SIAST Palliser Campus

Mathematics
STAT 220

Lecture Notes and Examples

Unit 2
Probability Theory and Distributions

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1 Basic Probability

1.1 Definitions of Probability

1.1.1 Subjective Probability

The **subjective** approach to probability is a method to express the strength of one's belief in the future occurrence of an event. The beliefs are usually based upon past experience or prior knowledge. Probability is assigned to an event as a number on a continuum between 0 and 1 in this fashion.

$$\underset{\text{Impossibility}}{0} \longleftarrow \text{P(event)} \longrightarrow \underset{\text{Certainty}}{1}$$

Example:
A biologist estimates that there is a 75% chance that if burrowing owls are reintroduced to an area of southern Saskatchewan that they will survive. What does the number mean and how would a biologist arrive at this number?

1.1.2 Experimental Probability

The **relative frequency** definition of probability relates the expected future rate of occurrence to what has happened in the past. The difference between this and subjective probability is that in the relative frequency definition an experiment is done and we refer to it therefore as **experimental probability**. To find the experimental probability of an event, repeat an experiment F times. Observe the number of times the event happened, the frequency f, out of the total number of times the experiment was repeated, F. The probability of the event is then

$$P(\text{event}) = \frac{\text{Frequency of occurrence of event}}{\text{Number of repetitions}} = \frac{f}{F}$$

Thinking back to our discussion of frequency distributions in Unit 1 we note that $F = \sum f$ where the sum counts the frequencies of all the possible outcomes, and hence the number of experiments done.

Example:
In a busy shopping mall, the question was put to 10 different people: "Are you in favour of extended store hours?" 7 people replied that they were. Based on these results, what is the probability that the next person asked would be in favour?

$$P(\text{in favour}) = \frac{f}{F} = \frac{7}{10} = 0.7$$

Notice that we have already done these calculations extensively in Unit 1. There we would have calculated the relative frequency or proportion P of an outcome. The calculation (and symbol) is identical for experimental probability but the interpretation is different. Now we use the proportion to infer the probability that an individual shopper chosen at random in the mall will be in favour of extended store hours. In the following exercise we illustrate the meaning of probability as a relative frequency of occurrence (experimental probability).

Example:
Take 4 coins and shake them fairly by allowing them to drop unobstructed on smooth surface. Repeat this experiment 64 times. In the 8 x 8 grid below record the number of heads that occur on a repetition of the experiment in each cell. Count the frequency of each value and record your results in the table on the right:

$\Rightarrow$

X(heads)	f	$P(X)$
0		
1		
2		
3		
4		
	$\sum f =$	$\sum P(X) =$

Graph the results below with 7 intervals on the horizontal axis and the number of heads as the midpoint of each interval. Plot P(X) on the vertical axis.

Interpretation:

1. If $P(X)$ means the relative frequency of the occurrence of X heads, what does $P(1)$ mean?
2. What is $P(1)$ specifically in your table?
3. How many times would you expect to get 1 head in the future, based on what has happened in this experiment, if you flipped 4 coins 200 times?
4. What is the shape of the distribution curve?
5. By examining the distribution curve, what is an estimate of the mean of the distribution of heads upon flipping 4 coins?
6. Calculate the mean and standard deviation of the distribution of heads from the data in the table. (Use $\bar{X}$ and s . Why?)
7. If 4 coins are flipped, what is the probability of getting 2 heads? Why isn't it 0.5?
8. What is the area under the P(X) histogram?
9. Would someone else's determination of the experimental probability be identical with yours?

1.1.3 Theoretical Probability

The **theoretical** or **classical** approach to probability is based upon determining the total number of possibilities in advance of taking the action. The set of possibilities is called the **sample space** or **universe**. In this case let n be the number of possibilities associated with the event while N is the total number of outcomes associated with the action to be taken (i.e. the size of the sample space). If the events in the sample space are **equally likely** then the theoretical probability of an event is:

$$P(\text{event}) = \frac{\text{Possible ways for event to happen}}{\text{Total number of possibilities}} = \frac{n}{N}$$

Example:
A group of four contains two males and two females. A committee of 2 is to be struck by drawing lots. What is the probability that the committee consists of two females? Count the possibilities.

N = 6

John, Bill	Bill, Sue
Bill, Mary	Mary, Sue
John, Sue	John, Mary

n = 1

Mary, Sue

$$P(\text{two female committee}) = \frac{n}{N} = \frac{1}{6} = 0.17$$

The theoretical probability is unique; assuming we do our calculations correctly we will all get the same answer. One could have calculated an experimental probability for this event by writing the six possible pairs on slips of paper and drawing them out of a hat repeatedly. In that case different experimental probabilities for the event would not likely have been exactly 1/6 but would have been close to the true theoretical probability. The experimental probability will converge to the theoretical probability as F, the number of experiments performed, increases. In symbols the actual (theoretical) probability satisfies

$$P(\text{event}) = \lim_{F\to\infty} \frac{f}{F}$$

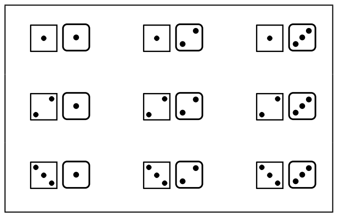
where f is the number of occurrences of the event in the F experiments and ∞ means positive infinity.[1]

If the probability is calculable theoretically, then this is preferable to an experimental probability. However many systems are too complicated to have probabilities calculated theoretically. For instance there is no way a theoretical calculation of probability may be done to find the probability a shopper is in favour of extended store hours.

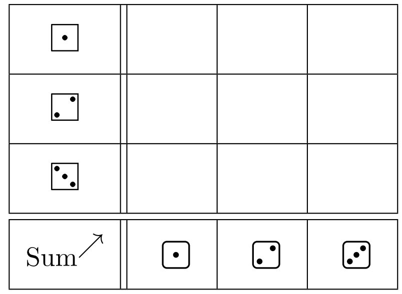
In Case Study 3 of Unit 1 a game designer was interested in determining the proportion (probability) of the sum on two 3-sided dice. The designer effectively calculated an experimental probability for the possible events (2,3,4,5,6). The object of the next exercise is to illustrate the theoretical/classical meaning of probability as a proportion in the sample space of all possibilities.

[1] Such statements regarding convergence of empirically measured values to the actual (theoretical) values is embodied in statistical theorems such as the **law of large numbers**.

Example:
Suppose two three-sided dice[2] are to be rolled so that they land fairly. We are interested in the probability of obtaining a specific sum on the two faces. The sample space consists of the following equally likely outcomes (left). List all of the possible sums on the faces of the dice by filling the appropriate sum inside the square on the grid to the right.



$\Rightarrow$



Summarize your findings in the table below and draw the graph of the observations to the right. You may want to compare your results to the relative frequencies (P) found experimentally in Case Study 3 of Unit 1.

X(Sum)	n	$P(X)$
2		
3		
4		
5		
6		
	$N =$	$\sum P(X) =$

$\Rightarrow$ P(X)

Interpretation:

1. If $P(X)$ means the probability of rolling a sum of X, what does $P(3)$ mean?
2. If two dice were rolled 180 times, how many times would you expect to see a sum of 3?
3. Describe the shape of this distribution.
4. What is the mean of the distribution of sums? Use μ not $\bar{X}$. (Here treat the n column as frequency f and use our formulae from Unit 1.)
5. What is the standard deviation of the distribution of sums? Use σ not s.
6. How does the area under the curve compare to the sum of the P(X) column?
7. What is wrong with the argument, “There are five possible sums (2, 3, 4, 5, 6) so the probability of any one of them is $\frac{1}{5}$” ?

[2]Recall our *three-sided* die is created by labelling two sides of a six-sided die with a 1, two sides with a 2, and two-sides with a 3. Would the above analysis be appropriate if three sides were labelled with a 1, two sides with a 2, and one side with a 3? How would the above analysis have to change?

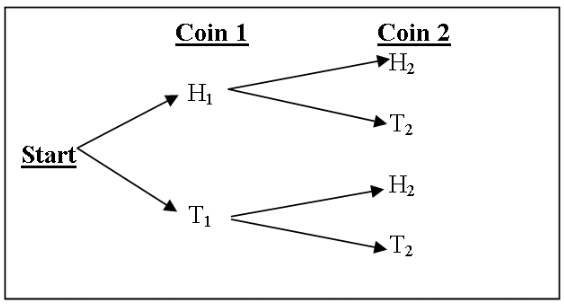
1.2 Counting Outcomes

We already know from Unit 1 how to determine experimental probability, it is just the relative frequency, P. To compute the probability of an event theoretically it is necessary to count the number of **outcomes** associated with both the **event** and the **sample space** of equally likely outcomes. There are various mathematical methods to assist in counting outcomes.

1.2.1 Tree Diagrams

Tree Diagrams list, in a tree structure, each of the steps in an experiment together with all possible outcomes at that step. Each path represents a possible outcome of the total experiment.

> **Example:**
> Two coins are flipped. Show all possible outcomes.



1.2.2 The Counting Theorem

The **Counting Theorem** gives the total number of possibilities, without listing each possibility, in a tree structure. This is useful in cases where it is not practical to draw the tree structure because it has too many paths. List each step in the experiment and determine the number of possible ways each step can be done. The number of ways of doing the whole experiment is found by multiplying the number of possibilities at each step in the experiment.

> **Example:**
> For the experiment of flipping two coins:
>
> $$\underbrace{(2)}_{\boxed{\text{Coin 1}}} \cdot \underbrace{(2)}_{\boxed{\text{Coin 2}}} = 4 \text{ possibilities in total}$$

1.2.3 Factorial

A **factorial** is a mathematical formula encountered in counting outcomes. It is denoted as $n!$ and is calculated as follows:

$$\boxed{n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 3 \cdot 2 \cdot 1}$$

The factorial $n!$ is the number of ways of **arranging** n items since, using the counting theorem, after the first slot is filled in one of n ways, the second slot can be filled in $(n-1)$ ways, etc.

Example:
In how many ways can five children be arranged in five different seats in a minivan?

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \text{ ways}$$

Find the $\boxed{n!}$ key on your calculator and show that $5! = 120$.

Notice that factorials increase in value very quickly. $20! = 2.4329020 \times 10^{18}$ on your calculator. (i.e. move the decimal 18 places to the right.) 800! exists but is too large for calculation purposes on your calculator. Finally note that by definition $0! = 1$ not 0 as may be verified on your calculator.

1.2.4 Permutations

An **arrangement** of the letters a to e could be written (c, e, b, a, d), where parentheses have been used, just as when we plot points, to remind us that the **order matters**. Such an arrangement is called a **permutation** and we have seen that $n!$ gives us the number of ways of arranging n objects, or in other words, the number of permutations of the n objects. Sometimes we want to count the number of ways of arranging fewer than all the objects. A permutation where we have selected only 3 of the 5 letters from a to e could be (c, e, b). The **permutation formula** ${}_nP_r$ counts the number of different permutations of r objects taken from n distinct objects. It equals[3]

$$\boxed{{}_nP_r = \frac{n!}{(n-r)!}} \leftarrow \text{ no repetitions are allowed}$$

No repetitions of the same element are allowed in a permutation of distinct objects.

If repetitions of the same element are allowed in an arrangement, then the number of arrangements of r objects taken from the n objects is, by the counting theorem,

$$\boxed{\underbrace{(n) \cdot (n) \cdot \ldots \cdot (n)}_{r \text{ times}} = n^r} \leftarrow \text{ repetitions are allowed}$$

In such a situation we would be allowed to count the arrangement (a, a, d) if we were counting arrangements of 3 objects taken from a to e with repetition.

Example:
Suppose two objects to be selected from the collection $\{a, b, c\}$. How many arrangements are possible?

$${}_3P_2 = \frac{3!}{(3-2)!} = \frac{3 \cdot 2 \cdot 1}{1} = 6 \text{ with no repetitions}$$

$$3^2 = 9 \text{ with repetitions}$$

Find your calculator key $\boxed{{}_nP_r}$ and evaluate the above expression directly. As a check here is the list of the possible arrangements:

[3]To see why this is the formula, note by the counting theorem that there are n ways to pick the first item, $(n-1)$ ways to pick the second, ..., and $(n-r+1)$ ways to pick the r^{th}. The number of ways to pick an arrangement is therefore $n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$ which equals the permutation formula since the denominator cancels the lower terms from $(n-r)$ to 1 in the $n!$.

(a, b)	(c, a)	(a, a)
(b, a)	(b, c)	(b, b)
(a, c)	(c, b)	(c, c)
no repetitions		
with repetitions		

It is always possible to bypass the permutation formula and use the counting theorem.

3 X 2 = 6 ← with no repetitions.
3 X 3 = 9 ← with repetitions.

1.2.5 Combinations

Sometimes order is not important in a counting problem. For instance, if you wanted to know in how many ways you could select three of your five friends to go on a trip with you, the order of the three selected would be meaningless. A **combination** is such a selection. A combination of 3 items selected from the letters a to e would be $\{b, c, e\}$. We use braces here to remind us that order is not meaningful. (The same combination could be written $\{c, b, e\}$.) The **combination formula** counts the number of different **selections** of r objects taken from n objects. It is written ${}_nC_r$ and calculated as[4]

$$\boxed{{}_nC_r = \frac{n!}{r!(n-r)!}}$$

Example:
Suppose two objects are to be selected from the collection {a, b, c}. How many selections are possible?

$${}_3C_2 = \frac{3!}{2!(3-2)!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 1} = 3$$

Find your calculator key $\boxed{{}_nC_r}$ and evaluate the above expression directly. As a check here is the list of the possible selections:

$$\boxed{\{a, b\},\ \{a, c\},\ \{b, c\}}$$

You cannot bypass the combination formula with the counting theorem.

1.2.6 Contingency Tables

Contingency Tables cross tabulate a data array where the outcomes consist of two factors into a summary table.

Example:
A die is rolled and a coin is tossed at the same time. A person interested in determining the probability of the joint result using relative frequency summarizes the results of 100 experiments in the following contingency table:

		DIE						
		1	**2**	**3**	**4**	**5**	**6**	Total
COIN	**Heads**	9	8	8	9	10	2	46
	Tails	9	10	7	6	12	10	54
	Total	18	18	15	15	22	12	100

[4] The extra $r!$ in the denominator of ${}_nC_r$ versus ${}_nP_r$ deals with the fact that the $r!$ permutations of the selected objects are equivalent as a combination where order is irrelevant.

Notes:

1. The totals at the side and bottom of the table are called **marginal frequencies**. The outcomes included in a marginal frequency have only one attribute. For example, of the hundred trials, heads occurred 46 times. 46 is a marginal frequency.

2. The numbers inside the table are called **joint frequencies**. The outcomes included in a joint frequency have two attributes. For example, of the hundred trials, on 9 occasions the coin turned up heads and the face on the die was a 4.

3. Note that we could have placed these results in a single column frequency table with 12 entries where the first variable value was $X = (\text{heads}, 1)$ with frequency 9 and the last was $X = (\text{tails}, 6)$ with frequency 10. Obviously the contingency table for displaying the frequencies is preferable.

4. Later we will use contingency tables for theoretical probability where the entries correspond to the number of elements of the sample space corresponding to the event. If we had done that here, what would the table have looked like?

Assignment:

1. A white jar and a blue jar each contain a collection of chips. Some of the chips are red and some are green. A jar is chosen at random and a chip is randomly chosen from it. Construct a tree diagram of the sample space of outcomes for this experiment.

2. A company is selecting a new president and vice president. There are 15 females and 100 males on staff. How many possible ways does the company have of selecting a president and vice president if the president must be female but there are no restrictions on the choice of a vice president?

3. Three newly hired people are to be assigned to 3 different regions. How many different arrangements are possible if a person can be assigned to any of the regions?

4. Six newly hired people are to be paired into teams. How many different pairs can be formed?

5. A high school science fair has 20 participants. In how many ways can
   (a) four students be chosen to move on to the regional science fair?
   (b) the four awards – 1^{st}, 2^{nd}, 3^{rd}, and 4^{th} place – be awarded?
   (c) the four awards – *best new idea*, *best use of math*, *best presentation*, and *best sweet-talking of judges* – be awarded?

6. A survey of 100 business people showed that 30 were for and 70 were against a bylaw. Ten of those for the bylaw were gas station operators, 20 against the bylaw were fast food outlet managers. If only gas station operators and fast food outlet owners were included in the survey, construct a contingency table of the survey results.

7. A builder builds 3 different models of homes. Each home has the option of two different floor plans and comes with or without an attached garage. Draw a tree diagram that lists all possible arrangements of homes built by the builder.

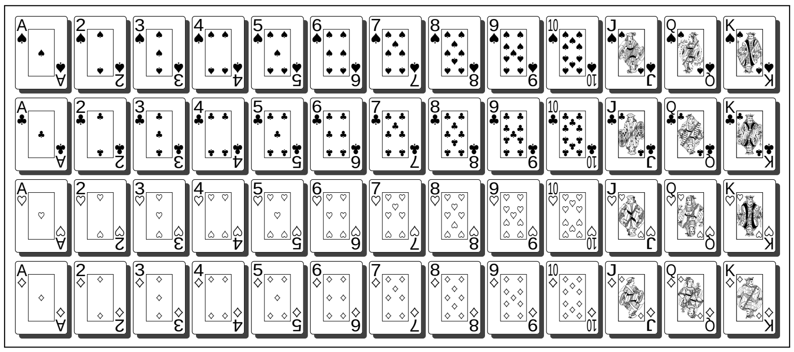
8. The integers from 1 to 49 are written on a sheet of paper. A person is asked to cross out 6 of the integers. In how many ways can this task be completed?

9. Calculate the value of: ${}_{900}C_{898}$

1.3 Set Theory and Counting Outcomes

1.3.1 Events are Sets!

Set Theory is a useful means of identifying events, combinations of events and numbers of outcomes in the sample space. A **set** is just a collection of objects called **elements**. For instance the set of playing cards in a standard deck would be, graphically,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and an element in the set would be the the king of spades, $K\spadesuit$. An **experiment** involves making a measurement or observation. The outcome of a single experiment is called a **simple event**. For example, if we consider the experiment of drawing a single card from the deck, then a **simple event** could be drawing the king of spades. Hence we can identify a simple event with a set element. The **sample space**, $\boldsymbol{S}$, is the set of all such simple events, in this case the set containing the entire deck.

A subset of S would also be a set. For example, the set $\boldsymbol{Q}$ of queens in a deck of cards is

$$\boldsymbol{Q} = \{Q\spadesuit, Q\clubsuit, Q\heartsuit, Q\diamondsuit\} \ .$$

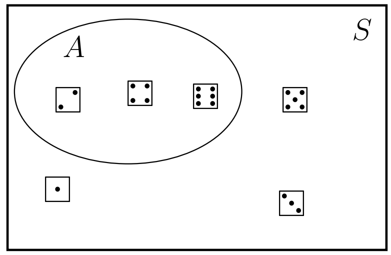
Since order is irrelevant for a set we use braces { } when writing it down here as we did for listing combinations. The connection between sets and probability is that we can represent the **event** of drawing a queen by its **set** of corresponding simple events in the sample space. The **number of elements** in a set we will write as $\boldsymbol{n(\ )}$. Here $n(Q) = 4$.

Pictorially it is useful to represent the set S of all simple events in the sample space with a rectangular box. For the experiment of drawing a single card from a deck of cards the sample space is just the deck shown in the box above. A particular event A (i.e. its set) will be some subset of these outcomes, which we can represent with a circle or oval inside the box. Such a diagram is called a **Venn Diagram**.

In a Venn Diagram one does not usually draw the simple events like the cards above. Rather one draws the events as circles and visualizes their contents and potential overlap with other sets. The dice in the following examples will be shown explicitly in this section to aid in understanding the concepts but a real Venn Diagram would not contain them.

Example:
Suppose a die is rolled. Let $\boldsymbol{S}$ represent the sample space. Let $\boldsymbol{A}$ **be the event the die is even**. Draw a Venn Diagram and count the number of simple events in the sample space and event A.

Solution:



$$\begin{aligned} \boldsymbol{S} &= \text{all outcomes in the sample space} \;=\; \{1,2,3,4,5,6\} \\ \boldsymbol{A} &= \text{all outcomes that are even} \;=\; \{2,4,6\} \\ \boldsymbol{n(S)} &= \text{number of outcomes in the sample space} \;=\; 6 \\ \boldsymbol{n(A)} &= \text{number of outcomes in event } A \;=\; 3 \end{aligned}$$

Visualize or shade in the event $\boldsymbol{A}$ in the Venn Diagram.

1.3.2 Complement of an Event

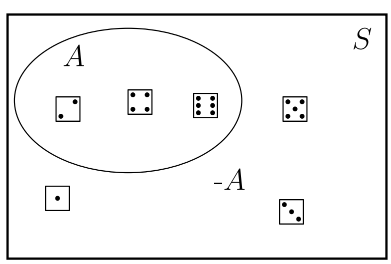
The **complement of an event**, written **-$\boldsymbol{A}$**, is the collection of all simple events in the sample space that are **not** in A. The number of simple events in the complement of an event must satisfy

$$\boxed{n(A) + n(\text{-}A) = n(S)}$$

since all outcomes in the sample space belong either to A or its complement, and these do not overlap.

Example:
Identify the complement -A of A of the previous example and find the number of its elements directly and by formula.

