X and Y in the equation, the translated centre and radius can be identified.
Exercise:
Find the centre and radius of the circle in the previous example.

$$x^2 + y^2 - 2x - 4y - 20 = 0 \text{ becomes } (x - 1)^2 + (y - 2)^2 = 25$$

Compare this with the translated form: $(x - h)^2 + (y - k)^2 = r^2$

The centre C(h, k) = (1, 2) and the radius is 5
This problem is solved.

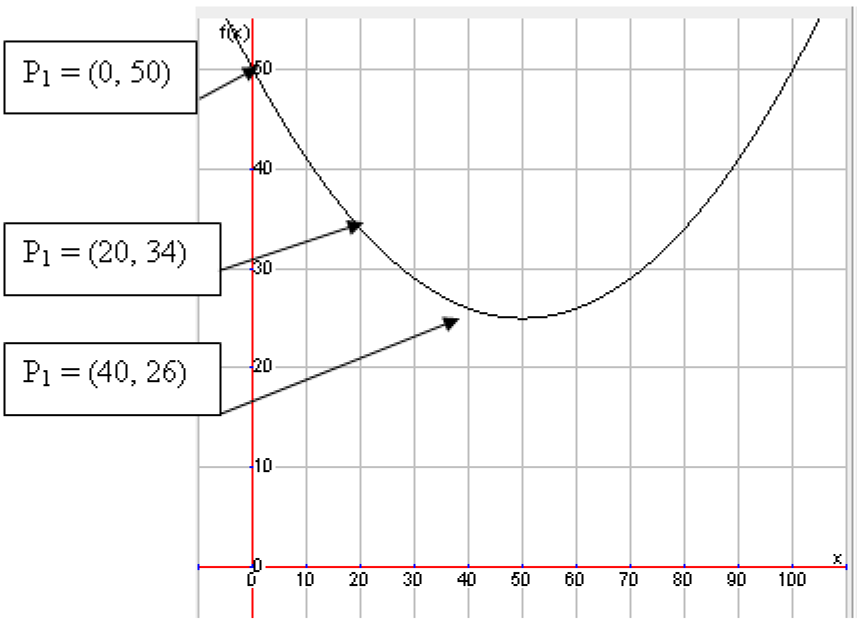


The Three Point Parabola Condition

The general conic equation $Ax^2 + By^2 + Cx + Dy + E = 0$ has B=0 for a vertical parabola or A=0 for a horizontal parabola. The general form of the equation of a parabola has three unknown constants. We'll call them A, B, C, again the **A, B, C** here are not the ones above. If 3 points are known, exactly one parabola of the vertical axis type or one of the horizontal axis type can be determined.

Example:
A vertical parabolic curve is to pass through three points on a terrain: $P_1(0, 50)$, $P_2(20, 34)$ and $P_3(40, 26)$. Find the equation which determines the elevation of all other points along the curve.

Form→ $\mathbf{x^2 + A\,x + B\,y + C = 0}$



Substitute the 3 known points into the general form of the equation of a parabola with a vertical axis. Solve the resulting 3 x 3 system using your calculator and matrix algebra.

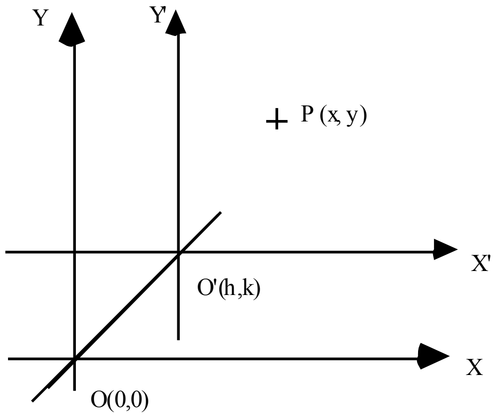
$$\begin{pmatrix} 0 & 50 & 1 \\ 20 & 34 & 1 \\ 40 & 26 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ -400 \\ -1600 \end{pmatrix} \xrightarrow{\text{By inverse matrix operations}} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} -100 \\ -100 \\ 5000 \end{pmatrix}$$

The equation is: $\mathbf{x^2 - 100\,x - 100\,y + 5000 = 0}$

A more useful form for this equation for purposes of finding elevations is to place it in explicit functional format, y = f(x). The equation transforms to: $\mathbf{y = 0.01\,x^2 - x + 50}$. Given any horizontal offset from the vertical reference axis, a vertical elevation on the curve can be found, a useful geomatic application.

Translations In The Plane

In mathematics, a translation is a movement along a straight line. If the centre of a coordinate system is translated, all points in the plane will have new coordinates relative to the new location of the axes. This motion can be described by matrix operations.



Relative to the X - Y axis system point P will have coordinates (x, y). If the X / Y axis system is translated so that the new origin is at O', the coordinates of P will no longer be (x, y).

Relative to the X' / Y' axis:

The x coordinate will be:

$x' = x - h$

The y coordinate will be:

$y' = y - k$

Using matrix algebra, the translation operations are:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} h \\ k \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$$

When the circle is not located with its centre at the origin, the axes are said to be translated.

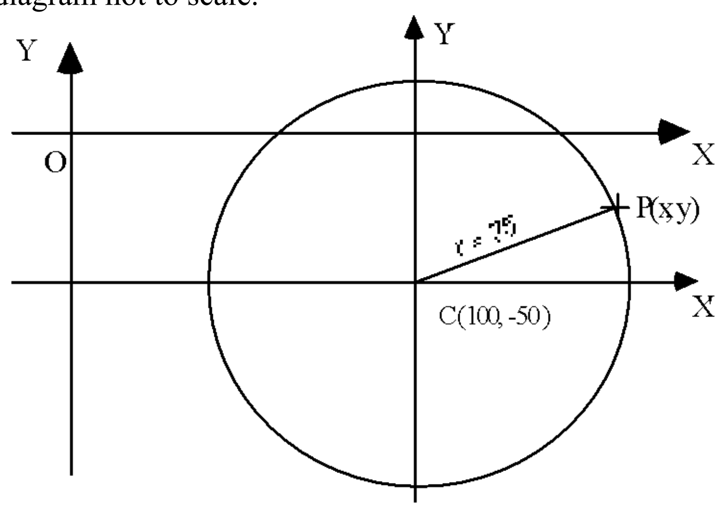
Problem:

A circle is constructed about a point 100 m East and 50 m South of a known point, O. The radius of the circle is 75 m.

(a) Find the equation of the circle with respect to its centre.

(b) Find the equation of the circle with respect to an axis system through point O.

Note: diagram not to scale.



With respect to an axis system through C, the equation of the circle is:

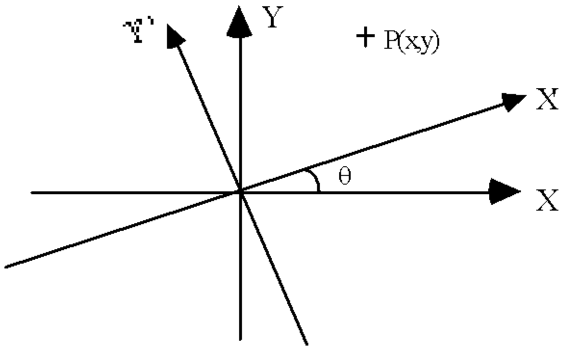
$$(x')^2 + (y')^2 = 75^2 = 5625 \quad \text{<------}$$

Since: $x' = x - 100$ and $y' = y + 50$, its equation with respect to O is:

$$(x - 100)^2 + (y + 50)^2 = 5625$$

Rotations in the Plane

Consider the following diagram which shows the rotation of an axes system:



Relative to the X/Y coordinate system the coordinates of point P are (x, y). Relative to a set of axes, X'/Y', that are rotated by angle of θ, the coordinates of P have changed. The new coordinates (x', y') can be determined by a transformation using a matrix operator.

Here are the transformation operations from one coordinate system to another:

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \text{Cos}(\theta) & \text{Sin}(\theta) \\ -\text{Sin}(\theta) & \text{Cos}(\theta) \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \text{Cos}(\theta) & \text{Sin}(\theta) \\ -\text{Sin}(\theta) & \text{Cos}(\theta) \end{bmatrix}^{-1} \begin{bmatrix} X' \\ Y' \end{bmatrix}$$

Example:

A coordinates axes system undergoes a rotation of $+30^{o}$, what is the new set of coordinates for a point that was located at (100, 200) under the new coordinate system?

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \text{Cos}(30^{\circ}) & \text{Sin}(30^{\circ}) \\ -\text{Sin}(30^{\circ}) & \text{Cos}(30^{\circ}) \end{bmatrix} \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 100\ \text{Cos}(30^{\circ}) + 200\ \text{Sin}(30^{\circ}) \\ -100\ \text{Sin}(30^{\circ}) + 200\ \text{Cos}(30^{\circ}) \end{bmatrix}$$

$$= \begin{bmatrix} 186.60 \\ 123.21 \end{bmatrix}$$

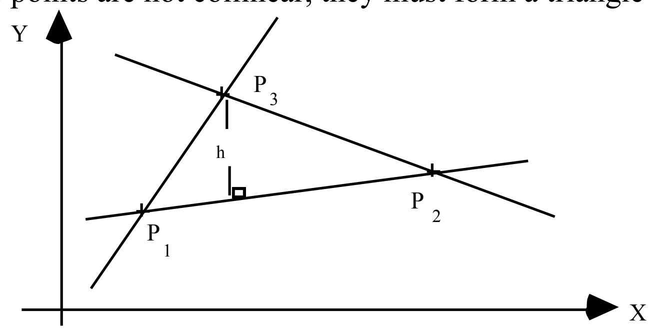
Notice that this matrix operator is easy to remember if we calculate the determinant of the array.