Solution:



$$\begin{aligned} \textbf{-}\boldsymbol{A} &= \text{the nonoccurrence of event } A \;=\; \{1,3,5\} \\ \boldsymbol{n(\text{-}A)} &= \text{number of outcomes in -}A \;=\; 3 \;\; \text{(counting directly)} \\ \boldsymbol{n(\text{-}A)} &= n(S) - n(A) \;=\; 6-3 \;=\; 3 \;\; \text{(by formula)} \end{aligned}$$

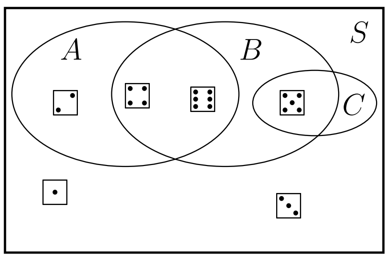
*Visualize or shade in the event **-$\boldsymbol{A}$** in the Venn Diagram.*

1.3.3 Intersection of Events

Just as we were able to create a new event by taking the complement of an event, we can also combine several events to create a new event. The **intersection of events** is the region of overlap of events. For two events it is written $\boldsymbol{A}$ **and** $\boldsymbol{B}$. It is sometimes called the **joint event** because two things have to happen at the same time for this event to happen.

Example:
Consider the previous problem of rolling the die. Suppose A is the event previously defined (an even number). Let $\boldsymbol{B}$ **be the event event of rolling a number greater than 3**. Let $\boldsymbol{C}$ **be the event of rolling a 5**. Draw a Venn Diagram of the sets, find their mutual intersections, and count the elements in each set.

Solution:



$$
\begin{aligned}
\boldsymbol{B} &= \text{all outcomes in event } B = \{4,5,6\} \\
\boldsymbol{C} &= \text{all outcomes in event } C = \{5\} \\
\boldsymbol{A} \textbf{ and } \boldsymbol{B} &= \text{all outcomes in both } A \text{ and } B = \{4,6\} \\
\boldsymbol{B} \textbf{ and } \boldsymbol{C} &= \text{all outcomes in both } B \text{ and } C = \{5\} \\
\boldsymbol{A} \textbf{ and } \boldsymbol{C} &= \text{all outcomes in both } A \text{ and } C = \{\}
\end{aligned}
$$

Note that the event $\boldsymbol{B}$ **and** $\boldsymbol{A}$ is the same as $\boldsymbol{A}$ **and** $\boldsymbol{B}$ so there are no more possible intersections between two events.

The counts in each set are:

$$
\begin{aligned}
\boldsymbol{n(B)} &= \text{number of outcomes in event } B = 3 \\
\boldsymbol{n(C)} &= \text{number of outcomes in event } C = 1 \\
\boldsymbol{n(A} \textbf{ and } \boldsymbol{B)} &= \text{number of outcomes in event } A \text{ and } B = 2 \\
\boldsymbol{n(B} \textbf{ and } \boldsymbol{C)} &= \text{number of outcomes in event } B \text{ and } C = 1 \\
\boldsymbol{n(A} \textbf{ and } \boldsymbol{C)} &= \text{number of outcomes in event } A \text{ and } C = 0
\end{aligned}
$$

*Visualize or shade in the event $\boldsymbol{A}$ **and** $\boldsymbol{B}$ and the event $\boldsymbol{B}$ **and** $\boldsymbol{C}$ in the Venn Diagram.*

1.3.4 Mutually Exclusive Events

In the last example the intersection of events A and C was empty, A and $C = \{\}$. If an event has no outcome, it is called the **empty set** and is denoted by the symbol $\varnothing$. If two events have as their intersection $\varnothing$ they are called **mutually exclusive**. The following are all equivalent:

1. Event A and event C are **mutually exclusive**.
2. Event A and event C are **disjoint**.
3. Event A and event C have no overlap.
4. Event A and event C cannot happen at the same time, $P(A \text{ and } C) = 0$.
5. A and $C = \{\} = \varnothing$
6. $n(A \text{ and } C) = 0$

1.3.5 Union of Events

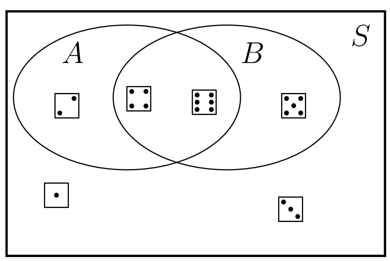
Another way to create new events from other events is to take their union. The **union of events** is the collection of all the simple events in the events combined. For two events it is written $\boldsymbol{A}$ **or** $\boldsymbol{B}$. It contains all the simple events in A together with all the simple events in B **including the joint simple events in both**. The number of outcomes in A or B can be written in terms of the number simple events in the events A, B, and A and B as follows:

$$\boxed{n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)}$$

Here we have to subtract the number of elements in A and B because otherwise the elements that are in both A and B get counted twice.

Example:
Consider the previous experiment of rolling a single die where A is rolling an even number and B is rolling a number greater than 3. Find the union of the events, the number of simple events it contains, and show that the above formula for calculating the number of simple events holds.

Solution:



$$\begin{aligned}
\boldsymbol{A} \textbf{ or } \boldsymbol{B} &= \text{all outcomes in either event } A \text{ or event } B \;=\; \{2,4,6,5\} \\
\boldsymbol{n(A} \textbf{ or } \boldsymbol{B)} &= \text{number of outcomes in } A \text{ or } B \;=\; 4 \;\; \text{(counting directly)} \\
\boldsymbol{n(A} \textbf{ or } \boldsymbol{B)} &= n(A) + n(B) - n(A \text{ and } B) \;=\; 3+3-2 \;=\; 4 \;\; \text{(by formula)}
\end{aligned}$$

Note that the event $\boldsymbol{B}$ **or** $\boldsymbol{A}$ is the same as $\boldsymbol{A}$ **or** $\boldsymbol{B}$.

*Visualize or shade in the event $\boldsymbol{A}$ **or** $\boldsymbol{B}$ in the Venn Diagram.*

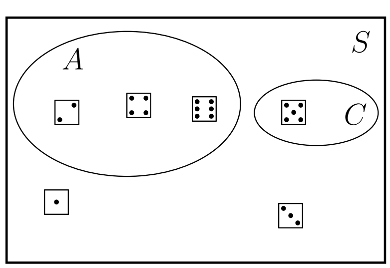
In the case of disjoint or mutually exclusive events we have the following special case formula, since $n(A \text{ or } C) = 0$,

$$\boxed{n(A \text{ or } C) = n(A) + n(C) \quad (A,\, C \text{ mutually exclusive})}$$

There is no overlap so we can just add the number of events.

Example:
Consider the previous experiment of rolling a single die where A is rolling an even number and C is rolling a 5. Find the union of the events, the number of simple events it contains, and show that the above formula for calculating the number of outcomes holds.

Solution:



$$\begin{aligned}
\boldsymbol{A} \textbf{ or } \boldsymbol{C} &= \text{all outcomes in either event } A \text{ or event } C \;=\; \{2,4,6,5\} \\
\boldsymbol{n(A} \textbf{ or } \boldsymbol{C)} &= \text{number of outcomes in } A \text{ or } C \;=\; 4 \;\; \text{(counting directly)} \\
\boldsymbol{n(A} \textbf{ or } \boldsymbol{C)} &= n(A) + n(C) = \; 3+1 \;=\; 4 \;\; \text{(by formula for m.e. events)}
\end{aligned}$$

*Visualize or shade in the event $\boldsymbol{A}$ **or** $\boldsymbol{C}$ in the Venn Diagram.*

Note that the union event A or B includes the outcomes for which both A and B occur. If one means to exclude the overlap of events, then the term **exclusive or** may be used. We will never mean this when we use the word *or* in this course. Thus if you have a BLT sandwich for lunch and you are asked if you had bacon or lettuce for lunch the answer is logically yes; you are included in ***B* or *L***.

1.3.6 Other Notation

We list here other common notation which occur equivalent to ours above:

complement	-A	A^c
intersection	A and B	$A \cap B$
union	A or B	$A \cup B$

The notation on the right for intersection and union comes from set theory. Because we wish to emphasize the logic in events we will use "and" and "or" directly in our notation.

Assignment:

Use proper set notation and Venn Diagrams to do the following questions.

1. In a standard deck of 52 playing cards:
   (a) How many cards are sixes?
   (b) How many cards are not sixes?
   (c) How many cards are spades and sixes?
   (d) How many cards are spades or sixes?
2. A graduate of a course is looking for a job. H is the event that the job has a high salary; P is the event that the job has a good pension and F is the event that the job has a good future.
   (a) Draw a Venn Diagram of all possible jobs and place an integer in each separate region within the Venn Diagram. (Check the answer at the back of the study guide to label your regions the same, and then proceed to do the next part of the question.)
   (b) List the numbers of the regions that correspond to the following combinations of events:

   $$\text{-}H, (F \text{ and } P), (H \text{ or } F), \text{-}(H \text{ or } F \text{ or } P), (F\text{and -}H), (H \text{ and } F \text{ and } P)$$
3. A survey of 100 tourists entering international customs shows the following: 60 visited London, 30 visited Rome and 10 visited London as well as Rome.
   (a) Draw a Venn Diagram of the situation. When you draw the diagram, keep in mind that some of the 60 who visited London could have also visited Rome, etc.
   (b) How many tourists visited London or Rome?
   (c) How many tourists visited neither place?
   (d) How many tourists visited only London?
   (e) What proportion of the tourists who visited Rome also visited London?

1.4 Rules Governing Probability

To calculate probabilities more easily, there are some relationships that can be used that are based upon the previous methods of counting outcomes.

1.4.1 Probability of An Event

Recall the basic rule for classical probability, written in terms of our new notation is:

$$\boxed{P(A) = \frac{n(A)}{n(S)}} \; ,$$

provided *all simple events in the sample space S are equally likely.*

Since the number of outcomes corresponding to an event must be at least zero and at most the entire sample space one has $0 \leq n(A) \leq n(S)$. Dividing each term by $n(S)$ and using the above formula gives

$$\boxed{0 \leq P(A) \leq 1} \; .$$

Make sure that when probabilities are calculated, the answer is in this range. While we have used equally likely simple events in deriving this result, this result and the others derived below do not require this to be the case.

1.4.2 Complement Rule

Dividing our complement counting rule,

$$n(A) + n(\text{-}A) = n(S) \; ,$$

by the number of outcomes in the sample space, $n(S)$, gives the following rule for the probability of a complement event:

$$\boxed{P(A) + P(\text{-}A) = 1}$$

Often the probability of the complement of an event is easier to calculate than the event itself. Calculate that event first and then use this rule to find the event of interest.

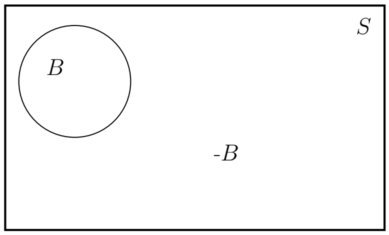
Example:
A biologist samples the fish found in the Frenchman River in south-western Saskatchewan with the following results:

Species	f
Northern Redbelly Dace (N)	2
Fathead Minnow (F)	6
White Sucker (W)	5
Brook Stickleback (B)	3
Iowa Darter (I)	1
	$\sum f = 17$

1. If a fish is selected at random from the sample, what is the probability it is a Brook Stickelback?
2. What is the probability it is not a Brook Stickelback?

Draw a Venn Diagram to identify and count outcomes.

Solution:



1. $\boxed{P(B)} = \dfrac{n(B)}{n(S)} = \boxed{\dfrac{3}{17}}$

2. $\boxed{P(\text{-}B)} = 1 - P(B) = \boxed{\dfrac{14}{17}}$

Notice that because we were asked the likelihood of drawing the fish **from the sample** and not **the river** this is actually a theoretical probability calculation.

1.4.3 Addition Rules

Calculating the probabilities of combinations of events, usually involves either adding or multiplying probabilities. Addition is required when the events are combined by the OR operation as will now be shown. Multiplication is required when the events are combined by the AND operation as will be shown in Section 1.5.

Our two addition rules for counting events

$$n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$$

$$n(A \text{ or } B) = n(A) + n(B) \quad (A,\, B \text{ mutually exclusive})$$

divided by $n(S)$ give the following two addition probability rules, one general and one special case:

$$\boxed{P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)}$$

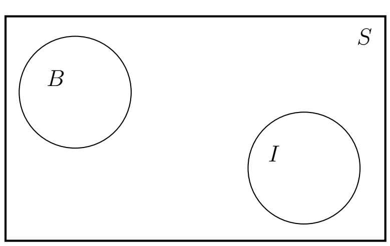
$$\boxed{P(A \text{ or } B) = P(A) + P(B) \quad (A,\, B \text{ mutually exclusive})}$$

These could be extended to include more than two events. For example the second rule for the union of three mutually exclusive events is $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)$.

Venn Diagrams and contingency tables are helpful in applying the above rules.

Example:
In the previous example, find the probability that a fish randomly selected from the sample is a Brook Stickleback (B) or an Iowa Darter (I). Draw a Venn Diagram for the situation.

Solution:



The two events are mutually exclusive as a selected fish cannot belong to both species at once. Using the special case addition rule gives:

$$\boxed{P(B \text{ or } I)} = P(B) + P(I) = \frac{n(B)}{n(S)} + \frac{n(I)}{n(S)} = \frac{3}{17} + \frac{1}{17} = \boxed{\frac{4}{17}}$$

Example:
The biologist in the previous example also examines the sex of each fish in the sample and presents the data the following contingency table:

		Species					
		N	F	W	B	I	Total
Sex	Male	1	4	3	1	0	9
	Female	1	2	2	2	1	8
	Total	2	6	5	3	1	17

If a fish is selected at random from the sample:

1. What is the probability that it is female or a Brook Stickelback (B)?
2. What is the probability that it is a Northern Redbelly Dace (N) or a White Sucker (W)?
3. Draw a Venn Diagram.

Use the symbols M and -M for Male and Female, since F is already being used for a fish species.
(Also it is desirable to use the complement in cases like this where there are only two possibilities.)

Solution:

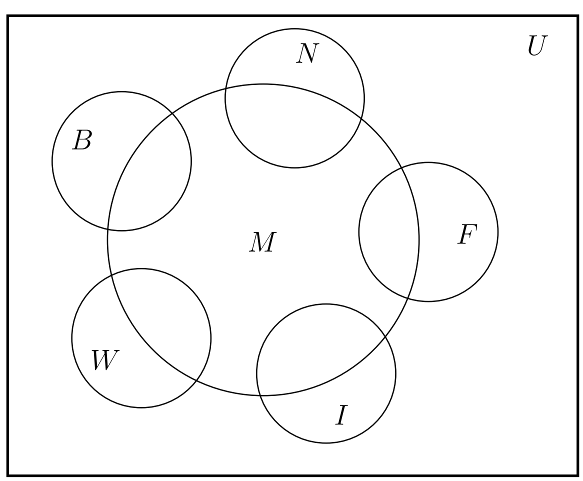
1. One notices that it is possible to draw a fish that is both a Brook Stickelback (B) and Female ($-M$) so the events are **not** mutually exclusive. Using the general addition rule gives:

$$\begin{aligned}\boxed{P(B \text{ or -}M)} &= P(B) + P(\text{-}M) - P(B \text{ and -}M)\\ &= \frac{3}{17} + \frac{8}{17} - \frac{2}{17}\\ &= \boxed{\frac{9}{17} = .529}\end{aligned}$$

2. As seen in the earlier example, different species are mutually exclusive events. (In general two rows or two columns on a contingency table will always describe mutually exclusive events.) Use the special case addition rule:

$$\boxed{P(N \text{ or } W)} = P(N) + P(W) = \frac{n(N)}{n(S)} + \frac{n(W)}{n(S)} = \frac{2}{17} + \frac{5}{17} = \boxed{\frac{7}{17} = .412}$$

3. The following is a Venn Diagram for the entire situation:



Note:

- Each event labelled is associated with the interior of a particular circle or rectangle. Its complement is the exterior region.
- The species circles have been drawn disjoint because we know all the species types are mutually exclusive. (For a general Venn Diagram where nothing is known about mutual exclusivity all the regions would be drawn to overlap.)
- The circles slice up the sample space into disjoint regions in which one should imagine there lie all possible outcomes (here fish). Label each such region with the number of outcomes it contains (the frequency) from the contingency table. Use the joint frequencies from inside the table, not the marginal frequencies. (Why?)
- Similarly label each disjoint region with the probability it contains.
- Any event of interest corresponds to some union of regions in the Venn Diagram for which you can now add up the probability. Consider the following event
  (not a male Fathead Minnow) and (not a female Brook Stickelback)
  Shade the region and find its probability.

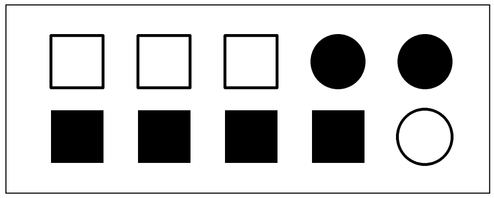
Assignment:

1. A bookkeeper records transactions from 40 invoices to a journal. 3 of the invoices were completed inaccurately.
   (a) An auditor selects one of these transactions at random. What is the probability that the transaction is in error?
   (b) What is the probability that the transaction is not in error?
2. 60% of all students at a school take Chemistry and 50% take Biology. 10% of all students take both Chemistry and Biology.
   (a) Draw a Venn Diagram of the situation.
   (b) Are the events mutually exclusive?
   (c) What is the probability that a student takes either Chemistry or Biology? (Shade the area)
   (d) What is the probability that a student takes neither Chemistry nor Biology? (Shade the area)
   (e) What is the probability that a student takes Chemistry but not Biology? (Shade the area)
3. A card is selected at random from a standard deck of 52 playing cards. What is the probability that the card is either a red five or a black six?

1.5 Conditional Probability and the Multiplication Rules

1.5.1 Conditional Probability

A container contains a group of black and white cubes and spheres:



One object is selected at random. Label the following events as shown:

B = The object is Black.

-B = The object is white (not Black).

C = The object is a Cube.

-C = the object is a sphere (not Cube).

Using our basic probability rules we see that the probability an object is black is:

$$P(B) = \frac{n(B)}{n(S)} = \frac{6}{10} = \frac{3}{5}$$

One may similarly verify that $P(-B) = \frac{2}{5}$, $P(C) = \frac{7}{10}$, $P(-C) = \frac{3}{10}$.

What is the probability the object is a black cube? This means the **joint** probability of it being black **and** a cube:

$$P(B \text{ and } C) = \frac{n(B \text{ and } C)}{n(S)} = \frac{4}{10} = \frac{2}{5}$$

Note that since it is logically the same to be black and a cube (B and C) as it is to be a cube and black (C and B), we have that it is always true that

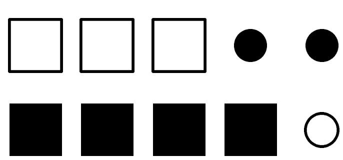
$$P(C \text{ and } B) = P(B \text{ and } C)$$

for any two events B, C. We will next develop a rule for evaluating joint probability, but first we must introduce the concept of conditional probability.

If we **know** that the object we have drawn is a cube, what is the probability that it is black? Such a probability is known as a **conditional probability** and we can write it in shorthand as:

$$P(B|C) = \text{Probability Black Given Cube}$$

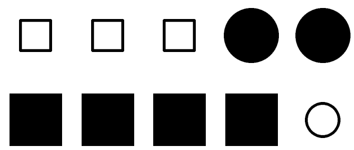
Notice the event that is known, here that the object is a cube (C), is placed second. The vertical bar " | " should be read as "*given*". To calculate the probability, observe that in this case the sample space of events is no longer all the objects, rather the sample space is restricted to the cubes (enlarged):



so that we have:

$$P(B|C) = \frac{n(B \text{ and } C)}{n(C)} = \frac{4}{7}$$

What is the probability that the object is a cube if we know it is black? This is the conditional probability of C given B, $P(C|B)$. Now the colour is known so the sample space is restricted to the black objects:



$$P(C|B) = \frac{n(B \text{ and } C)}{n(B)} = \frac{4}{6} = \frac{2}{3}$$

Comparing $P(C|B)$ to $P(B|C)$ we see that unlike joint probability (and), it is **not** true that $P(C|B) = P(B|C)$. Furthermore note that neither of these equal $P(B \text{ and } C)$.

1.5.2 Multiplication Rule (General)

In general, for two events A and B, we have, as just shown, that

$$P(B|A) = \frac{n(A \text{ and } B)}{n(A)}$$

By dividing the numerator and denominator on the right hand side each by the total number of events in the sample space, $n(S)$ we have

$$P(B|A) = \frac{\frac{n(A \text{ and } B)}{n(S)}}{\frac{n(A)}{n(S)}} = \frac{P(A \text{ and } B)}{P(A)}$$

A useful formula for conditional probability is therefore:

$$\boxed{P(B|A) = \frac{P(A \text{ and } B)}{P(A)}}$$

Solving for the joint probability (and) we get the **multiplication rule**:

$$\boxed{P(A \text{ and } B) = P(A) \cdot P(B|A)}$$

For our example above we can verify that our joint probability result $P(C \text{ and } B) = \frac{2}{5}$ can be obtained using the conditional probability:

$$P(C \text{ and } B) = P(C) \cdot P(B|C) = \frac{7}{10} \cdot \frac{4}{7} = \frac{4}{10} = \frac{2}{5}$$

Since $P(C \text{ and } B) = P(B \text{ and } C)$ we could have also used $P(C|B)$ instead to get the same answer:

$$P(B \text{ and } C) = P(B) \cdot P(C|B) = \frac{3}{5} \cdot \frac{2}{3} = \frac{2}{5}$$

1.5.3 Multiplication Rule (Independent Events)

Our general multiplication rule has a special case which can be used if the two events A, B are **independent**. Two events are independent if the probability of one happening is not changed by the occurrence of the other (or vice versa). This is the case when two coins are flipped – the probability of getting heads on one is not affected by the result (heads or tails) that occurred on the other. The individual coin flip events are independent. In symbols we can define two events A, B to be independent if $P(B|A) = P(B)$. This makes sense because in words it means that the probability of B occurring did not change if A had occurred. In the special case of independent events we see, using this definition in the general multiplication rule, that:

$$\boxed{P(A \text{ and } B) = P(A) \cdot P(B) \quad (A,\, B \text{ independent}) \;\; .}$$

If two events are not independent we say they are **dependent**. For dependent events, or in the case where one does not know whether the events are independent or not, the general multiplication rule must be used.

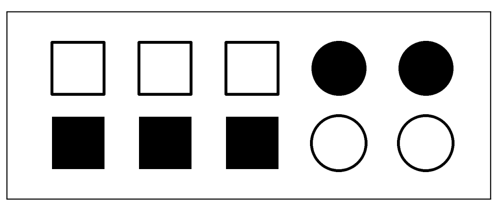
To test whether two events are independent we can test the conditional probability expression above, $P(B|A) = P(B)$, or equivalently we can check if $P(A|B) = P(A)$. Finally if we can show that the multiplication rule for independent events actually holds this also proves independence.[5] In summary

$$\boxed{\text{Events } A,\, B \text{ are } \textbf{independent} \text{ if and only if } \begin{cases} P(B|A) = P(B), \; \textit{or} \\[1em] P(A|B) = P(A), \; \textit{or} \\[1em] P(A \text{ and } B) = P(A) \cdot P(B) \end{cases} .}$$

Example:

1. Are the events of drawing a black object (B) and drawing a cube (C) from the previous example dependent or independent?

2. Repeat the last question if the original set of objects had been:



Solution:

1. We saw above for the original box of objects that $P(B) = 3/5$ while $P(B|C) = 4/7$. Thus $P(B) \neq P(B|C)$ so the events B, C by the first definition are not independent, and hence are dependent. (Try testing the other two definitions of independence and show they also fail.)

2. With the new objects lets check the third definition of independence. We have

$$P(B \text{ and } C) = \frac{n(B \text{ and } C)}{n(S)} = \frac{3}{10} \; ,$$

[5]Why? If $P(A \text{ and } B) = P(A) \cdot P(B)$, then $P(B) = \frac{P(A \text{ and } B)}{P(A)} = P(B|A)$ where the last equality holds as we saw earlier for conditional probability. Hence $P(B|A) = P(B)$ and the events are independent.

while

$$P(B) \cdot P(C) = \frac{5}{10} \cdot \frac{6}{10} = \frac{30}{100} = \frac{3}{10}\ .$$

Hence $P(B \text{ and } C) = P(B) \cdot P(C)$ and the third definition proves the two events B, C are now actually independent. (Try testing the other two definitions of independence and show they also hold.)

The last example shows a case where two events are independent by definition which perhaps was not obvious. Typically we can recognize two events as not having an effect on each other's likelihood and hence being independent so that we can use the simplified multiplication rule. Unlike mutually exclusivity, independence of events cannot be illustrated by a Venn Diagram. Tree diagrams are helpful for sequencing the events when applying the multiplication rules as the next example shows. Finally note that we can generalize the special case multiplication rule for 3 (or more) events that are all independent as:

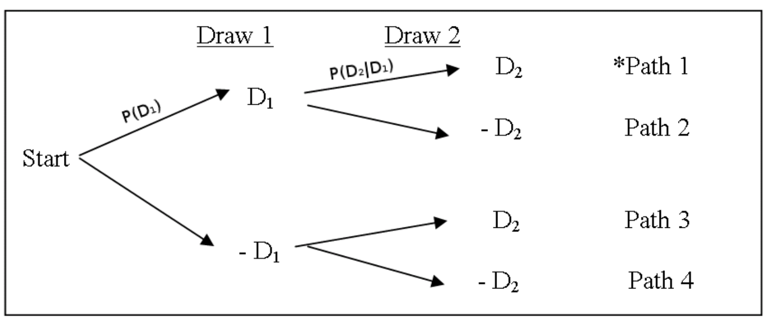
$$P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B) \cdot P(C) \qquad (A,\, B,\, C \text{ independent}).$$

The general multiplication rule can also be generalized to more than two events as shown in the examples below.

Example:
A box holds 20 items. 5 of the items are defective.

1. If an item is drawn from the box and **not replaced** before the next is drawn, what is the probability of selecting two defective items?

   The following tree diagram illustrates the experiment with its possible outcomes being each of the four paths.



   Selecting two defective items corresponds to Path 1 (D_1 and D_2). Since the draws are being done **without replacement**, each time a draw is made the proportion of defective items in the box are dependent upon what was drawn on previous draws. This is the case of **dependent** events. Using the general Multiplication Rule:

$$\begin{aligned}\boxed{P(D_1 \text{ and } D_2)} &= P(D_1) \cdot P(D_2|D_1) \\ &= \frac{5}{20} \cdot \frac{4}{19} = \frac{20}{380} = \boxed{\frac{1}{19}}\end{aligned}$$

   Notice on the tree diagram that we can associate probabilities with each branch, those of Path 1 having been labelled. $P(D_2|D_1)$ is associated with the second branch because at that point we know a defective item has been drawn on the first draw. (D_1 has occurred.) The entire Path 1 corresponds to the event (D_1 and D_2) and its probability is just the product of the probabilities of the branches of which it is composed.

2. If an item is drawn from the box and then **replaced** before the next item is drawn, what is the probability of selecting two defective items?

   The branch of interest in the tree remains the same. Since the items are drawn **with replacement** each time a draw is made the proportion of defective items in the box remains the same and consequently the probability of drawing a defective item on the second draw is the same as on the first. This is the case of **independent events**. Using the Multiplication Rule for independent events.

$$\boxed{P(D_1 \text{ and } D_2)} = P(D_1) \cdot P(D_2) = \frac{5}{20} \cdot \frac{5}{20} = \frac{25}{400} = \boxed{\frac{1}{16}}$$

   Notice that this joint event is slightly more likely to occur when the items are drawn with replacement as there are more defective items available on the second draw in this case.

3. If two items are drawn from the box **at once**, what is the probability that exactly one is defective?

   Drawing two at once is equivalent to drawing two items without replacement because once the first item is chosen it is not available to be picked as the second item. The second and third paths from the top both represent the occurrence of exactly one defective item. So the joint event could happen by path two **or** path three. There individual probabilities are:

$$P(\text{Path 2}) = P(D_1 \text{ and -}D_2) = P(D_1) \cdot P(\text{-}D_2|D_1) = \frac{5}{20} \cdot \frac{15}{19} = \frac{15}{76}$$

$$P(\text{Path 3}) = P(\text{-}D_1 \text{ and } D_2) = P(\text{-}D_1) \cdot P(D_2|\text{-}D_1) = \frac{15}{20} \cdot \frac{5}{19} = \frac{15}{76}$$

   Paths on a tree diagram are mutually exclusive outcomes so using the M.E. addition rule:

$$\begin{aligned}\boxed{P(\text{exactly one defective})} = P(\text{Path 2 or Path 3}) &= P(\text{Path 2}) + P(\text{Path 3}) \\ &= \frac{15}{76} + \frac{15}{76} = \frac{30}{76} = \boxed{\frac{15}{38}}\end{aligned}$$

4. If three items are drawn **without replacement**, what is the probability that all three are defective?

   Imagine a tree diagram with three steps. This joint event corresponds to one path in the tree where a defective item is picked at each each step. The steps in this case are dependent since we are picking without replacement. The general multiplication rule for two events generalizes to three events as follows:[6]

$$\boxed{P(D_1 \text{ and } D_2 \text{ and } D_3)} = P(D_1) \cdot P(D_2|D_1) \cdot P(D_3|D_1 \text{ and } D_2) = \frac{5}{20} \cdot \frac{4}{19} \cdot \frac{3}{18} = \boxed{\frac{1}{114}}$$

5. If three items are drawn **with replacement**, what is the probability that all three items are defective?

   This is represented by the same path in the tree as in the last example. The steps are independent since we are drawing with replacement.

   The multiplication rule for two independent events generalizes to three events as follows:[7]

$$\boxed{P(D_1 \text{ and } D_2 \text{ and } D_3)} = P(D_1) \cdot P(D_2) \cdot P(D_3) = \frac{5}{20} \cdot \frac{5}{20} \cdot \frac{5}{20} = \boxed{\frac{1}{64}}$$

6. If three items are drawn **with replacement**, what is the probability of at least one good item?

   In this case we could calculate the probability of all the paths with at least one not defective (-D) item, which is seven paths! The quick way is to notice this event is the complement of the event that all are defective (last example) and use the complement rule.

$$\boxed{P(\text{at least one not defective})} = P(\,\text{-}\underbrace{[D_1 \text{ and } D_2 \text{ and } D_3]}_{A}\,) \quad = \quad 1 - P(\,\underbrace{D_1 \text{ and } D_2 \text{ and } D_3}_{A}\,)$$
$$= \quad 1 - \frac{1}{64} \;=\; \boxed{\frac{63}{64}}$$

Assignment:

1. In the following experiments, identify the steps in the sequence and determine if the steps are dependent or independent.
   (a) The outcome is observed on each occasion when a coin is flipped 3 times.
   (b) A carton of bolts has 4 that are defective. A handful of three bolts are randomly selected and inspected for defects.
   (c) An opinion poll is conducted in a large shopping centre. People are selected at random and asked their opinion about the type of service they prefer in a restaurant.
2. A shipment of eggs contains 25 cartons of eggs. 6 of the cartons contain cracked eggs. A customer buys some of the cartons without inspecting the contents. What is the probability that 2 of the cartons contain cracked eggs if:
   (a) The customer buys two cartons.
   (b) The customer buys three cartons.
3. A card is dealt from a standard deck of 52 playing cards.
   (a) What is the probability that the card is black and a six?
   (b) Are the events of drawing a black card and drawing a six independent?
4. Of the 20 members who attended an organizational meeting, 5 are women. A secretary and a treasurer are to be chosen from the group by a random draw.
   (a) If the same person can hold both offices, what is the probability that both offices are held by women?
   (b) If the same person cannot hold both offices, what is the probability that both offices are held by women?
5. In the example above, if three items are drawn **without replacement**, what is the probability of at least one good item?

[6] $P(\underbrace{D_1 \text{ and } D_2}_{A} \text{ and } \underbrace{D_3}_{B}) = P(\underbrace{D_1 \text{ and } D_2}_{A}) \cdot P(\underbrace{D_3}_{B} \,|\, \underbrace{D_1 \text{ and } D_2}_{A}) = P(D_1) \cdot P(D_2|D_1) \cdot P(D_3|D_1 \text{ and } D_2)$

[7] $P(\underbrace{D_1 \text{ and } D_2}_{A} \text{ and } \underbrace{D_3}_{B}) = P(\underbrace{D_1 \text{ and } D_2}_{A}) \cdot P(\underbrace{D_3}_{B}) = P(D_1) \cdot P(D_2) \cdot P(D_3)$ (independent events)

1.6 Probability and Odds

Probability and odds are related but not the same. Odds are often quoted in connection with situations that involve *risk*. In decision theory taking a risk means making a decision where a possible outcome of that decision is a loss or an injury. Odds are often associated with gambling activities because it is evident that the risk taken is a loss of money. Many business decisions involve risk. The loss could represent a real financial loss or a loss of opportunity because a certain course of action was followed instead of another more favourable course of action.

Odds are quoted as the following ratio:

$$\boxed{\text{Odds in Favour} = \frac{P(\text{Favour})}{P(\text{Against})}} \qquad \boxed{\text{Odds Against} = \frac{P(\text{Against})}{P(\text{Favour})}}$$

** The fractions are reduced to lowest terms before expressing the ratio. **

Example:

1. A die is rolled. What are the odds in favour of a six?

   Let S = the event of rolling a six and -S = the event of not rolling a six.

   Then $P(S) = 1/6$ and $P(\text{-}S) = 5/6$ so the odds in favour of a six are quoted as:

   $$P(S) : P(-S) = 1 : 5$$

   Notice the connection to the concept of gambling. The integer ratios are the wagers that would have to be placed by two people who were betting on the outcome of the roll of the die. In order that the bet is *fair* to both parties in the wager, the person betting for the 6 should put up $1 for every $5 put up by the person betting against the six. This also makes sense when the spots on the die are examined. One side corresponds to a six and five sides do not.

2. A businessperson assigns odds of 5 : 3 to the success of a newly developed product in the marketplace. What probability does the person place upon the success of the product?

   The total *pot* wagered is 5 + 3 = 8 of which 5 is bet on success.

   $$P(\text{Success}) = \frac{5}{8} = 0.625$$

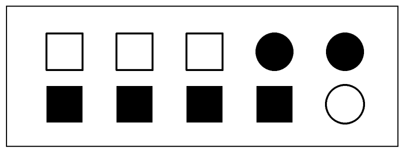
Assignment:

1. What are the odds in favour of drawing a face card from a deck of 52 standard playing cards on the first draw?
2. What are the odds against rolling a sum of seven on the roll of two six sided dice?
3. A weather forecaster gives the probability of precipitation as 10%. What are the odds against precipitation?
4. A stockbroker quotes the odds as 3 : 2 in favour of the rise in the bond market this week. What probability is the broker assigning to a rise in the bond market?

1.7 Contingency Tables and Bayes' Rule

1.7.1 Frequency Contingency Tables

Contingency tables can be used to represent certain frequency distributions when outcomes involve multiple events, as shown in Section 1.2. Similarly they are useful for the frequency of occurrence of events in theoretical probability calculations as shown in Section 1.4. The notion of conditional probability and the multiplication rules can often be made clearer by constructing a contingency table of the observable outcomes. For example, we can represent our probability problem of Section 1.5 involving the random selection of a single item from among black and white cubes and spheres



with a contingency table:

		Shape		
		C	-C	
Colour	B	4	2	6
	-B	3	1	4
		7	3	10

		Shape		
		C	-C	
Colour	B	$n(B \text{ and } C)$	$n(B \text{ and -}C)$	$n(B)$
	-B	$n(\text{-}B \text{ and } C)$	$n(\text{-}B \text{ and -}C)$	$n(\text{-}B)$
		$n(C)$	$n(\text{-}C)$	$n(S)$

On the right are shown the symbols corresponding to the table entries.

Joint probabilities, such as finding the probability of a white cube (-B and C), are easily interpreted, since we restrict ourselves to the row and column of the events in question. We can read the answer directly from the table because within the table are the joint frequencies of all combinations of pairs of outcomes.

	C	-C	
B	4	2	6
-B	$\boxed{3}$	1	4
	7	3	$\boxed{10}$

$$P(\text{white cube}) = P(\text{-}B \text{ and } C) = \frac{n(\text{-}B \text{ and } C)}{n(S)} = \frac{3}{10}$$

Similarly interpreting a conditional probability is straightforward. The known constraint restricts consideration to its row or column. For instance, what is the conditional probability of getting a cube if you know the picked object is black?

	C	-C	
B	$\boxed{4}$	2	$\boxed{6}$
-B	3	1	4
	7	3	10

$$P(\text{cube given black}) = P(C|B) = \frac{n(B \text{ and } C)}{n(B)} = \frac{4}{6} = \frac{2}{3}$$

Using the contingency table, it is possible to see exactly what it means to say that the sample space is limited to the condition under conditional probability. Note that the denominator in the proportion is reduced to 6 and does not contain the entire group of objects observed since not all 10 objects are black.

Assignment:

Parts are allocated to bins in a warehouse by people on two different shifts. Shift one allocates 2000 parts while shift two allocates 3500 parts. 200 of the parts allocated by shift one are placed in the wrong bins. 250 of the parts allocated by shift two are placed in the wrong bins. Suppose a part is randomly selected from those allocated by both shifts. What is the probability that it was:

1. allocated by shift one and is in the correct bin?
2. allocated by shift two and is in the wrong bin?
3. in the wrong bin given that it was allocated by shift one?
4. allocated by shift two given it was in the correct bin?

1.7.2 Probability Contingency Tables

Up to this point we have used frequency contingency tables whose entries are either experimental frequencies or counts of possible outcomes. We can create a **probability contingency table** by dividing every entry in an experimental frequency contingency table by the sum of frequencies or by dividing every entry in a theoretical probability calculation by the number of elements in the sample space, $n(S)$. As an example of the second case, divide every entry in the contingency table of the last section by $n(S) = 10$ and use $P(A) = n(A)/n(S)$ to get:

		Shape		
		C	-C	
Colour	B	$\frac{4}{10}$	$\frac{2}{10}$	$\frac{6}{10}$
	-B	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{4}{10}$
		$\frac{7}{10}$	$\frac{3}{10}$	1

		Shape		
		C	-C	
Colour	B	$P(B$ and $C)$	$P(B$ and -$C)$	$P(B)$
	-B	$P($-B and $C)$	$P($-B and -$C)$	$P($-$B)$
		$P(C)$	$P($-$C)$	1

On the right are shown the symbols corresponding to the table entries. Note the following:

1. Joint probabilities lie inside the table. For example $P(\text{black sphere}) = P(B \text{ and -}C) = \frac{2}{10}$. Recall that $P(B \text{ and -}C) = P(\text{-}C \text{ and } B)$ for a joint probability.
2. Marginal probabilities lie on the sides of the table. For example $P(\text{cube}) = P(C) = \frac{7}{10}$.
3. The bottom right entry for a probability contingency table is always 1 since the total probability is 1.
4. All rows and columns add up as usual for a contingency table.
5. Conditional probabilities **do not appear** within a probability contingency table. To get a conditional probability one uses the inverted multiplication rule:

$$\boxed{P(B|A) = \frac{P(A \text{ and } B)}{P(A)}}$$

and reads the probabilities on the right hand side from the contingency table. For instance, in the above example the conditional probability of getting a cube given the object is black is:

$$P(C|B) = \frac{P(B \text{ and } C)}{P(B)} = \frac{\frac{4}{10}}{\frac{6}{10}} = \frac{4}{10} \cdot \frac{10}{6} = \frac{4}{6} = \frac{2}{3}$$

We will next turn to problems where the probabilities are what is given and we can use probability contingency tables to simplify complicated problems.

1.7.3 Bayes' Rule for Revising Probabilities

This rule is used to revise the probability of an event in the light of additional evidence for the existence or nonexistence of the event. Bayes' Rule is used to compute a **conditional probability** where the condition is additional evidence about the occurrence of the event.

As a theoretical example, suppose you and a friend set up a game of chance based on betting whether or not an object from among our black and white cubes and spheres is black or not. As we have seen $P(B) = \frac{6}{10} = .6$ and using the odds one could show that for the game to be fair the person betting on black should have to put up \$6 for every \$4 put up by the person betting on white. Next suppose that after the object is drawn, but before the bets are made and the object is is revealed, a third person drawing the object is able to secretly communicate to you the shape of the object. (Say they have the opportunity to feel the object as they are drawing it out blindfolded.) Based on this shape information you could improve your betting strategy for if you know the object is a cube then $P(B|C) = \frac{4}{7} = .57$ which is less than .6 so you should bet on it being white. On the other hand if you know the object is a sphere then $P(B|-C) = \frac{2}{3} = .67$ which is greater than .6 and you should bet on the object being black. The additional evidence you have (shape) allows you to revise your probabilities which in turn allows you to more likely win money than lose it over time.[8]

For a second example, if the probability that a transaction is in error in a journal is 5% the inference is that 5% of all transactions in the journal are incorrect. We could probably revise this figure upward if we had additional evidence about the circumstances of the particular entry. For example, if we knew that the transaction was entered by a new employee, we would likely revise the probability upward from 5%.

Bayes' Rule is for calculating conditional probabilities can be quite complicated. Ultimately it grows out of our simpler formula,

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

as we will see below. Rather than use the Bayes' Rule formula directly, however, such calculations are often made easier by using a contingency table or by sequencing events with a tree diagram.

> **Example:**
> Suppose it is known that a store receives shirts from two different suppliers. Supplier A provides the store with 60% of its shirts. Supplier B provides the rest. 1% of all the shirts supplied by A are missing buttons. 2% of all the shirts supplied by B are missing buttons. A shirt is examined for flaws. It is observed to be missing a button. What is the probability that B supplied the shirt?
>
> **Solution:**
> The first step in solving any of the these problems is to correctly identify the information given with symbols. Here it makes sense to let A and B be the events that the shirt is supplied by the respective supplier. Secondly we can let M be the event a button is missing, while I is the event

[8] This example is for illustration only — neither gambling with friends, nor cheating your friends at gambling for that matter, is being encouraged! See problem 2 in Section 13 for how much you would win each game on average.

the buttons are intact. (Using -A for B and -M for I would also have worked.) In terms of these symbols one identifies the given probabilities:

1. $P(A) = 60\% = .60$
2. $P(B) = P(\text{-}A) = 1 - P(A) = .40$, since *the rest* means the complement.
3. $P(M|A) = 1\% = .01$, since this is probability of a missing button if the supplier A is **known**.
4. $P(M|B) = 2\% = .02$, since here the supplier B is known.

Finally one needs to identify the probability being asked for in our symbols. In this case, if we **know** that the shirt is missing a button (M), the probability that B supplies it is $P(B|M)$. We now show two solutions:

Contingency Table Solution

For a contingency table solution follow these steps:

Step 1) Identify all events with symbols (as above).

Step 2) Identify all given probabilities in terms of the symbols (as above). Some complement event probabilities, such as $P(B)$, may be implied.

Step 3) Convert all given conditional probabilities to joint probabilities using the general multiplication rule. This is required because, as shown earlier, a probability contingency table contains only joint (and) and marginal probabilities, but **no conditional probabilities**. For example, since we know $P(M|A)$ we can get $P(A \text{ and } M)$:

$$P(A \text{ and } M) = P(A) \cdot P(M|A) = (0.60) \cdot (.01) = .006$$

Similarly $P(B \text{ and } M) = (.02) \cdot (.40) = .008$.