$$\begin{vmatrix} \text{Cos}(\theta) & \text{Sin}(\theta) \\ -\text{Sin}(\theta) & \text{Cos}(\theta) \end{vmatrix} = \text{Cos}^2(\theta) + \text{Sin}^2(\theta) = 1$$

These translations and transformations will enable you to work in a revised reference frame and perform the translation and rotation back to the original reference frames by using a matrix operator to restate your results.

The Area Of A Point Triangle

If three points are not collinear, they must form a triangle of nonzero area.



The area of this triangle can be written compactly using a 3 x 3 determinant.

$$\textbf{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{<------}$$

The Collinearity Test

The only way for the area bounded by three points in a plane to be zero is if the three points are in a straight line. So we can use this test to determine if three given points are in a straight line, or collinear.

If the matrix solution to the three-point triangle problem does not exist, it is because the 3 x 3 array does not possess an inverse. Matrices like this are called **singular** matrices. Recall from the transpose method of deriving an inverse that a division by the determinant of the array takes place when an inverse is found. $A^{-1} = \frac{1}{|A|} A^T$. A noninvertible matrix occurs because of a division by 0.

In general 3 points, P_1, P_2, P_3, lie on the same straight line if:

$$|A| = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

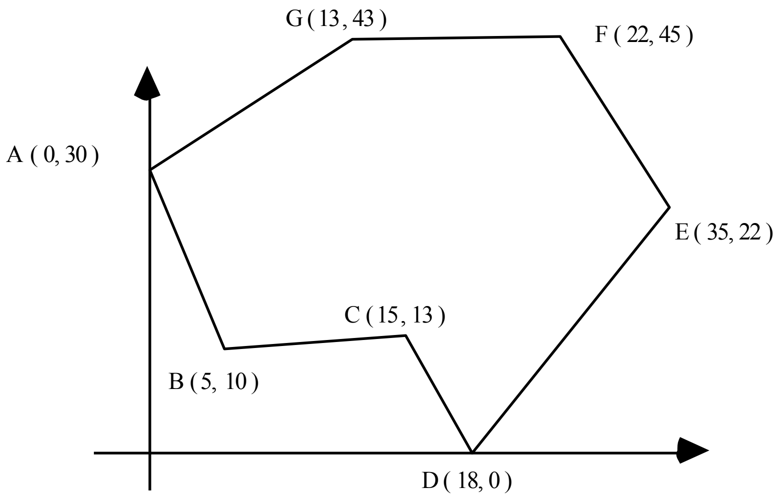
Example:

Are (-1.25, -3.75), (5, 2.5), (11.25, 8.75) collinear?

Check: By calculator $\begin{vmatrix} -1.25 & -3.75 & 1 \\ 5 & 2.5 & 1 \\ 11.25 & 8.75 & 1 \end{vmatrix} = 1.5625\ E - 11 \approx 0$

So the points must be collinear.

Example:
If a course is run from B to G, what is the area of the triangle ABG cut from the traverse below?



$$Area\ of\ triangle\ (ABG) = \pm\frac{1}{2}\begin{vmatrix} 0 & 30 & 1 \\ 5 & 10 & 1 \\ 13 & 43 & 1 \end{vmatrix} = 162.5$$ by calculator.

Example:
Find the area of the entire traverse ABCDEFGA above.

If you select the most interior point on the traverse, any traverse can be divided into a series of triangles whose areas can be summed. The traverse above can be divided into 5 triangles. Choose the sign on each area so that all areas are positive and then sum the areas of all triangles.

Area A**BCDEFGA** **= Area [ABG + BFG + BCF + FCE + ECD]**
= (1/2) [325 + 281 + 299 + 577 + 287]
= 884.5

What is the area of a square? of a hexagon?

For the square, you can select the origin as one point to make the calculation almost trivial. This gives a determinant with a triangle of zeros, which is evaluated by multiplying by the elements on the diagonal.

For the hexagon, break the figure down into its constituents. You should be able to find the area of the hexagon by evaluating the area of a square and of a triangle, then multiplying by the counts of these figures.

http://www.cs.mtu.edu/~shene/COURSES/cs3621/NOTES/geometry/geo-tran.html