Step 4) Create a probability contingency table for the events filling out any known marginal and joint probabilities. Play "contingency table sudoku" by completing the table using the fact that all rows and columns must add up. Remember to put a 1 in the bottom right corner since it is a probability contingency table. Our completed table is as follows:

		Supplier A	Supplier B	
Buttons	Buttons Missing	0.01 x .60 = 0.006	0.02 x 0.40 = 0.008	0.014
	Buttons Intact	0.594	0.392	0.986
		0.600	0.400	1.000

Step 5) Identify the probability of interest. If it is a joint (and) probability or a marginal probability just read it off the table. If, more likely, it is a conditional probability then it does not appear in the table. We calculate it using the inverted multiplication rule. In our case (see the circled row in the contingency table):

$$P(B|M) = \frac{P(M \text{ and } B)}{P(M)} = \frac{.008}{.014} = \frac{4}{7}$$

Notice it is the known evidence, here M, that appears in the denominator of the formula.

With this contingency table most of the problems about joint and conditional probabilities can be answered. In this instance the probability that B supplied the shirt is analyzed in light of the evidence that it is missing a button. In words, if we examined only the shirts that are missing buttons this is the proportion of those that were supplied by B. Notice that B's probability of having supplied the shirt is no longer 40%; once it is known that there is a button missing, it rises to $4/7 = 57\%$. The proportion of outcomes with a certain attribute in the population in this context is called the **prior** probability. The revised probability based on the proportion of these same outcomes in a sample space limited to a given condition is called the **posterior** probability.

Tree Diagram Solution

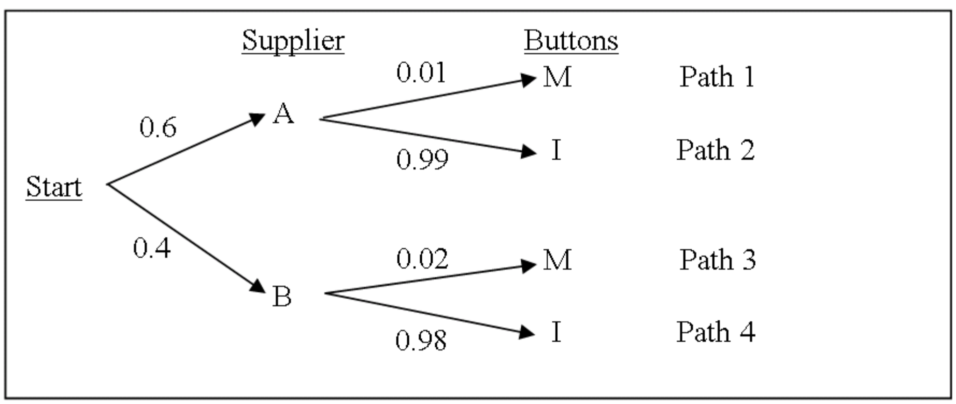
In this method we construct a tree diagram. Follow the following steps:

Step 1) Identify all events with symbols (as above).

Step 2) Identify all given probabilities in terms of the symbols (as above).

Step 3) Create the tree diagram. The first step of the diagram should be selected based upon the category for which you have the total probabilities. In this case we know $P(A)$ and $P(B)$ so the first step is the supplier.

Step 4) Fill in the probabilities. In this case it is the conditional probabilities that appear in the later branches of the tree. (Joint probabilities do not appear on the branches, they are the probabilities of the total paths.) Finally we can use the fact that all the branches emanating from a given vertex must add to get 1 to get remaining branch probabilities. A tree diagram for the current problem is as follows:



Step 5) Evaluate the desired probability. A joint probability (and) can be evaluated by identify the path to which it corresponds and multiplying the probabilities along it. For a conditional probability such as our case start with the inverted multiplication rule:

$$P(B|M) = \frac{P(M \text{ and } B)}{P(M)}$$

Here the numerator is just the probability of Path 3, namely the product of the probabilities along the path, or, in symbols using the multiplication rule:

$$P(M \text{ and } B) = P(B \text{ and } M) = P(B) \cdot P(M|B) = (.4) \cdot (.02) = .008$$

For the denominator, notice both paths 1 and 3 produce missing buttons. The event of a missing button could happen by either path. These are mutually exclusive, so

$$\begin{aligned} P(M) &= P(\text{Path 1}) + P(\text{Path 3}) \\ &= P(A \text{ and } M) + P(B \text{ and } M) \\ &= P(A) \cdot P(M|A) + P(B) \cdot P(M|B) \\ &= (0.6) \cdot (0.01) + (0.4) \cdot (0.02) \\ &= 0.014 \end{aligned}$$

Substituting the numerator and denominator in the original expression gives our final result:

$$P(B|M) = \frac{0.008}{0.014} = \frac{4}{7}$$

Notice that this is exactly the same result as in the first solution.

Problems for which Bayes' Rule applies can be done either with contingency tables or tree diagrams. Some problems may be more easily illustrated by tree than by table and vice versa. There is no need to memorize formulae here if you realize that this is a problem of limiting a sample space to a required condition when calculating a probability. The contingency table method is usually the most straightforward.

So what is Bayes' rule? In the tree diagram of the last problem, if we had just substituted symbols we would get:

$$P(B|M) = \frac{P(B) \cdot P(M|B)}{P(A) \cdot P(M|A) + P(B) \cdot P(M|B)}$$

This is Bayes' rule for our specific problem written in our symbols. It can be written more generally, for instance, in the case of more suppliers. We will not use it directly in this course. More complicated conditional probability problems may similarly be solved as shown next.

Example:
A large tender for supplying gravel was awarded to three different companies. Firm X supplied 20% of the tender, firm Y supplied 30% of the tender and firm Z supplied the rest. A technician tests the gravel supplied by each firm and determines that a certain proportion of gravel supplied by each firm does not meet the specifications laid down in the tender agreement. He observed that 2% from X, 2% from Y, and 1.2% from Z are faulty materials. A portion of road failed, due to faulty gravel that was built with gravel from the stockpile supplied by these three firms. A judge in a lawsuit that arises from the failure wishes to apportion damages to the three firms. What is the probability that each firm supplied the failed gravel?

Solution:
Let P be the event that the gravel is poor (faulty) and G that it is good. Let X, Y, and Z be the event that the grave was supplied by a particular firm. Then $P(X) = .2$, $P(Y) = .3$, and $P(Z) = 1 - .2 - .3 = .5$. One is also given the conditional probabilities $P(P|X) = .02$, $P(P|Y) = .02$, $P(P|Z) = .012$. Converting the latter to their respective joint probabilities and completing the contingency table results in

		Firm			
		X	Y	Z	
Quality	Poor	**0.02** x 0.2= 0.004	**0.02** x 0.3 = 0.006	**0.012** x 0.5 = 0.006	0.016
	Good	0.196	0.294	0.494	0.984
		0.2	**0.3**	**0.5**	1.0

Using the inverted multiplication rule gives:

$$P(X|P) = \frac{P(X \text{ and } P)}{P(P)} = \frac{0.004}{0.016} = \frac{1}{4}$$

$$P(Y|P) = \frac{P(Y \text{ and } P)}{P(P)} = \frac{0.006}{0.016} = \frac{3}{8}$$

$$P(Z|P) = \frac{P(Z \text{ and } P)}{P(P)} = \frac{0.006}{0.016} = \frac{3}{8}$$

So a judge, wise in the ways of statistics, could reasonably make each firm responsible for these fractions of the supplier damages claim. One notices that despite Z having better gravel, that firm also supplied more of it, resulting in the same probability as Y.

Repeat this problem using a tree diagram.

Assignment:

** Show all work by drawing a contingency table or a tree diagram. Use the correct symbols and formulae of probability theory. **

1. A statistics instructor knows from past experience that a student who does all his homework has an 80% chance of passing the course. A student who does not regularly do his homework has only a 30% chance of passing. The instructor notes that in his current class only 55% of the students regularly do their homework.
   (a) What proportion of all the people in the class pass?
   (b) What proportion of all those in the class who passed regularly did their homework?
2. A company has two suppliers for a certain chemical that it uses in its production process. Firm X supplies 80% of the chemical and firm Y supplies the remainder of the chemical. 95% of the chemical from Y is good while only 85% of the chemical from X is good. A certain production run is spoiled due to poor chemical.
   (a) What is the probability that X supplied the chemical?
   (b) What is the probability that Y supplied the chemical?
3. Four ticket sellers are employed at a department store. Betty sells 30% of the tickets. Don sells 30% of the tickets. Sally sells 20% of the tickets and Joe sells 20% of the tickets. They all forget to give the customer their change on occasion. Betty forgets 2% of the time, Don forgets 3% of the time, Sally forgets 5% of the time and Joe forgets 4% of the time.
   (a) What proportion of all customers go away without receiving their change?
   (b) Given that a customer returns because they were not given their change what is the probability that Joe sold them their ticket?
4. (Optional: This is a fairly complex problem.) A marble is to be randomly drawn out of one of three urns. Urn A has 10 black and 10 white marbles. Urn B has 5 white and 15 black marbles. Urn C has 15 white and 5 black marbles. The urn is to be chosen by flipping two coins. If the coins are both heads, urn A is chosen. If the coins are both tails urn B is chosen. If the coins are of unlike sides, urn C is chosen.
   (a) What proportion of the time will urn B be chosen?
   (b) What proportion of the time will a white marble be chosen?
   (c) Given that a white marble was chosen what is the probability that it came from urn B?

1.8 Review Exercises on Basic Probability

Use proper probability symbols to solve the following questions.

1. An analysis of a collection of loans made to 28 agriculture industries, (A), and 32 food stores (F), revealed the following loan assessments. (low risk loans = L)

		LOAN ASSESSMENT	
		Low Risk	Moderate to High Risk
Industry	Agriculture	6	22
	Food Retailers	15	17

Interpret the meaning and give the value of the following probability symbols: (a) $P(A)$ (b) $P(\text{-}A)$ (c) $P(A \text{ and } L)$ (d) $P(A|L)$ (e) $P(\text{-}A \text{ and } -L)$ (f) $P(L|A)$ (g) $P(A|\text{-}L)$

2. A card is drawn from a deck of 52 playing cards. What is the probability that the card is black or a six?

3. A card is drawn from a deck and then replaced. The deck is shuffled and another card is drawn.
   (a) What is the probability that both cards are face cards? (i.e., Jack, Queen, King)
   (b) If both cards are drawn at the same time, what is the probability that both cards are face cards?

4. If 3 cards are drawn from a deck, what is the probability that they are all face cards?

5. A card is drawn from a deck, what is the probability that it is a face card given that it is a spade?

6. In a recent survey of merchants in the downtown core, 80% favoured greater powers of arrest be given to police, 60% favoured a curfew for persons under 16 years of age, 50% favoured both proposals.
   (a) What % of the merchants favoured at least one of the proposals?
   (b) What % of the merchants favoured none of the proposals? Hint: Draw a Venn Diagram.

The following probability exercises are more difficult

7. To be hired by a company, applicants granted interviews must pass two separate tests. Based on experience, only 80% of those interviewed pass the first test. To take the second test an applicant must pass the first test. Only 30% of those who take the second test pass it.
   (a) What proportion of the interviewees are hired?
   (b) What proportion of the people not hired failed test two?
   (c) What are the odds of being hired if an interview is granted?

   Hint: Draw a tree diagram.

8. Twenty percent of homebuyers who have mortgages with a certain bank default on their payments. 45% of those who default were initially given a good credit rating. 80% of those who do not default initially received a good credit rating. Based on this analysis:

(a) What is the probability that a person will default given that they were given a good credit rating initially?

(b) Comment on the validity of the initial credit rating.

Hint: Draw a contingency table.

9. A company has two suppliers for a certain chemical that it uses in its production process. Firm X supplies 80% of the chemical while firm Y supplies the remainder. 95% of the chemical from Y is good while 85% of the chemical from X is good. A certain production run is spoiled because of poor chemical.

   (a) Who most likely supplied the chemical given that it is poor?

   (b) What are the odds that X supplied it?

   Hint: Use a contingency table.

10. Transport Canada estimates that seat belt were worn by drivers in 60% of all reported accidents. In 8% of the accidents where seat belts were worn, the accident was fatal. In 20% of the accidents where seat belts were not worn the accident was fatal.

    (a) What proportion of all reported accidents involved drivers who were wearing seat belts and were killed?

    (b) What proportion of all reported accidents involved drivers who were not wearing seat belts and were killed?

    Hint: Draw a tree diagram.

11. A community is serviced by a major air carrier. 80% of all its flights arrive on time. On three separate occasions during the year, a person travels via this carrier.

    (a) What is the probability that the airline is late on all three occasions?

    (b) What is the probability that the airline is late on exactly one occasion?

    Hint: Draw a tree diagram.

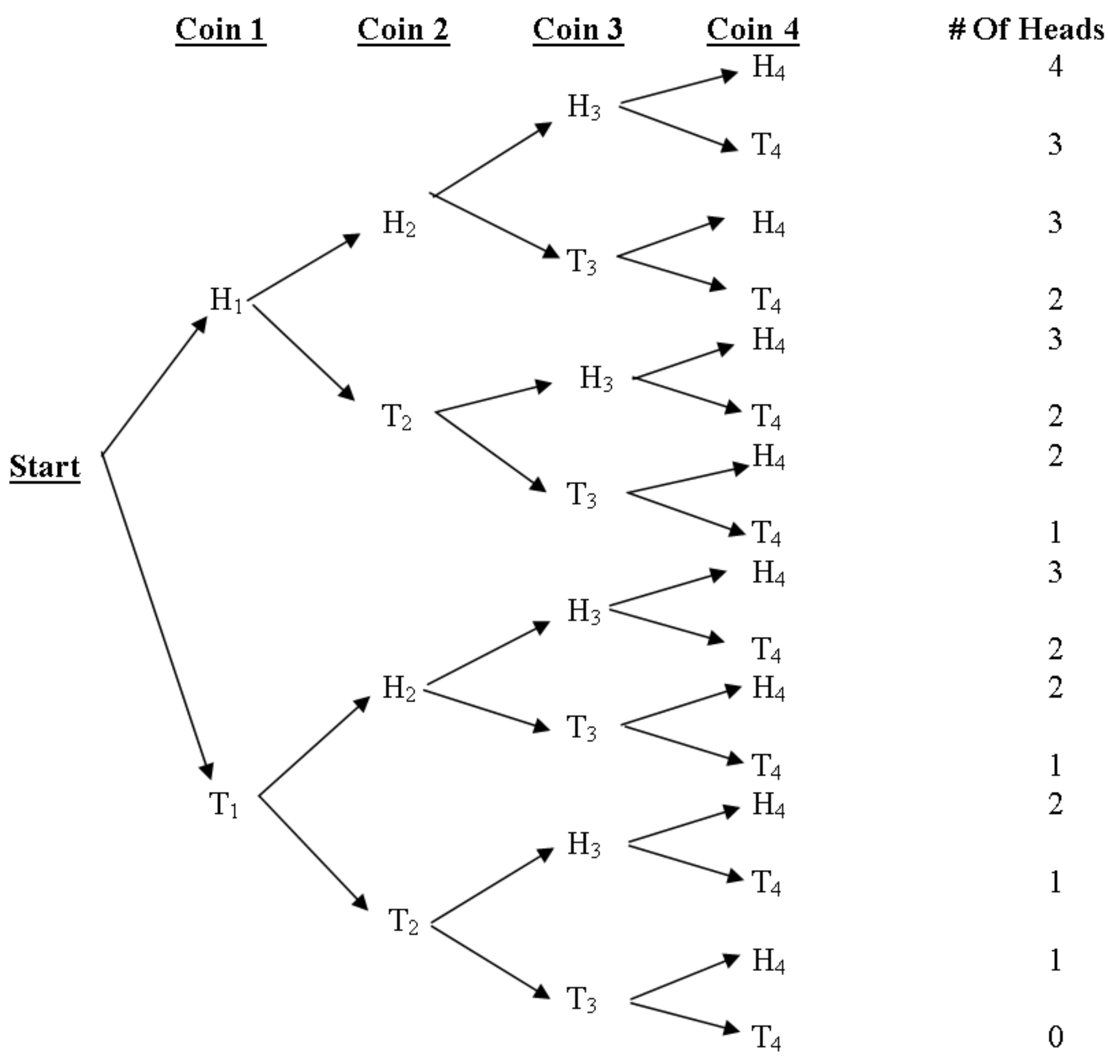
2 Discrete Probability Distributions

2.1 Definition

Probability distributions list all possible values of the random variable together with their probability of occurrence. If the observations result from a count the distribution is called a **discrete probability distribution**.

Example:
4 coins are tossed. Construct a probability distribution of the number of heads observed among the 4 coins.

Here is a tree diagram with all the equally likely possibilities:

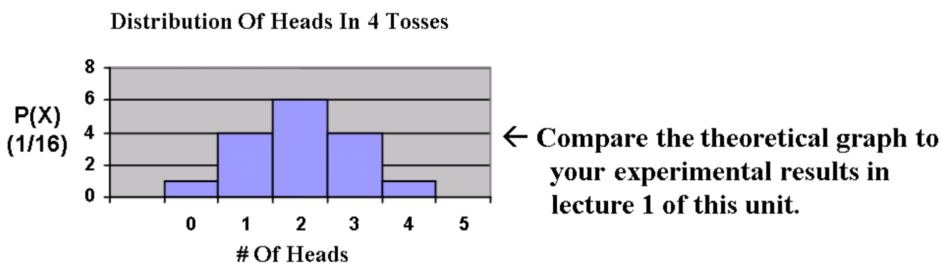


Here is the probability distribution. Because each path is equally likely divide the sum of the number of outcomes that make up an event by 16 to find the probability of the event.

X(heads)	P(X)
0	1/16
1	4/16
2	6/16
3	4/16
4	1/16
	$\sum P(X) = 16/16 = 1$

In Unit 1 of this course we learned that every statistical distribution has three important characteristics. Probability distributions are theoretical distributions with these same features.

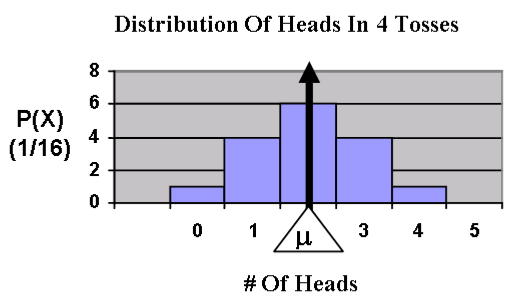
1. **Shape**



The distribution of number of heads has a symmetrical shape.

2. **Centre**

The centre of a probability distribution is called its **expected value**. The expected value is located at the balance point of the distribution.



The formula for finding the mean is:[9]

$$\boxed{\mu = \sum X \cdot P(X)}$$

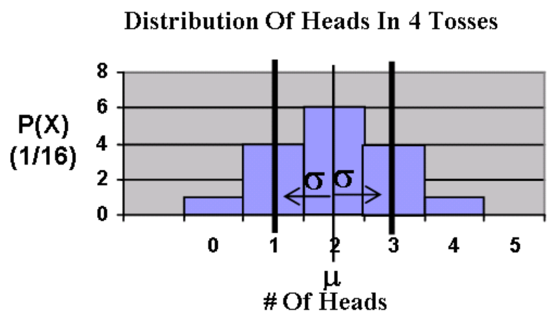
[9] In Unit 1 we could have written the formula for μ in terms of relative frequency P:

$$\mu = \frac{\sum fX}{\sum f} = \left(\frac{1}{\sum f}\right) \sum fX = \sum \left(\frac{1}{\sum f}\right) fX = \sum \left(\frac{f}{\sum f}\right) X = \sum P \cdot X = \sum X \cdot P \ .$$

Since relative frequency is just experimental probability, the formula has to be the same. A similar proof can be made for the computing formula for σ below.

3. **Variability**

The variance of a distribution is defined as its expected squared deviation from the mean and the square root of the variance is the standard deviation. The standard deviation measures the distance from centre in which the majority of possible outcomes are located.



The formulae for finding the variance and standard deviation in a probability distribution are:

$$\sigma^2 = \sum (X-\mu)^2 \cdot P(X) \ \leftarrow \text{ Distribution Variance based on the } \textbf{Definition}$$

$$\boxed{\sigma = \sqrt{\sum (X-\mu)^2 \cdot P(X)}} \ \leftarrow \text{ Distribution Standard Deviation based on the } \textbf{Definition}$$

$$\sigma^2 = \sum X^2 \cdot P(X) - \left(\sum X \cdot P(X)\right)^2 \ \leftarrow \text{ Distribution Variance by the } \textbf{Computing Formula}$$

$$\boxed{\sigma = \sqrt{\sum X^2 \cdot P(X) - \left(\sum X \cdot P(X)\right)^2}} \ \leftarrow \text{ Distribution Standard Deviation by the } \textbf{Computing Formula}$$

Example:
Here are the calculations done in a tabular format for the mean and standard deviation on the coin flipping example

X (heads)	$P(X)$	$XP(X)$ (hds)	$(X-\mu)^2$ (hds^2)	$(X-\mu)^2 P(X)$ (hds^2)	
0	1/16	0	4	4/16	
1	4/16	4/16	1	4/16	
2	6/16	12/16	0	0	
3	4/16	12/16	1	4/16	
4	1/16	4/16	4	4/16	
	$\sum P(X) = 1$	$\sum XP(X) = 2$		$\sum (X-\mu)^2 P(X) = 1$	

Calculate the mean: $\mu = \sum X \cdot P(X) = 2.0$ heads

Calculate the standard deviation: $\sigma = \sqrt{\sum (X-\mu)^2 \cdot P(X)} = \sqrt{1 \text{ (heads}^2)} = 1.0$ head

Refer back to Section 1.1 on probability definitions where we estimated this probability experimentally. Notice that when the theoretical results are compared to what was observed experimentally, the theoretical parameters closely match the experimental statistics. We say that the probability distribution models the physical situation. On the basis of the probability distribution, predictions can be made about a physical situation.

Calculator Note

Some modern calculators allow you to calculate means and standard deviations of discrete probability distributions. Try entering your data like you would for a frequency distribution but in place of the frequency enter the probability associated with the value.

Assignment:

1. Using the computing formula, recalculate the standard deviation for the last example. Use the remaining space on the table for your calculation.

2. A shoe storeowner has kept track of the proportion of times a customer at the store buys X pairs of shoes.

X	0	1	2	3	4
$P(X)$	.17	.56	.20	.05	.02

   (a) If this pattern continues to hold, how many pairs of shoes can the next customer to the store be expected to buy?

   (b) Will there be much variation in this value from customer to customer?

3. A drive-in restaurant owner maintains records of the proportion of vehicles containing a given # of occupants.

X(occupants)	Proportion of Vehicles
1	.05
2	.45
3	.02
4	.30
5	.01
6	.17

   (a) Is this a probability distribution? How can we verify this? What is the significance of the sum of the $P(X)$ column?

   (b) What is the mean of this distribution?

   (c) What is the standard deviation of this distribution?

   (d) By examining the probability column, what is the shape of this distribution?

   (e) If in the future 100 cars pass through the gate, how many people can we expect to have passed through the gates?

4. A game is played with a 10-sided die. 5 of the sides are red, 4 are green and 1 is blue. If a blue turns up the player wins \$10. If a red turns up the player wins \$1. **Hint to do this problem set up a probability distribution.**

   (a) What should the player pay if a green turns up in order that this be a fair game? (i.e. one in which the expected value over the long run is \$0)

   (b) If the operator charges \$5 if a green turns up, how much will the operator gain per play in the long run?

5. A businessperson wishes to start a business venture. The person estimates that there is a 60% chance of making a \$15,000 profit, a 30% chance of earning an \$8,000 profit and a 10% chance of losing \$5,000 on the venture. What is the expected profit from this venture?

2.2 The Binomial Distribution

2.2.1 Definition

The **binomial probability distribution** is a discrete distribution defined by the following conditions:

1. The **experiment** consists of a **fixed sequence of trials**.
2. Each **trial** results in either a **success** or a **failure**. A success means: "What we were looking for happened".
3. The **probability of a success in each trial** is **constant**.
4. The variable $\boldsymbol{X}$ is the discrete **number of successes** in the sequence of trials.
5. The **trials** in the sequence are **independent**.

Note: These criteria are equivalent to a sequence using replacement sampling.

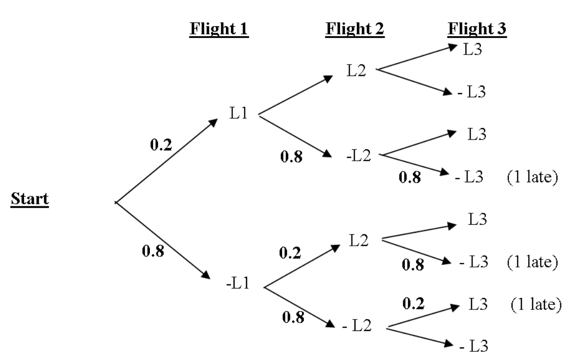
When these conditions exist, a mathematical equation exists to calculate the probabilities in the probability table.

Example:
Verify that these conditions hold for question 11 of Section 1.8 .

Solution:
Recall: A community is serviced by a major air carrier. 80% of all its flights arrive on time. On three separate occasions during the year, a person travels via this carrier. Find the probability that the airliner is late on exactly 1 occasion.

1. The experiment has 3 trials (=flights) in it.
2. Each flight is a success (=late) or a failure (=on time)
3. The probability of success on a trial (a late flight) is fixed at 0.2 .
4. The variable is the number of successes (late flights) among three flights and that is discrete.
5. Whether any flight is late or not does not depend on previous flights so the trials are independent.

This situation satisfies all five conditions so it is called a **binomial experiment**. We will use this example to show how the mathematical equation is developed to compute binomial probabilities. The tree diagram for this experiment shows that not all outcomes for the 3 flights are equally likely.



On a path with 1 late, a joint sequence of 3 things happens. Each of these paths has 1 late flight and 2 not late flights. To find the probability of any of these paths, we must use the multiplication rule for independent events. $P(\text{Path 4}) = (.2) \cdot (.8) \cdot (.8)$ and $P(\text{Path 6}) = (.8) \cdot (.2) \cdot (.8)$ and $P(\text{Path 7}) = (.8) \cdot (.8) \cdot (.2)$. Notice that all of these calculations have one factor of .2 and two of .8 so they produce the same value on multiplication, $(0.2)^1 \cdot (0.8)^2$. Adding the probabilities of the mutually exclusive paths gives

$$\begin{aligned}\boxed{P(\text{exactly 1 late})} &= P(\text{Path 4 or Path 6 or Path 7}) \\ &= P(\text{Path 4}) + P(\text{Path 6}) + P(\text{Path 7}) \\ &= \underbrace{(3)}_{\boxed{\substack{\text{Number of} \\ \text{paths with} \\ \text{1 late}}}} \cdot \underbrace{(0.2)^1 \cdot (0.8)^2}_{\boxed{\substack{\text{Probability} \\ \text{of path} \\ \text{with 1 late}}}} = \boxed{.384}\end{aligned}$$

We can generalize this into the **binomial probability function**:

The probability of **X successes** among a sequence of **n independent trials** where the **probability of a success on a trial** is $\boldsymbol{\pi}$ and the probability of a failure is $\mathbf{1 - \boldsymbol{\pi}}$ can be calculated as:[10]

$$\boxed{P(X) = {}_nC_X \cdot \pi^X \cdot (1-\pi)^{n-X}}$$

You may have encountered the symbol π in previous mathematics courses with respect to circle calculations where π was about 3.14. π is a lower case Greek letter roughly equivalent to the letter p in our alphabet. We will use π to represent a parameter, the population proportion.

Example:
In the experiment of observing the number of heads in the toss of 4 coins, verify that the binomial probability rule could be used to calculate probabilities in this. Calculate the probability of observing 2 heads in the flip of 4 coins.

Solution:
This is a binomial experiment because 1) we have a sequence of four trials (coin flips), 2) each coin turns up in one of two states, heads (success) or tails (failure), 3) the probability of a success remains fixed at 0.5, 4) the number of successes is what is being counted, and 5) the outcome on each coin is independent from the outcomes on the other coins.

Because this is a binomial experiment we can use the binomial probability function to do the calculation. Substitute $n = 4$, $\pi = 0.5$, $1 - \pi = 0.5$, and $X = 2$.

$$P(2) = {}_4C_2 \cdot 0.5^2 \cdot (1 - 0.5)^{4-2} = 0.375$$

We have calculated the value for $X = 2$, but X can have any integer value from 0 to 4. If we applied this equation to the other X values, we would have the probability distribution for this experiment without resorting to drawing the tree diagram or applying the basic probability rules.

[10]The combination ${}_nC_X$ counts the number of paths with exactly X successes. To see this note that identifying a specific path with X successes requires stating which X of the n branches have successes. This is done by selecting X numbers without replacement from $1 \ldots n$, which can be done in ${}_nC_X$ ways.

X	$P(X)$
0	0.0625
1	0.2500
2	0.3750
3	0.2500
4	0.0625
	$\sum P(X) = 1.0000$

Now refer to the binomial probability tables. Notice these tables give the complete probability distribution for any binomial experiment where the number of trials, n is 25 or less and the probability of a success falls into one of the selected values listed in the captions of the table.

Example:
Use the binomial probability tables to verify the probability distribution for the coin flipping experiment.

Solution:
Find the location in the table where n = 4, identify the caption where probability is 0.5 and check the values in the table against the values in the table above.

If the coins being flipped were not fair so that the chance of getting heads on an individual coin is .6, how does this change the probability distribution for the experiment? (Use the tables to complete the table below.) Is the probability distribution still symmetric?

X	$P(X)$
0	
1	
2	
3	
4	
	$\sum P(X) = 1.0000$

Example:
Compute the probability of getting exactly 3 sixes in 5 rolls of a six-sided die.

Solution:
First verify that this is a binomial experiment. In this case $n = 5$, but $\pi = 1/6$ (which is not one of the probability values listed in the table), $1 - \pi = 5/6$, and $X = 3$. Substitute these values into the binomial function:

$$P(3) = {}_5C_3 \cdot \left[\frac{1}{6}\right]^3 \left[\frac{5}{6}\right]^2 = 0.0322 \leftarrow \text{ Must be calculated by calculator}$$

** Notice here that at first glance this may not have seemed to be a binomial problem since each roll of the die has six possible outcomes not two. The problem becomes binomial once it is recognized that success is getting a six while failure is not getting a six. **

2.2.2 Binomial Parameters

In Section 2 we learned that once a probability distribution is constructed, the mean and standard deviation parameters can be calculated by the formulae:

$$\boxed{\mu = \sum X \cdot P(X)} \qquad \boxed{\sigma = \sqrt{\sum (X-\mu)^2 \cdot P(X)}}$$

This is a time consuming procedure especially when n is large. In the case of the binomial distribution, because there is a known function for calculating individual probabilities, these formulae can be evaluated once and for all to find a formula for finding the mean and the standard deviation of a binomial distribution.

$$\boxed{\mu = n\pi} \qquad \boxed{\sigma = \sqrt{n \cdot \pi \cdot (1-\pi)}}$$

Example:
Calculate the mean and standard deviation for the coin flipping experiment and compare the results with those found in Section 2.

Solution:
In this case $n = 4$, $\pi = 0.5$, and $1 - \pi = 0.5$, so

$$\mu = (4) \cdot (0.5) = 2.0 \text{ heads}$$

$$\sigma = \sqrt{(4) \cdot (0.5) \cdot (0.5)} = 1.0 \text{ head}$$

These results are identical to those found previously but with considerably less effort. These shortcut formulae are only applicable for binomial probability distributions.

Assignment:

1. A class of students wrote a 25 question true and false test. Each student guessed at the answer to the questions, what is the probability that a student achieved exactly:
   (a) none correct
   (b) 13 correct
   (c) What would the mean mark in the class be?
   (d) What would the standard deviation of the set of marks in the class be?
2. If the above test in Problem 1 had been a multiple-choice test with 5 selections on each question where only one selection is correct, what would the answers to question 1 be?
3. 70% of all drivers wear seat belts. A random sample of 4 cars are stopped.
   (a) Construct the probability distribution for the number of drivers wearing seat belts. Use the binomial tables.
   (b) Compute the mean and standard deviation of the distribution.
   (c) What is the probability that 2 or fewer drivers are wearing seat belts?
4. (a) In February 2011 Microsoft released a software update for Windows mobile phones and it was found that the update had problems for 1 in 10 users. If Microsoft tested the update on 30 randomly selected phones running the software before its release, what is the probability at least one would have shown problems? (*Hint: Use the complement event.*)
   (b) Out of the 30 phones how many would you expect to fail?
   (c) Are the 30 trials *really* independent so a binomial distribution is justified?
   (d) Assuming Microsoft did not notice the problem before the release, what might account for this given the high probability of detection?

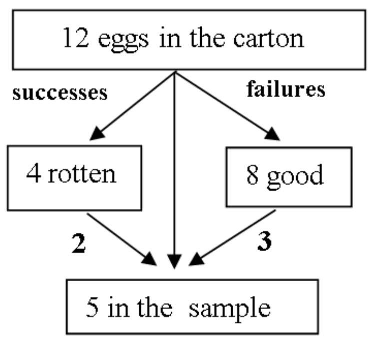
2.3 The Hypergeometric Distribution

2.3.1 Definition

If the binomial criteria are met except that sampling is done without replacement, the binomial function cannot be used. Under these conditions the consecutive trials are dependent and the probability of a success varies from stage to stage in the sequence. The hypergeometric probability distribution applies in this case.

Example:
A carton of 12 eggs contain four that are rotten. 5 eggs are randomly selected from the carton **without replacement**. What is the probability of finding exactly 2 rotten eggs among the 5 selected?

Solution:
This is the hypergeometric case and not the binomial case because the trials are dependent. As in the case of the binomial distribution we will identify success to be what we are counting, namely a rotten egg. It is easiest to visualize the calculation with a sampling diagram.



$$\boxed{P(2\text{ rotten})} = \frac{n(2\text{ rotten})}{n(5\text{ eggs})} = \frac{\begin{bmatrix}\text{Number of ways of}\\ \text{selecting 2 from 4 rotten}\end{bmatrix} \cdot \begin{bmatrix}\text{Number of ways of}\\ \text{selecting 3 from 8 good}\end{bmatrix}}{\begin{bmatrix}\text{Number of ways of}\\ \text{selecting 5 from 12 eggs}\end{bmatrix}}$$

$$= \frac{{}_4C_2 \cdot {}_8C_3}{{}_{12}C_5} = \frac{(6)\cdot(56)}{792} = \boxed{0.424}$$

Generalizing our results we can write down a formula for the hypergeometric distribution. It is computed as

$$P(X) = \frac{\begin{bmatrix}\text{Number of ways of}\\ \text{selecting the successes}\\ \text{from the successes}\end{bmatrix} \cdot \begin{bmatrix}\text{Number of ways of}\\ \text{selecting the failures from}\\ \text{the failures}\end{bmatrix}}{\begin{bmatrix}\text{Number of ways of}\\ \text{selecting the sample from}\\ \text{the population}\end{bmatrix}}$$

We therefore get the **hypergeometric probability function:**

$$\boxed{P(X) = \frac{({}_S C_X) \cdot ({}_{N-S} C_{n-X})}{{}_N C_n}}$$

where

S = # of successes in the population

N = # in the population

X = # of successes required

n = # in the sample

In the example above we have $N = 12$, $n = 5$, $S = 4$, $X = 2$. By inspecting the sampling diagram you can identify the necessary combinations without memorizing the formula.

2.3.2 Large Population Limit

If the population is very large compared to the size of the sample drawn from it, drawing the sample does not significantly change the proportion of successes within the population from one draw to the next. In this case the binomial distribution can be used to approximate probabilities with very little error. Statisticians judge a population to be large when the sample size n is less than 5% of the size of the population N (so $n/N < .05$). The binomial calculation only involves 1 combination calculation as opposed to 3 in the hypergeometric case.

Example:
Of the 50,000 voters in a constituency, 35,000 would vote for candidate Smith. In a random sample of 10 voters selected from the constituency what is the probability that only 2 voters would vote for Smith?

Solution:
This is really the hypergeometric distribution because sampling is done without replacement but as the sample of 10 is drawn from the 50,000 voters the proportion of voters in favour of Smith would stay very close to 70% on each draw. Look the answer up in the binomial tables with $n = 10$, $\pi = 0.7$ and $X = 2$. The answer by the binomial approximation is $P(2) = 0.001$. On the calculator the binomial distribution formula gives $P(2) = .0014467005$.

Assignment:

1. A committee of 6 is to be struck from a group of 18 people. 7 of the 18 are female. If the 6 are to be chosen randomly, what is the probability that:
   (a) no males are chosen?
   (b) exactly half the committee is male?
2. Suppose a handful of 3 marbles is to be randomly drawn from a box containing 200 marbles. If 1/5 of the marbles are white and the rest are coloured, what is the probability of drawing 1 white marble in the sample? **Hint: 3 is less than 5% of 200.**
3. A committee of 10 is to be chosen at random from a community of 1000. 80 members of the community are receiving social assistance. The probability that 2 of the committee members are on social assistance is to be found. Is this a hypergeometric or a binomial case? Use the appropriate rule to compute this probability.
4. A parts bin contains 10 boxes of bolts from supplier A and 30 from supplier B. 10 boxes are selected at random from the parts bin. What is the probability that 5 are from supplier A?
5. A committee of 6 is to be struck from an association of 20 people. There are 12 males and 8 females in the association. If the committee is chosen randomly, what is the probability of choosing:
   (a) a committee that is evenly split, male and female
   (b) a committee with less than 3 females?
6. A parts bin contains 20 bolts. The threads are stripped on half of the bolts. A random sample of 5 bolts is selected from the bin. What is the probability that the sample contains 2 bolts with stripped threads if:
   (a) a bolt is replaced before the next is drawn?
   (b) a bolt is not replaced before the next is drawn?
7. The labour force in a city is comprised of 12,000 workers of whom 2000 are unemployed. A random sample of 25 workers is to be surveyed regarding their opinion about job market strategies.
   (a) What is the probability that there will be 2 unemployed workers chosen in the sample?
   (b) For a truly representative sample, how many unemployed workers should it contain?
8. (a) Using the binomial distribution calculate the probability of $X = 2$ for the egg example if the eggs are drawn with replacement and compare it to the hypergeometric value.
   (b) Calculate the voter opinion example using the hypergeometric formula and compare it to the binomial distribution estimate.

2.4 The Poisson Distribution

2.4.1 Definition

The **Poisson probability distribution** is sometimes referred to as the **law of improbable events** because it is used to compute discrete probabilities where the number of trials is very large and the probability of a success on a trial is very small. In many of these cases the binomial or hypergeometric formulae are the correct formulae to use but are impractical to calculate. Recall what happens to the value of a factorial in a combination when the number of trials is very large. Most computing devices soon run out of computing power for these very large values

If the distribution is discrete, the number of trials very large and the probability of a success very small, probabilities are sometimes calculated by the **Poisson probability function**:

$$\boxed{P(X) = \frac{\mu^X \cdot e^{-\mu}}{X!}}$$

Here we have:

- $\boxed{\mu = n \cdot \pi}$ **Just as in the binomial case.**
- $e = 2.718\ldots$ found with the $\boxed{e^x}$ key on the calculator

Example:
The accident rate for a certain type of industrial accident has been observed to be 1 per 10,000 workers in the industry. An insurance company has 8000 workers insured for this type of accident. What is the probability that the insurance company will have to pay out exactly 2 claims?

Solution:

Step 1) Find the population parameter, $\boldsymbol{\pi}$, for the proportion of workers from this industry who experience this type of accident.

$$\pi = \frac{1}{10,000} = 0.0001$$

Step 2) Since π is small and n is large we can now calculate the probability requested using the Poisson formula. We need the **expected number** μ of workers who will experience this type of accident from among those insured.

$$\mu = n\pi = 8000 \cdot \frac{1}{10,000} = 0.8 \text{ accidents}$$

Step 3) Calculate the probability.

$$P(X = 2) = \frac{\mu^X \cdot e^{-\mu}}{X!} = \frac{0.8^2 \cdot e^{-0.8}}{2!} = 0.143785268 = 0.1438$$

Note:

- If the expected number of accidents had been given to us we could have used the Poisson probability function directly (Step 3).

- This problem is a limiting case of a binomial problem and so could be done using the binomial probability function. (See problem 5.)
- Only from the worker's perspective is the likelihood that he or she individually will be injured is what is "improbable". The likelihood that at least one of the 8000 workers is injured is quite probable. Calculate it! (Answer: $P(X \geq 1) = .5507$)

2.4.2 Queuing Problems

The Poisson distribution is often used to solve problems involving line ups or queues, as the only requirement to do the calculation once certain criteria are met is the average of the distribution.

> **Example:**
> The number of people in an express line was observed to follow a Poisson distribution with an average of 5 waiting in the line at any time. What is the probability that a random check finds there are 7 people waiting in line?
>
> The average value $\mu = 5$ persons is given and we are told that probabilities obey the Poisson rule so substitute $X = 7$ and $\mu = 5$ into the Poisson equation.
>
> $$P(X = 7) = \frac{\mu^X \cdot e^{-\mu}}{X!} = \frac{5^7 \cdot e^{-5}}{7!}$$
>
> Calculate the answer by calculator. You should get .1044 .

Because the a Poisson Distribution depends only upon a single parameter, μ, it is possible to tabulate it upon a single page (compared to the binomial distribution). Confirm on the formula sheet the answers for the last two examples using $\mu = .8$ and $\mu = 5$ respectively.

Assignment:

1. The error rate in typesetting at a shop has been observed to be 1 character in 2000 for a certain type of job. A job of this type contains 18000 characters.
   (a) How many typesetting errors should they expect on the job?
   (b) What is the probability of finding 0 typesetting errors?
   (c) What is the probability of finding 10 typesetting errors?
2. Following a particular office procedure, it has been observed that the error rate in entering transactions in a journal are 7 per 1000. An auditor selects 300 entries and finds 10 errors in transactions entered in the journal by a new employee.
   (a) Is this likely to occur given the error rate of the office?
   (b) If the auditor observes 10 errors is chance the probable cause of the error? Explain.
3. The average number of people in line at the wicket of a parcel post counter is 3. What is the probability that the next time you observe this line up that there will be 6 people waiting?
4. The IT helpdesk at a large institution receives on average 2 calls in any given 15 minute period. (For simplicity, assume that the helpdesk employee is occupied for the full fifteen minutes dealing with the issue.) Use the Poisson table to answer the following.
   (a) What is the probability exactly four helpdesk employees will be answering a call in any given 15 minute period?
   (b) If there are always four employees at the IT switchboard, what is the probability that someone will not be able to get help immediately when they call the helpdesk?
   (c) What is the "improbable event" in this problem that allows us to use the Poisson distribution?
5. Recalculate the probability of paying 2 claims for industrial accidents using the binomial probability function. Compare your answer to the Poisson approximation.

2.5 Review Exercises on Discrete Probability Distributions

1. A grocery store keeps track of the proportion of times that a given number of rotten oranges are found in boxes supplied by a certain produce wholesaler. They observed:

X	0	1	2	3	4	5	6	7
$P(X)$	0.08	0.12	0.16	0.26	0.17	0.10	0.06	0.05

   (a) On the average, how many rotten oranges are found per box?

   (b) How much variation is there among boxes in the number of rotten oranges found per box?

2. Consider the rigged betting game involving black and white spheres and cubes from page 27. What is your expected winnings per game following the revised betting strategy based on knowledge of the shape if it costs \$4 to bet on white and \$6 to bet on black? Remember the strategy is to bet on white (-B) if it is a cube (C) and black (B) if it is a sphere (-C).

   ** Hint: Create a discrete probability distribution with X being the possible winnings (-\$4, \$4, -\$6, \$6). The respective probabilities of each event turn out to be the 4 possible joint probabilities ($P(B$ and $C)$, $P(B$ and -$C)$, P(-B and -$C)$, P(-B AND $C)$). Fill in the table and calculate the expected value μ. **

3. Of all the members of a large credit union, 20% have loans with the credit union. A random sample of 6 members are polled regarding their credit agency. What is the probability that of the 6:

   (a) None have loans with the credit union?

   (b) All six have loans with the credit union?

   (c) At least 1 has a loan with the credit union?

4. A stack of 30 bills contains 10 that are counterfeit. If 2 different bills are examined, what is the probability that:

   (a) One is counterfeit?

   (b) At least one is counterfeit?

5. An auditor knows that in a certain situation standard office procedure is not followed about 3% of the time. The auditor examines 300 transactions made under this condition. What is the probability that:

   (a) in 3 transactions procedure was not followed?

   (b) in less than 5 transactions procedure was not followed?

6. An assembly line is packaging bolts in lots of 100 to a carton. The line is set so that the error rate in packaging is 1%. That is 1% of all packages do not have 100 bolts. Suppose a random sample of 30 cartons of bolts are examined for content. What is the probability that 5 cartons do not have the required number of bolts?

3 Continuous Probability Distributions

3.1 Definition

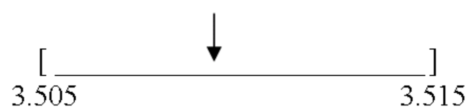
Continuous distributions model situations where the variable's value is obtained by a measuring process. In the cases examined so far, the variable's value has always been obtained by counting and that is why the binomial, hypergeometric and Poisson distributions were referred to as discrete distributions.

A continuous variable might represent a quantity such as:

Length, Weight, Test scores, Volume, Fuel Economy, etc.

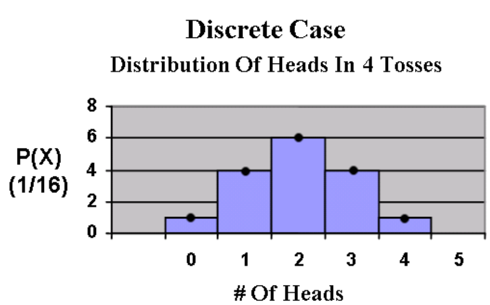
Any measured value, by the nature of the measuring process, is inaccurate. A measuring process is incapable of coming up with an exact value unlike a counting process that always comes up with an exact value.

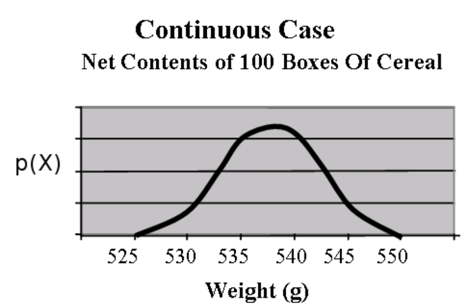
If an object has a reported length of 3.51 m, this is interpreted as a length being in the interval:



The true value for the length is unknown. This notion applies to every measurement. The reported value is only as accurate as the device used to take the measurement. The probability that a continuous variable assumes some exact value is 0. So, for instance, $P(X = 3.51000000\ldots) = 0$. Probabilities for continuous variables must be therefore be specified over an **interval**. For example we wish to know $P(3.505 \text{ m} < X < 3.515 \text{ m})$.

On a discrete distribution curve, individual probabilities P are plotted at distinct locations X on the horizontal axis. We often choose to plot these as bars with width one, both for appearance but also so that the area of the bar also equals the probability (Area $= P \cdot 1 = P$). However, in the discrete case, the intermediate values between the discrete outcomes are not possible measurements. For instance, in our discrete coin-toss histogram below a value of $X = 1.23$ is not a possible outcome. On a continuous distribution curve, there is a point plotted for every value on the horizontal axis since the value of the variable could be anywhere on the interval.

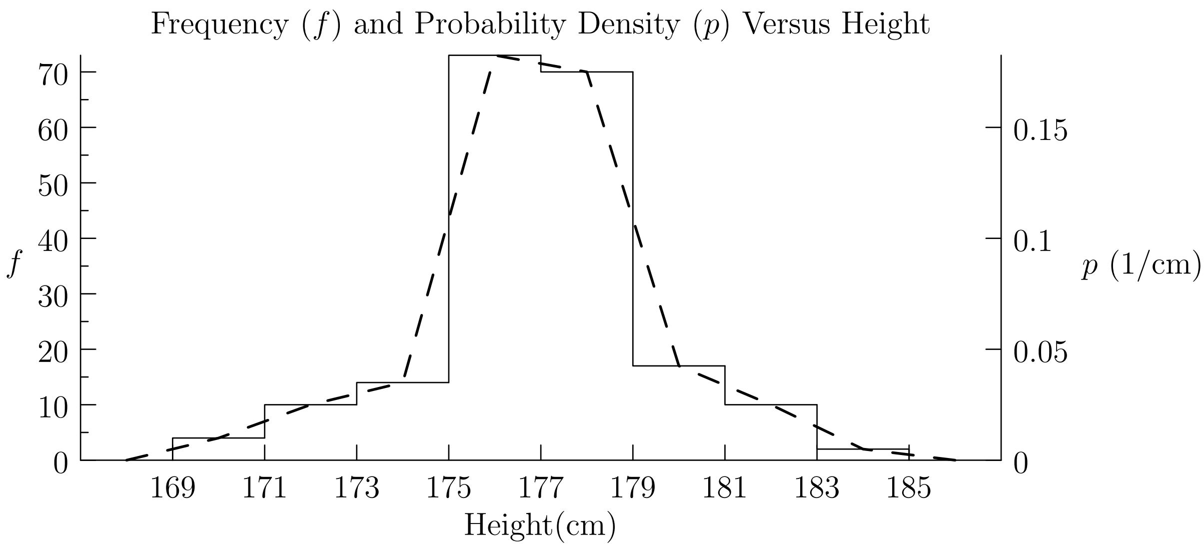




Smooth continuous curves can be represented by a mathematical expression called a **probability density function**, $\mathbf{p(X)}$. The function does not calculate probabilities but calculates the y-values on the curve. It generates the plotted points on the curve. The $p(X)$ curve has the property that the **area under the curve is 1**. The probability of an event is found by determining the **area under the curve** and above the X-axis associated with the **interval** of the event.

3.2 Experimental Probability Densities

It should be note that in Unit 1 we already created our own probability density functions when we plotted the relative frequency density histogram p versus X, or even better the smoothed out relative frequency polygon $p(X)$ versus X for a continuous variable. For example if we created these plots for the distribution of basketball player heights introduced in Section 6.2 of Unit 1 we have:



In Unit 1 if we were interested in measuring the proportion of observations between the two variable values $X = 173$ cm and $X = 179$ cm we would have calculated the area under the relative frequency density polygon (using the heights measured off the p-axis) over the interval $(173 \text{ cm}, 179 \text{ cm})$. To consider $p(X)$ as a probability density one could simply rephrase the question as, "What is the probability that a basketball player in the sample, chosen at random, lies within interval $(173 \text{ cm}, 179 \text{ cm})$?" The answer would be the same. Finally, the more useful probabilistic interpretation, however, is one of inference, namely, if the height of any high-school basketball player in Western Canada is measured, what is the likelihood it lies within $(173 \text{ cm}, 179 \text{ cm})$? Since the sample relative frequency density should be a good approximation of that for the population, the answer, once again, is just the area under the curve over the interval. As such our work in constructing relative frequency density polygons $p(X)$ amounts to creating probability density functions experimentally; only a change of interpretation is required.

Example:
Find the probability a randomly selected basketball player is between 173 cm and 179 cm by estimating the area under the curve. Shade in the area. (Remember to use the probability density axis, p, for the height. Using the histogram line rather than the frequency polygon will be easier since it gives rectangles with area base times height.) (Answer: $\approx .785$)

Solution:
$P(173 \text{ cm} < X < 179 \text{ cm}) =$

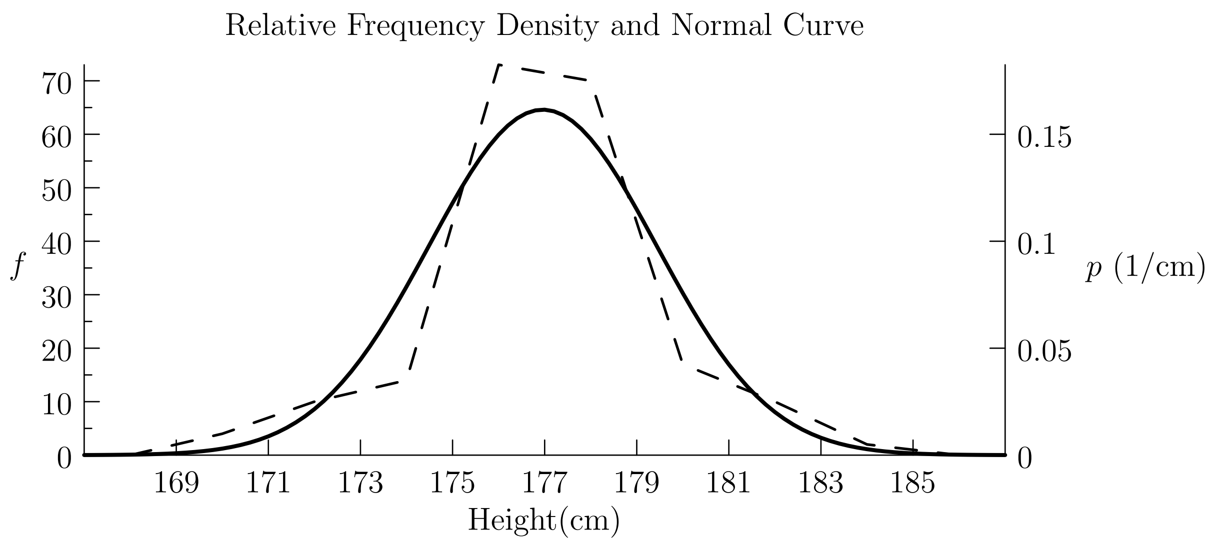
3.3 The Normal Curve or Bell Curve

3.3.1 Definition

Any function of a variable with total area under the curve equalling one could be interpreted as a probability density function, and, as seen above, we can create our own experimentally that have this property. We have already seen that special cases of discrete probability distributions occur: the binomial, hypergeometric, and Poisson distributions. Similarly the **normal curve** or **bell curve** is a special case of a probability density function. The probability density function for the normal curve is:

$$\boxed{p(X) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}}$$

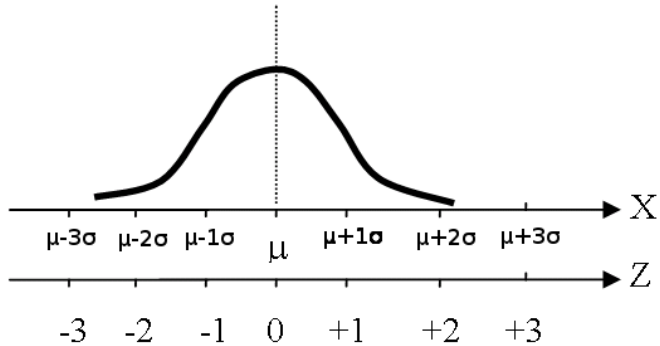
It is not necessary to know this equation to work with the distribution. The normal curve is useful for modeling certain types of measurement distributions and also shows up theoretically. As an example of the former if we take the parameters $\mu = 177.0$ cm and $\sigma = 2.5$ cm, which are the actual mean and standard deviation of the basketball player data, a plot of normal curve (solid line) is as follows:



We see that our frequency polygon (dashed line) derived from the experimental data is quite close to the normal curve and we would say our data is approximately **normally distributed**. It is important to note that the graph of the normal curve is completely specified by its mean μ and standard deviation σ. Different values of μ and σ will produce different curves but all of these curves have essential properties in common. If we made a table of values for X and $y = p(X)$ for any normal distribution with a given mean and standard deviation and plotted the ordered pairs, the curve that results always has some standard features in common with our example:

1. The curve is bell-shaped.
2. The curve is symmetrical about μ.
3. The curve is asymptotic to the X-axis.
4. A fixed proportion of observations (probability) lie within a given number of standard deviations of the mean. The distance from the mean is plotted in units of standard score Z, already introduced in Unit 1, Section 16.2: $\boxed{Z = \frac{X-\mu}{\sigma}}$

Because of this last property, it is useful to add a standard score axis Z below the X-axis:



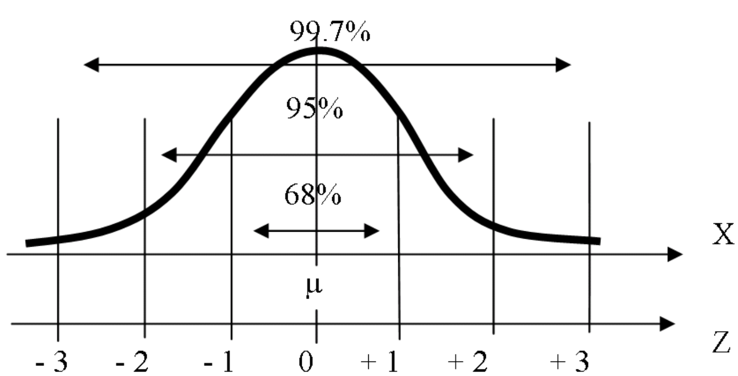
3.3.2 The Empirical Rule

In order to evaluate probabilities involving normally distributed variables we need to be able to find the area under the curve for intervals of interest. Because of standard feature 4 of normal variables detailed in the last section we can state the following known as the **Empirical Rule**:

If a variable is **normally distributed** a measurement has a

- **68%** or roughly **2/3** chance of falling within **1** standard deviation of the mean.
- **95%** chance of falling within **2** standard deviations of the mean.
- **99.7%** chance of falling within **3** standard deviations of the mean. (Effectively all the measurements fall within this interval.)

The following diagram illustrates a curve with this dispersion about average.



In symbols the Empirical Rule states:

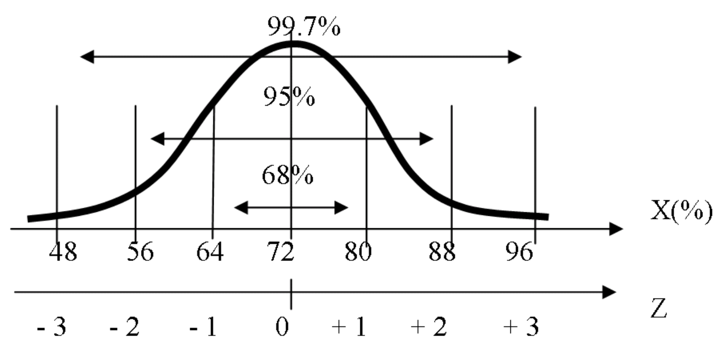
- $P(\mu - 1\sigma < X < \mu + 1\sigma) = P(-1 < Z < 1) \approx .68$
- $P(\mu - 2\sigma < X < \mu + 2\sigma) = P(-2 < Z < 2) \approx .95$
- $P(\mu - 3\sigma < X < \mu + 3\sigma) = P(-3 < Z < 3) \approx .997$

The Empirical Rule should be memorized to allow quick interpretations of normally distributed variables. Note that the Empirical Rule can be used to interpret the likelihood of a measurement of a normally distributed variable in the same way we used Chebyshev's Theorem before in Unit 1 but now more specifically.[11] A value of X with $|Z| \approx 0$ is approximately equal to the mean, with $|Z| \approx 1$ is only slightly removed from the mean, with $|Z| \approx 2$ is moderately removed from the mean while greater than three standard deviations different ($|Z| \gtrsim 3$) is extremely far from the mean (and hence improbable).

Example:
If a set of exam marks are normally distributed with a mean of 72% and a standard deviation of 8%.

1. What is the range of scores achieved?
2. Between what two scores will the middle 2/3 of the class fall?
3. Above what value do the top 2.5% of the scores lie?
4. Is a grade of 40% a typical mark?

Solution:
A diagram illustrating the Empirical Rule for this problem is as follows:



Here we labelled the X-axis just by starting at the mean (72%) and going up and down in units of the standard deviation (8%).

With the aid of the diagram one has:

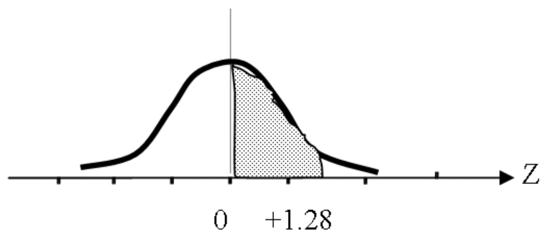
1. $R = X_n - X_1 \approx (\mu + 3\sigma) - (\mu - 3\sigma) = 96\% - 48\% = 48\%$
2. Between $\mu - 1\sigma = 64\%$ and $\mu + 1\sigma = 80\%$
3. We need a tail of area 2.5%, so above $\mu + 2\sigma = 88\%$
4. $Z = \dfrac{X - \mu}{\sigma} = \dfrac{40\% - 72\%}{8\%} = -4$, which is not typical ($|Z| > 3$)

[11]See problem 9 page 60.

3.3.3 Areas Under the Normal Curve

Sometimes we are interested in the probability associated with an arbitrary interval of a normal variable X so the Empirical Rule is of no use. If X has mean μ and standard deviation σ, the procedure used to calculate the area under the normal curve above such an interval is to first transform it into its equivalent interval in terms of the standard score Z which has mean 0 and standard deviation 1. The normal curve with this transformed axis is called the **standard normal curve**. Areas under this curve can be calculated[12] and are tabulated on the formula sheets. Our tables tabulate the area from 0 to Z. More complicated areas are built out of these areas using the properties of the normal distribution.

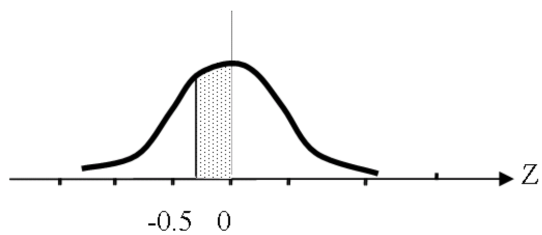
Example 1:



$$P(0 < Z < 1.28) = 0.3997$$

(directly from the table)

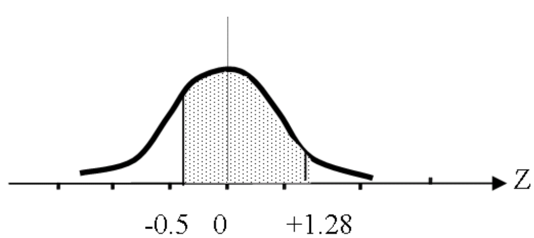
Example 2:



$$P(-0.50 < Z < 0) = 0.1915$$

(since, by symmetry of the normal distribution, $P(-0.50 < Z < 0) = P(0 < Z < 0.50)$)

Example 3:

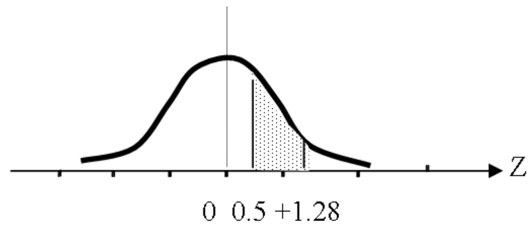


$$P(-0.50 < Z < +1.28) = 0.1915 + 0.3997 = 0.5912$$

(the total shaded region is the sum of the two previous regions.)

[12]The student familiar with calculus will recognize that the problem is one of evaluating a definite integral of the form $\int_a^b e^{-Z^2/2}\,dZ$. Despite its simplicity, the integrand has no antiderivative in terms of simple functions and the integral must be evaluated numerically.

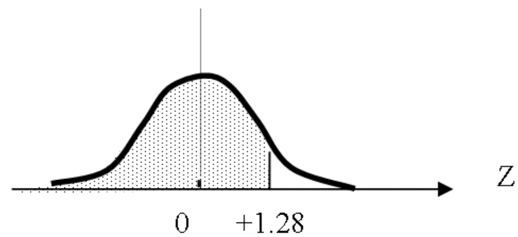
Example 4:



$$P(+0.50 < Z < +1.28) = 0.3997 - 0.1915 = 0.2082$$

(the shaded region is the difference of the first two regions.)

Example 5:



$$P(Z < +1.28) = 0.5 + 0.3997 = .8997$$

(half the normal curve, by symmetry, is 0.5 area)

In summary:

1. The table of normal curve gives the area between centre ($Z = 0$) and a positive Z value.
2. The area between centre and a negative Z value is identical to the area between centre and a positive Z value because of the symmetry of the curve.
3. The area in the table may not be the area on the interval in question. These areas might have to be added together or subtracted. If a tail area is involved remember half the normal curve is 0.5 in area.

Example:
Continuing the previous example, if test scores on a personnel test are normally distributed with a mean of 72% and a standard deviation of 8%.

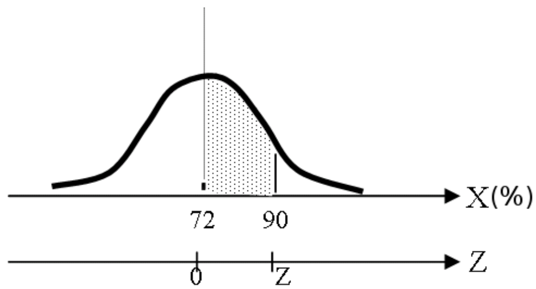
1. What proportion of scores fall between 72% and 90%?
2. What proportion of scores exceed 90%?
3. Below what grade do the lowest 10% of the exams fall?

Solution:

For all normal distribution problems a properly labelled diagram with correct scales on the X and Z-axes allows easy checking of the reasonability of solutions.

Variable X is the test score so draw a bell curve with a test score axis as we did in the original Empirical Rule diagram. To find areas under this curve, a second horizontal axis with the standard score must be included. (Note the scale ticks have been omitted here for clarity.)

1.



Since $X = 72\%$ is the mean, it corresponds to $Z = 0$. Transform $X = 90\%$ to its corresponding Z-score:

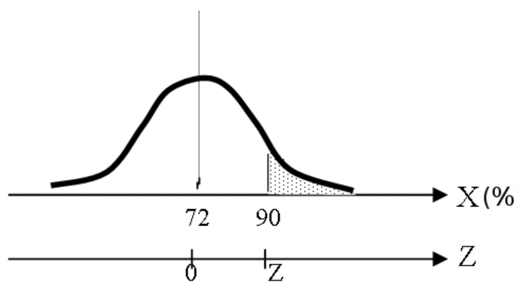
$$Z = \frac{X - \mu}{\sigma} = \frac{90\% - 72\%}{8\%} = 2.25$$

The area in the shaded region is:

$$P(72\% < X < 90\%) = P(0 < Z < 2.25) = 0.4878$$

from the tables.

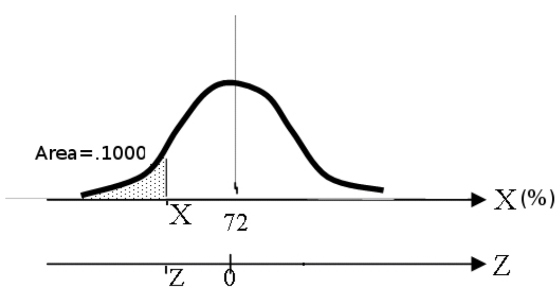
2.



The area in the right tail is:

$$P(90\% < X) = P(2.25 < Z) = 0.5 - 0.4878 = 0.0122$$

3.



In this case we work in reverse. The tail has an area of 10%=.1000, which corresponds to a table area of $.5 - .1000 = .4000$ The closest value inside the table is 0.3997 which corresponds to a Z-value of 1.28. Since we are to the left of the mean, we use $Z = -1.28$. Inverting our Z-score formula gives:

$$\boxed{X = \mu + Z\sigma} = 72\% + (-1.28) \cdot (8\%) = 61.76\% = 62\%$$

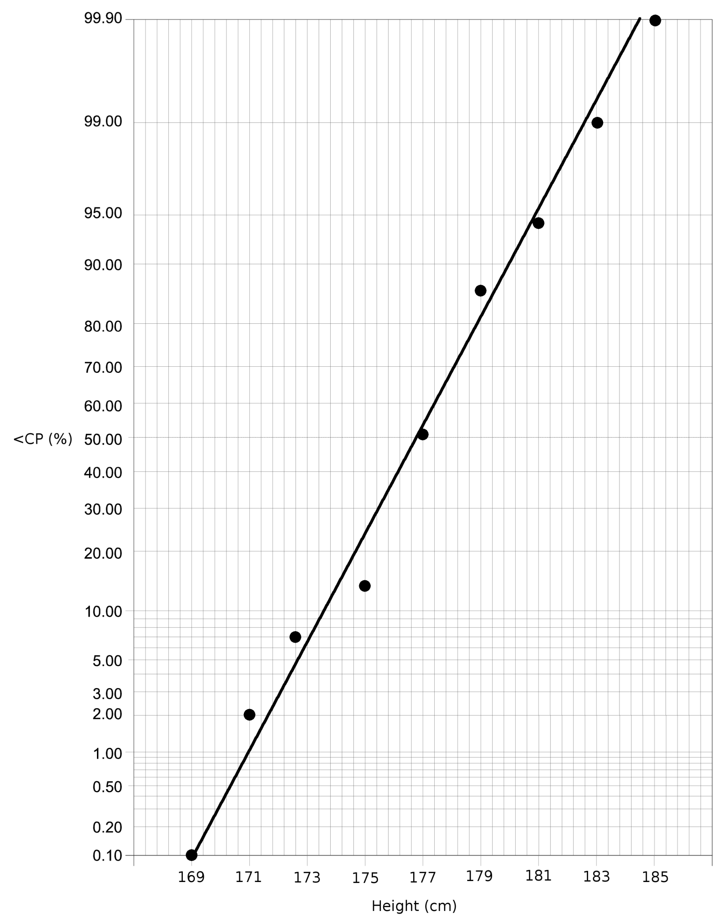
3.4 Tests for Normality

One of the most important objectives of statistics is to determine a goodness of fit of a set of data to a theoretical model. Once we fit a set of data to a particular theoretical model in statistics, we can use that model to solve the problems pertaining to the analysis of that data.

The normal probability model is the most commonly applied model. In most applied problems it would be good to know that the normal distribution is a close approximation to the actual unknown distribution we are working with, particularly because it is easy to use. We have already considered some methods to compare a distribution to the normal distribution:

1. Plot histograms/frequency polygons and compare the appearance observed to that of the normal curve (using the data's mean and standard deviation) for shape.
2. Compute the descriptive summary measures (mean, median, mode, standard deviation) and compare the characteristics of the data with those of the theoretical properties of the normal probability distribution.
3. Compare the proportions of data within units of standard deviations from the mean with those prescribed by the empirical rule.

Another approach to checking for normality is to plot a cumulative relative frequency ogive ($<CP$ versus X) on **normal probability paper**. This type of graph paper has a regular arithmetic horizontal axis for X, but the vertical axis indicates cumulative probabilities associated with a normal distribution. It is a transformed normal scale. If the plot of data makes a relatively straight line, the normal distribution is a good approximation and may be used for solving the problem at hand. Note that the idea of linearizing data is a common method for recognizing trends in observations. As an example if we plot an ogive for the basketball player heights of Section 6.2 of Unit 1 on normal probability paper we see that the points lie quite close to a straight line, suggesting the data is well-described by a normal distribution. This was also suggested by our earlier frequency polygon plot.



Other methods exist for testing the normality of data but are beyond the scope of this presentation.

The normal curve will be used repeatedly in Unit 3. Students are encouraged to master this topic before proceeding.

Assignment:

For each question part below make sure to draw a labeled diagram of the normal curve with the appropriate region shaded in. Pay attention to the proper use of symbols.

1. The distribution of wages in a community is approximately normal with a mean of \$37,500 and a standard deviation of \$3,000. ** The following questions require only the use of the Empirical Rule **
   (a) Draw a labeled diagram to illustrate this information.
   (b) What are the standard scores of incomes of i) \$42,000 ii) \$28,000 iii) \$60,000? Which of these wages would you typically find in the community?
   (c) What proportion of all wage earners in the community earn between \$31,500 and \$43,500?
   (d) What is the probability a wage earner in the community earns more than \$31,500?
   (e) What is the minimum wage likely to be observed in the community?

2. The heights of males in the general population is approximately normal with a mean of 172 cm and a standard deviation of 10 cm. If a male is selected at random, what is the probability his height falls within the range
   (a) 172 cm $< X <$ 192 cm ?
   (b) $X >$ 180 cm ?
   (c) 160 cm $< X <$ 180 cm ?

3. Scores on an aptitude test have been observed to be approximately normal with a mean of 76% and a standard deviation of 5%.
   (a) What proportion of all scores exceed the mean?
   (b) What proportion of the scores are between 65% and 85%?
   (c) Below what score do the bottom 10% of all scores fall?
   (d) Above what score do the top 15% of all scores fall?
   (e) If 1000 people took the test, how many would you expect to score above 80?

4. Delinquent accounts for a chain store are normally distributed with a mean of \$308.45 and a standard deviation of \$43.09. If an account is selected at random from among those that are delinquent, what is the probability that it falls in the range?
   (a) $X >$ \$250
   (b) $X <$ \$400
   (c) \$250 $< X <$ \$400

5. A personnel office hires only those persons who score in the top 5% of an applicant test. Scores on the test vary about an average of 65% with a standard deviation of 10%. What is the cutoff score for being hired?

6. A machine produces parts that are circular in cross section. The diameter of the cross section can be controlled by the production process to vary normally about a mean of 10 cm with a standard deviation of 2 mm. Parts whose diameters differ from the average too significantly are rejected as scrap. If the extreme 10% of the parts are rejected, what is an acceptable range for the diameter of a part?

7. A set of test scores are normally distributed with a mean of 70% and a standard deviation of 10%. Suppose a person receives a percentile of 62. What was the person's actual score on the test?

8. Confirm the Empirical Rule results using areas from the normal distribution table.

9. Using the Empirical Rule, verify that Chebyshev's Theorem applies for the normal distribution with $k = 2$ and $k = 3$.

10. In Section 3.2 you found the probability a basketball player was between 173 cm and 179 cm. In Section 3.3.1 that probability distribution was then modelled with a normal distribution with standard deviation $\mu = 177.0$ cm and standard deviation $\sigma = 2.5$ cm. Recalculate the probability using the normal model and compare it to your original result.

11. You are interested in going south for a week during Spring Break. The following table gives a sample of past years data (up to 2011) for the high temperature on March 1^{st} for various locations.

Destination	March 1^{st} Daily High (°C)									
Cancun, Mexico	28	27	28	28	31	30	28	31	31	29
Los Angeles, California	17	22	29	14	19	19	19	17	18	21
Malibu, California	13	17	22	14	14	15	17	15	15	21
Mazatlan, Mexico	25	26	28	26	27	29	26	23	25	27
Miami, Florida	31	21	29	26	28	26	23	26	29	24
Phoenix, Arizona	24	19	30	27	18	24	22	19	16	22
Puerto Vallarta, Mexico	24	27	29	26	26	27	27	25	27	27
Varadero, Cuba	29	22	28	33	28	24	28	33	29	29

Pick one of the above destinations and do the following calculations. (You can compare the destinations by looking at the answer key when you're done.)

(a) Calculate the mean and standard deviation for the March 1^{st} high temperature for your location. (Check your answer before proceeding to the next question.)

(b) A good day at the beach or lying by an outdoor pool absolutely requires a daily high of at least 25°C. If we assume the daily high temperature for your location is **normally distributed** with the mean and standard deviation you calculated from the last question:

  i. Calculate and interpret the Z-score for 25°C at your location.

  ii. What is the probability that a given day at the beginning of March will have beach weather (high at least 25°C) at your location? (Check your answer before proceeding to the next question.)

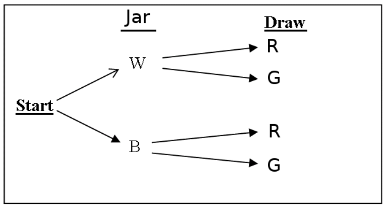
(c) Next, assume you will be staying for one week. If the weather on each day could be considered to be independent – a bad approximation (why?) – use your probability from the last question and the **binomial distribution** to calculate:

  i. The expected (mean) number of days out of the seven that will have beach weather (high at least 25°C).

  ii. The probability that you will have at least 5 days of beach weather for your week.

Answers

page 8:

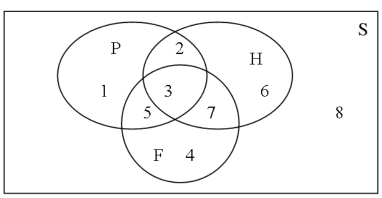
1. 

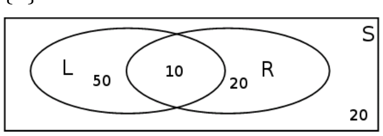
2. 1710
3. 6
4. 15
5. (a) 4845
   (b) 116,280
   (c) 160,000
6.

	Gas Station Operators	Fast Food Outlets	Totals
For Bylaw	10	20	30
Against Bylaw	50	20	70
Totals	60	40	100

7. The tree has 12 paths
8. 13,983,816
9. 404,550

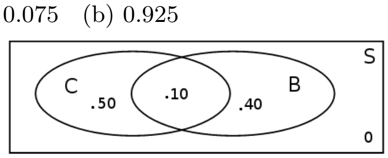
page 13:

1. (a) $n(Sx) = 4$
   (b) $n(\text{-}Sx) = 48$
   (c) $n(Sp \text{ and } Sx) = 1$
   (d) $n(Sp \text{ or } Sx) = 16$
2. (a) 

   (b) {1,5,4,8}, {5,3}, {2,3,4,5,6,7}, {8}, {4,5}, {3}
3. (a) 

   (b) 80
   (c) 20
   (d) 50
   (e) $\frac{1}{3}$

page 17:

1. (a) 0.075 (b) 0.925
2. (a) 

   (b) no since $P(C \text{ and } B) \neq 0$
   (c) 1.00
   (d) 0.00
   (e) 0.5
3. 0.0769

page 23:

1. (a) independent
   (b) dependent
   (c) independent
2. (a) $\frac{1}{20}$ (b) $\frac{57}{460}$
3. (a) $\frac{1}{26}$ (b) yes
4. (a) $\frac{1}{16}$ (b) $\frac{1}{19}$
5. $\frac{113}{114}$

page 24:

1. 3 : 10
2. 5 : 1
3. 9 : 1
4. 0.6

page 26:

1. $\frac{18}{55}$
2. $\frac{1}{22}$
3. $\frac{1}{10}$
4. 0.6436

page 31:

1. (a) 0.575 (b) 0.7652
2. (a) 0.9231 (b) 0.0769
3. (a) 0.0330 (b) 0.2424
4. (a) 0.25 (b) 0.5625 (c) 0.1111

page 32:

1. (a) $P(A) = \frac{7}{15}$ (b) $P(\text{-}A) = \frac{8}{15}$
   (c) $P(A \text{ and } L) = \frac{1}{10}$ (d) $P(A|L) = \frac{2}{7}$
   (e) $P(\text{-}A \text{ and } \text{-}L) = \frac{17}{60}$ (f) $P(L|A) = \frac{3}{14}$
   (g) $P(A|\text{-}L) = \frac{22}{39}$
2. $\frac{7}{13}$
3. $\frac{9}{169}$
4. $\frac{11}{221}$
5. $\frac{11}{1105}$
6. $\frac{3}{13}$
7. (a) 90% (b) 10%
8. (a) 0.24 (b) 0.74 (c) 6 : 19
9. (a) 0.123
   (b) about 12% of the bad credit risks slip through the screening process.
10. (a) $P(X|\text{-}G) = \frac{12}{13}$ while $P(Y|\text{-}G) = \frac{1}{13}$
    (b) 12 : 1
11. (a) 0.048 (b) 0.08

12. (a) 0.008 (b) 0.384

page 37:

1. $\sigma = 1.0$ head
2. (a) $\mu = 1.19$ pair
   (b) $\sigma = 0.84$ pairs; this produces a high $C.V.$
3. (a) Yes, all possible outcomes and the probability of each are listed because $\sum P(X) = 1$.
   (b) 3.28
   (c) 1.56
   (d) trimodal
   (e) 328
4. (a) \$3.75 (b) \$0.50
5. \$10,900

page 42:

1. (a) 2.9802×10^{-8}
   (b) 0.1550
   (c) 12.5
   (d) 2.5
2. (a) 0.0038
   (b) 0.0003
   (c) 5
   (d) 2.0
3. (a)

X	0	1	2	3	4
$P(X)$	0.008	0.076	0.265	0.412	0.240

   (b) $\mu = 2.8$
   (c) $\sigma = 0.92$
   (d) 0.349
4. (a) $P(X \geq 1) = .9576$
   (b) $\mu = 3.0$ failures.
   (c) In theory the trials are dependent assuming the phones were tested without replacement. However given the large number of phones to choose from in the population the probability of failure would not change appreciably after a phone was removed from it and so a binomial probability calculation is justified. The theoretically accurate calculation would require a hypergeometric distribution but the answer would be the same.
   (d) The issue may have been related to Microsoft having not used a random sample. (The problem was reported to have affected largely the Omnia 7 model of Samsung headsets. Microsoft also suggested that those affected may have had a bad Internet connection or too little storage on the computer from which the update was installed, which also may not have been reflected in the testing lab.)

page 45:

1. (a) 0.0004
   (b) 0.3111
2. 0.384
3. In theory hypergeometric but in practice use the binomial equation. 0.1478
4. 0.0424
5. (a) 0.3179
   (b) 0.5449
6. (a) 0.3125
   (b) 0.3483
7. (a) 0.1258
   (b) 4
8. (a) .329, significant difference from hypergeometric .424 since small population ($n/N = 5/12 > .05$).
   (b) .001445261037, well approximated by binomial value .0014467005 since $n/N = 10/50000 < .05$)

page 48:

1. (a) 9
   (b) 0.00012
   (c) 0.1186
2. (a) no, $P(X = 10) = .000056287$. Even $P(X \geq 10)$ is still tiny.
   (b) no
3. 0.05041
4. (a) 0.090
   (b) 0.053
   (c) The improbable event is that a particular person in the institution phones the helpdesk in any given 15 minute period. If, for example, there were $n = 10,000$ people at the institution, since $2 = \mu = n\pi$, the (im)probability of this event would be $\pi = 2/10,000 = .0002$.
5. 0.143790301 well approximated by Poisson's 0.143785268 since n large and π small.

page 49:

1. (a) 3.11
   (b) 1.80
2. \$0.40
3. (a) 0.2621
   (b) 0.0001
   (c) 0.7379
4. (a) 0.4598

(b) 0.5632

5. (a) 0.015

   (b) 0.0549

6. 1.1×10^{-5}

page 59:

1. (a) Range $\approx \$46,500 - \$28,500 = \$18,000$, approx 68% between \$34,500 and \$40,500, approx 95% between \$31,500 and \$43,500

   (b) i) $Z = +1.50$ (typical)
   ii) $Z = -3.17$ (unlikely)
   iii) $Z = +7.50$ (very unlikely)

   (c) 95%

   (d) 97.5%

   (e) \$28,500.00

2. (a) 0.4772

   (b) 0.2119

   (c) 0.6730

3. (a) 0.50

   (b) 0.9502

   (c) 69.6%

   (d) 81.2%

   (e) 212 people

4. (a) 0.9131

   (b) 0.9830

   (c) 0.8961

5. 81.45%

6. 96.71 mm to 103.29 mm

7. 73%

8. $P(-1.00 < Z < 1.00) = .6826$,
   $P(-2.00 < Z < 2.00) = .9544$,
   $P(-3.00 < Z < 3.00) = .9974$

9. $k = 2$: 95% > 75%, $k = 3$: 99.7% > 89%

10. $P(173 \text{ cm} < X < 179 \text{ cm}) = .7333$ (normal) $\approx$ .785 (original)

11. (a)

Destination	$\bar{X}$ (°C)	s (°C)
Cancun, Mexico	29.1	1.5
Los Angeles, California	19.5	4.0
Malibu, California	16.3	3.0
Mazatlan, Mexico	26.2	1.7
Miami, Florida	26.3	3.1
Phoenix, Arizona	22.1	4.3
Puerto Vallarta, Mexico	26.5	1.4
Varadero, Cuba	28.3	3.4

(b)

Destination	Z	Interpretation of Z	$P(X \geq 25°\text{C})$
Cancun, Mexico	-2.73	extremely low	0.9968
Los Angeles, California	1.38	slightly high	0.0838
Malibu, California	2.90	extremely high	0.0019
Mazatlan, Mexico	-0.71	slightly low	0.7611
Miami, Florida	-0.42	approximately average	0.6628
Phoenix, Arizona	0.67	slightly high	0.2514
Puerto Vallarta, Mexico	-1.07	slightly low	0.8577
Varadero, Cuba	-0.97	slightly low	0.8340

Here X is the daily high temperature on March 1$^{\text{st}}$.

(c)

Destination	μ (beach days)	$P(X \geq 5)$
Cancun, Mexico	7.0	$0.999998864 \approx 1$
Los Angeles, California	0.6	$7.5098 \times 10^{-05} \approx 0$
Malibu, California	0.0	$5.1836 \times 10^{-13} \approx 0$
Mazatlan, Mexico	5.3	0.7791
Miami, Florida	4.6	0.5617
Phoenix, Arizona	1.8	0.0132
Puerto Vallarta, Mexico	6.0	0.9354
Varadero, Cuba	5.8	0.9052

Here X is the number of beach days and $P(X \geq 5) = P(5) + P(6) + P(7)$. Note independence from day to day for weather is a bad approximation since weather usually comes in systems that go for multiple days. As such you might expect the distribution for X to be more polarized, with all bad and all good days being more likely than indicated by the binomial distribution.

Binomial Distribution

n=1

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.990	0.950	0.900	0.800	0.700	0.600	0.500	0.400	0.300	0.200	0.100	0.050	0.010
1	0.010	0.050	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	0.950	0.990

n=2

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.980	0.902	0.810	0.640	0.490	0.360	0.250	0.160	0.090	0.040	0.010	0.003	0.000
1	0.020	0.095	0.180	0.320	0.420	0.480	0.500	0.480	0.420	0.320	0.180	0.095	0.020
2	0.000	0.003	0.010	0.040	0.090	0.160	0.250	0.360	0.490	0.640	0.810	0.902	0.980

n=3

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.970	0.857	0.729	0.512	0.343	0.216	0.125	0.064	0.027	0.008	0.001	0.000	0.000
1	0.029	0.135	0.243	0.384	0.441	0.432	0.375	0.288	0.189	0.096	0.027	0.007	0.000
2	0.000	0.007	0.027	0.096	0.189	0.288	0.375	0.432	0.441	0.384	0.243	0.135	0.029
3	0.000	0.000	0.001	0.008	0.027	0.064	0.125	0.216	0.343	0.512	0.729	0.857	0.970

n=4

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.961	0.815	0.656	0.410	0.240	0.130	0.062	0.026	0.008	0.002	0.000	0.000	0.000
1	0.039	0.171	0.292	0.410	0.412	0.346	0.250	0.154	0.076	0.026	0.004	0.000	0.000
2	0.001	0.014	0.049	0.154	0.265	0.346	0.375	0.346	0.265	0.154	0.049	0.014	0.001
3	0.000	0.000	0.004	0.026	0.076	0.154	0.250	0.346	0.412	0.410	0.292	0.171	0.039
4	0.000	0.000	0.000	0.002	0.008	0.026	0.062	0.130	0.240	0.410	0.656	0.815	0.961

n=5

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.951	0.774	0.590	0.328	0.168	0.078	0.031	0.010	0.002	0.000	0.000	0.000	0.000
1	0.048	0.204	0.328	0.410	0.360	0.259	0.156	0.077	0.028	0.006	0.000	0.000	0.000
2	0.001	0.021	0.073	0.205	0.309	0.346	0.312	0.230	0.132	0.051	0.008	0.001	0.000
3	0.000	0.001	0.008	0.051	0.132	0.230	0.312	0.346	0.309	0.205	0.073	0.021	0.001
4	0.000	0.000	0.000	0.006	0.028	0.077	0.156	0.259	0.360	0.410	0.328	0.204	0.048
5	0.000	0.000	0.000	0.000	0.002	0.010	0.031	0.078	0.168	0.328	0.590	0.774	0.951

n=6

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.941	0.735	0.531	0.262	0.118	0.047	0.016	0.004	0.001	0.000	0.000	0.000	0.000
1	0.057	0.232	0.354	0.393	0.303	0.187	0.094	0.037	0.010	0.002	0.000	0.000	0.000
2	0.001	0.031	0.098	0.246	0.324	0.311	0.234	0.138	0.060	0.015	0.001	0.000	0.000
3	0.000	0.002	0.015	0.082	0.185	0.276	0.312	0.276	0.185	0.082	0.015	0.002	0.000
4	0.000	0.000	0.001	0.015	0.060	0.138	0.234	0.311	0.324	0.246	0.098	0.031	0.001
5	0.000	0.000	0.000	0.002	0.010	0.037	0.094	0.187	0.303	0.393	0.354	0.232	0.057
6	0.000	0.000	0.000	0.000	0.001	0.004	0.016	0.047	0.118	0.262	0.531	0.735	0.941

n=7

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.932	0.698	0.478	0.210	0.082	0.028	0.008	0.002	0.000	0.000	0.000	0.000	0.000
1	0.066	0.257	0.372	0.367	0.247	0.131	0.055	0.017	0.004	0.000	0.000	0.000	0.000
2	0.002	0.041	0.124	0.275	0.318	0.261	0.164	0.077	0.025	0.004	0.000	0.000	0.000
3	0.000	0.004	0.023	0.115	0.227	0.290	0.273	0.194	0.097	0.029	0.003	0.000	0.000
4	0.000	0.000	0.003	0.029	0.097	0.194	0.273	0.290	0.227	0.115	0.023	0.004	0.000
5	0.000	0.000	0.000	0.004	0.025	0.077	0.164	0.261	0.318	0.275	0.124	0.041	0.002
6	0.000	0.000	0.000	0.000	0.004	0.017	0.055	0.131	0.247	0.367	0.372	0.257	0.066
7	0.000	0.000	0.000	0.000	0.000	0.002	0.008	0.028	0.082	0.210	0.478	0.698	0.932

n=8

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.923	0.663	0.430	0.168	0.058	0.017	0.004	0.001	0.000	0.000	0.000	0.000	0.000
1	0.075	0.279	0.383	0.336	0.198	0.090	0.031	0.008	0.001	0.000	0.000	0.000	0.000
2	0.003	0.051	0.149	0.294	0.296	0.209	0.109	0.041	0.010	0.001	0.000	0.000	0.000
3	0.000	0.005	0.033	0.147	0.254	0.279	0.219	0.124	0.047	0.009	0.000	0.000	0.000
4	0.000	0.000	0.005	0.046	0.136	0.232	0.273	0.232	0.136	0.046	0.005	0.000	0.000
5	0.000	0.000	0.000	0.009	0.047	0.124	0.219	0.279	0.254	0.147	0.033	0.005	0.000
6	0.000	0.000	0.000	0.001	0.010	0.041	0.109	0.209	0.296	0.294	0.149	0.051	0.003
7	0.000	0.000	0.000	0.000	0.001	0.008	0.031	0.090	0.198	0.336	0.383	0.279	0.075
8	0.000	0.000	0.000	0.000	0.000	0.001	0.004	0.017	0.058	0.168	0.430	0.663	0.923

n=9

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.914	0.630	0.387	0.134	0.040	0.010	0.002	0.000	0.000	0.000	0.000	0.000	0.000
1	0.083	0.299	0.387	0.302	0.156	0.060	0.018	0.004	0.000	0.000	0.000	0.000	0.000
2	0.003	0.063	0.172	0.302	0.267	0.161	0.070	0.021	0.004	0.000	0.000	0.000	0.000
3	0.000	0.008	0.045	0.176	0.267	0.251	0.164	0.074	0.021	0.003	0.000	0.000	0.000
4	0.000	0.001	0.007	0.066	0.172	0.251	0.246	0.167	0.074	0.017	0.001	0.000	0.000
5	0.000	0.000	0.001	0.017	0.074	0.167	0.246	0.251	0.172	0.066	0.007	0.001	0.000
6	0.000	0.000	0.000	0.003	0.021	0.074	0.164	0.251	0.267	0.176	0.045	0.008	0.000
7	0.000	0.000	0.000	0.000	0.004	0.021	0.070	0.161	0.267	0.302	0.172	0.063	0.003
8	0.000	0.000	0.000	0.000	0.000	0.004	0.018	0.060	0.156	0.302	0.387	0.299	0.083
9	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.010	0.040	0.134	0.387	0.630	0.914

n=10

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.904	0.599	0.349	0.107	0.028	0.006	0.001	0.000	0.000	0.000	0.000	0.000	0.000
1	0.091	0.315	0.387	0.268	0.121	0.040	0.010	0.002	0.000	0.000	0.000	0.000	0.000
2	0.004	0.075	0.194	0.302	0.233	0.121	0.044	0.011	0.001	0.000	0.000	0.000	0.000
3	0.000	0.010	0.057	0.201	0.267	0.215	0.117	0.042	0.009	0.001	0.000	0.000	0.000
4	0.000	0.001	0.011	0.088	0.200	0.251	0.205	0.111	0.037	0.006	0.000	0.000	0.000
5	0.000	0.000	0.001	0.026	0.103	0.201	0.246	0.201	0.103	0.026	0.001	0.000	0.000
6	0.000	0.000	0.000	0.006	0.037	0.111	0.205	0.251	0.200	0.088	0.011	0.001	0.000
7	0.000	0.000	0.000	0.001	0.009	0.042	0.117	0.215	0.267	0.201	0.057	0.010	0.000
8	0.000	0.000	0.000	0.000	0.001	0.011	0.044	0.121	0.233	0.302	0.194	0.075	0.004
9	0.000	0.000	0.000	0.000	0.000	0.002	0.010	0.040	0.121	0.268	0.387	0.315	0.091
10	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.006	0.028	0.107	0.349	0.599	0.904

n=11

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.895	0.569	0.314	0.086	0.020	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.099	0.329	0.384	0.236	0.093	0.027	0.005	0.001	0.000	0.000	0.000	0.000	0.000
2	0.005	0.087	0.213	0.295	0.200	0.089	0.027	0.005	0.001	0.000	0.000	0.000	0.000
3	0.000	0.014	0.071	0.221	0.257	0.177	0.081	0.023	0.004	0.000	0.000	0.000	0.000
4	0.000	0.001	0.016	0.111	0.220	0.236	0.161	0.070	0.017	0.002	0.000	0.000	0.000
5	0.000	0.000	0.002	0.039	0.132	0.221	0.226	0.147	0.057	0.010	0.000	0.000	0.000
6	0.000	0.000	0.000	0.010	0.057	0.147	0.226	0.221	0.132	0.039	0.002	0.000	0.000
7	0.000	0.000	0.000	0.002	0.017	0.070	0.161	0.236	0.220	0.111	0.016	0.001	0.000
8	0.000	0.000	0.000	0.000	0.004	0.023	0.081	0.177	0.257	0.221	0.071	0.014	0.000
9	0.000	0.000	0.000	0.000	0.001	0.005	0.027	0.089	0.200	0.295	0.213	0.087	0.005
10	0.000	0.000	0.000	0.000	0.000	0.001	0.005	0.027	0.093	0.236	0.384	0.329	0.099
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.004	0.020	0.086	0.314	0.569	0.895

n=12

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.886	0.540	0.282	0.069	0.014	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.107	0.341	0.377	0.206	0.071	0.017	0.003	0.000	0.000	0.000	0.000	0.000	0.000
2	0.006	0.099	0.230	0.283	0.168	0.064	0.016	0.002	0.000	0.000	0.000	0.000	0.000
3	0.000	0.017	0.085	0.236	0.240	0.142	0.054	0.012	0.001	0.000	0.000	0.000	0.000
4	0.000	0.002	0.021	0.133	0.231	0.213	0.121	0.042	0.008	0.001	0.000	0.000	0.000
5	0.000	0.000	0.004	0.053	0.158	0.227	0.193	0.101	0.029	0.003	0.000	0.000	0.000
6	0.000	0.000	0.000	0.016	0.079	0.177	0.226	0.177	0.079	0.016	0.000	0.000	0.000
7	0.000	0.000	0.000	0.003	0.029	0.101	0.193	0.227	0.158	0.053	0.004	0.000	0.000
8	0.000	0.000	0.000	0.001	0.008	0.042	0.121	0.213	0.231	0.133	0.021	0.002	0.000
9	0.000	0.000	0.000	0.000	0.001	0.012	0.054	0.142	0.240	0.236	0.085	0.017	0.000
10	0.000	0.000	0.000	0.000	0.000	0.002	0.016	0.064	0.168	0.283	0.230	0.099	0.006
11	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.017	0.071	0.206	0.377	0.341	0.107
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.014	0.069	0.282	0.540	0.886

n=13

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.878	0.513	0.254	0.055	0.010	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.115	0.351	0.367	0.179	0.054	0.011	0.002	0.000	0.000	0.000	0.000	0.000	0.000
2	0.007	0.111	0.245	0.268	0.139	0.045	0.010	0.001	0.000	0.000	0.000	0.000	0.000
3	0.000	0.021	0.100	0.246	0.218	0.111	0.035	0.006	0.001	0.000	0.000	0.000	0.000
4	0.000	0.003	0.028	0.154	0.234	0.184	0.087	0.024	0.003	0.000	0.000	0.000	0.000
5	0.000	0.000	0.006	0.069	0.180	0.221	0.157	0.066	0.014	0.001	0.000	0.000	0.000
6	0.000	0.000	0.001	0.023	0.103	0.197	0.209	0.131	0.044	0.006	0.000	0.000	0.000
7	0.000	0.000	0.000	0.006	0.044	0.131	0.209	0.197	0.103	0.023	0.001	0.000	0.000
8	0.000	0.000	0.000	0.001	0.014	0.066	0.157	0.221	0.180	0.069	0.006	0.000	0.000
9	0.000	0.000	0.000	0.000	0.003	0.024	0.087	0.184	0.234	0.154	0.028	0.003	0.000
10	0.000	0.000	0.000	0.000	0.001	0.006	0.035	0.111	0.218	0.246	0.100	0.021	0.000
11	0.000	0.000	0.000	0.000	0.000	0.001	0.010	0.045	0.139	0.268	0.245	0.111	0.007
12	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.011	0.054	0.179	0.367	0.351	0.115
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.010	0.055	0.254	0.513	0.878

n=14

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.869	0.488	0.229	0.044	0.007	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.123	0.359	0.356	0.154	0.041	0.007	0.001	0.000	0.000	0.000	0.000	0.000	0.000
2	0.008	0.123	0.257	0.250	0.113	0.032	0.006	0.001	0.000	0.000	0.000	0.000	0.000
3	0.000	0.026	0.114	0.250	0.194	0.085	0.022	0.003	0.000	0.000	0.000	0.000	0.000
4	0.000	0.004	0.035	0.172	0.229	0.155	0.061	0.014	0.001	0.000	0.000	0.000	0.000
5	0.000	0.000	0.008	0.086	0.196	0.207	0.122	0.041	0.007	0.000	0.000	0.000	0.000
6	0.000	0.000	0.001	0.032	0.126	0.207	0.183	0.092	0.023	0.002	0.000	0.000	0.000
7	0.000	0.000	0.000	0.009	0.062	0.157	0.209	0.157	0.062	0.009	0.000	0.000	0.000
8	0.000	0.000	0.000	0.002	0.023	0.092	0.183	0.207	0.126	0.032	0.001	0.000	0.000
9	0.000	0.000	0.000	0.000	0.007	0.041	0.122	0.207	0.196	0.086	0.008	0.000	0.000
10	0.000	0.000	0.000	0.000	0.001	0.014	0.061	0.155	0.229	0.172	0.035	0.004	0.000
11	0.000	0.000	0.000	0.000	0.000	0.003	0.022	0.085	0.194	0.250	0.114	0.026	0.000
12	0.000	0.000	0.000	0.000	0.000	0.001	0.006	0.032	0.113	0.250	0.257	0.123	0.008
13	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.007	0.041	0.154	0.356	0.359	0.123
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.007	0.044	0.229	0.488	0.869

n=15

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.860	0.463	0.206	0.035	0.005	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.130	0.366	0.343	0.132	0.031	0.005	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.009	0.135	0.267	0.231	0.092	0.022	0.003	0.000	0.000	0.000	0.000	0.000	0.000
3	0.000	0.031	0.129	0.250	0.170	0.063	0.014	0.002	0.000	0.000	0.000	0.000	0.000
4	0.000	0.005	0.043	0.188	0.219	0.127	0.042	0.007	0.001	0.000	0.000	0.000	0.000
5	0.000	0.001	0.010	0.103	0.206	0.186	0.092	0.024	0.003	0.000	0.000	0.000	0.000
6	0.000	0.000	0.002	0.043	0.147	0.207	0.153	0.061	0.012	0.001	0.000	0.000	0.000
7	0.000	0.000	0.000	0.014	0.081	0.177	0.196	0.118	0.035	0.003	0.000	0.000	0.000
8	0.000	0.000	0.000	0.003	0.035	0.118	0.196	0.177	0.081	0.014	0.000	0.000	0.000
9	0.000	0.000	0.000	0.001	0.012	0.061	0.153	0.207	0.147	0.043	0.002	0.000	0.000
10	0.000	0.000	0.000	0.000	0.003	0.024	0.092	0.186	0.206	0.103	0.010	0.001	0.000
11	0.000	0.000	0.000	0.000	0.001	0.007	0.042	0.127	0.219	0.188	0.043	0.005	0.000
12	0.000	0.000	0.000	0.000	0.000	0.002	0.014	0.063	0.170	0.250	0.129	0.031	0.000
13	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.022	0.092	0.231	0.267	0.135	0.009
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.005	0.031	0.132	0.343	0.366	0.130
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.005	0.035	0.206	0.463	0.860

n=16

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.851	0.440	0.185	0.028	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.138	0.371	0.329	0.113	0.023	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.010	0.146	0.275	0.211	0.073	0.015	0.002	0.000	0.000	0.000	0.000	0.000	0.000
3	0.000	0.036	0.142	0.246	0.146	0.047	0.009	0.001	0.000	0.000	0.000	0.000	0.000
4	0.000	0.006	0.051	0.200	0.204	0.101	0.028	0.004	0.000	0.000	0.000	0.000	0.000
5	0.000	0.001	0.014	0.120	0.210	0.162	0.067	0.014	0.001	0.000	0.000	0.000	0.000
6	0.000	0.000	0.003	0.055	0.165	0.198	0.122	0.039	0.006	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	0.020	0.101	0.189	0.175	0.084	0.019	0.001	0.000	0.000	0.000
8	0.000	0.000	0.000	0.006	0.049	0.142	0.196	0.142	0.049	0.006	0.000	0.000	0.000
9	0.000	0.000	0.000	0.001	0.019	0.084	0.175	0.189	0.101	0.020	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.006	0.039	0.122	0.198	0.165	0.055	0.003	0.000	0.000
11	0.000	0.000	0.000	0.000	0.001	0.014	0.067	0.162	0.210	0.120	0.014	0.001	0.000
12	0.000	0.000	0.000	0.000	0.000	0.004	0.028	0.101	0.204	0.200	0.051	0.006	0.000
13	0.000	0.000	0.000	0.000	0.000	0.001	0.009	0.047	0.146	0.246	0.142	0.036	0.000
14	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.015	0.073	0.211	0.275	0.146	0.010
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.023	0.113	0.329	0.371	0.138
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.028	0.185	0.440	0.851

n=17

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.843	0.418	0.167	0.023	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.145	0.374	0.315	0.096	0.017	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.012	0.158	0.280	0.191	0.058	0.010	0.001	0.000	0.000	0.000	0.000	0.000	0.000
3	0.001	0.041	0.156	0.239	0.125	0.034	0.005	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.008	0.060	0.209	0.187	0.080	0.018	0.002	0.000	0.000	0.000	0.000	0.000
5	0.000	0.001	0.017	0.136	0.208	0.138	0.047	0.008	0.001	0.000	0.000	0.000	0.000
6	0.000	0.000	0.004	0.068	0.178	0.184	0.094	0.024	0.003	0.000	0.000	0.000	0.000
7	0.000	0.000	0.001	0.027	0.120	0.193	0.148	0.057	0.009	0.000	0.000	0.000	0.000
8	0.000	0.000	0.000	0.008	0.064	0.161	0.185	0.107	0.028	0.002	0.000	0.000	0.000
9	0.000	0.000	0.000	0.002	0.028	0.107	0.185	0.161	0.064	0.008	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.009	0.057	0.148	0.193	0.120	0.027	0.001	0.000	0.000
11	0.000	0.000	0.000	0.000	0.003	0.024	0.094	0.184	0.178	0.068	0.004	0.000	0.000
12	0.000	0.000	0.000	0.000	0.001	0.008	0.047	0.138	0.208	0.136	0.017	0.001	0.000
13	0.000	0.000	0.000	0.000	0.000	0.002	0.018	0.080	0.187	0.209	0.060	0.008	0.000
14	0.000	0.000	0.000	0.000	0.000	0.000	0.005	0.034	0.125	0.239	0.156	0.041	0.001
15	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.010	0.058	0.191	0.280	0.158	0.012
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.017	0.096	0.315	0.374	0.145
17	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.023	0.167	0.418	0.843

n=18

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.835	0.397	0.150	0.018	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.152	0.376	0.300	0.081	0.013	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.013	0.168	0.284	0.172	0.046	0.007	0.001	0.000	0.000	0.000	0.000	0.000	0.000
3	0.001	0.047	0.168	0.230	0.105	0.025	0.003	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.009	0.070	0.215	0.168	0.061	0.012	0.001	0.000	0.000	0.000	0.000	0.000
5	0.000	0.001	0.022	0.151	0.202	0.115	0.033	0.004	0.000	0.000	0.000	0.000	0.000
6	0.000	0.000	0.005	0.082	0.187	0.166	0.071	0.015	0.001	0.000	0.000	0.000	0.000
7	0.000	0.000	0.001	0.035	0.138	0.189	0.121	0.037	0.005	0.000	0.000	0.000	0.000
8	0.000	0.000	0.000	0.012	0.081	0.173	0.167	0.077	0.015	0.001	0.000	0.000	0.000
9	0.000	0.000	0.000	0.003	0.039	0.128	0.185	0.128	0.039	0.003	0.000	0.000	0.000
10	0.000	0.000	0.000	0.001	0.015	0.077	0.167	0.173	0.081	0.012	0.000	0.000	0.000
11	0.000	0.000	0.000	0.000	0.005	0.037	0.121	0.189	0.138	0.035	0.001	0.000	0.000
12	0.000	0.000	0.000	0.000	0.001	0.015	0.071	0.166	0.187	0.082	0.005	0.000	0.000
13	0.000	0.000	0.000	0.000	0.000	0.004	0.033	0.115	0.202	0.151	0.022	0.001	0.000
14	0.000	0.000	0.000	0.000	0.000	0.001	0.012	0.061	0.168	0.215	0.070	0.009	0.000
15	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.025	0.105	0.230	0.168	0.047	0.001
16	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.007	0.046	0.172	0.284	0.168	0.013
17	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.013	0.081	0.300	0.376	0.152
18	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.018	0.150	0.397	0.835

n=19

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.826	0.377	0.135	0.014	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.159	0.377	0.285	0.068	0.009	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.014	0.179	0.285	0.154	0.036	0.005	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	0.001	0.053	0.180	0.218	0.087	0.017	0.002	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.011	0.080	0.218	0.149	0.047	0.007	0.001	0.000	0.000	0.000	0.000	0.000
5	0.000	0.002	0.027	0.164	0.192	0.093	0.022	0.002	0.000	0.000	0.000	0.000	0.000
6	0.000	0.000	0.007	0.095	0.192	0.145	0.052	0.008	0.001	0.000	0.000	0.000	0.000
7	0.000	0.000	0.001	0.044	0.153	0.180	0.096	0.024	0.002	0.000	0.000	0.000	0.000
8	0.000	0.000	0.000	0.017	0.098	0.180	0.144	0.053	0.008	0.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.005	0.051	0.146	0.176	0.098	0.022	0.001	0.000	0.000	0.000
10	0.000	0.000	0.000	0.001	0.022	0.098	0.176	0.146	0.051	0.005	0.000	0.000	0.000
11	0.000	0.000	0.000	0.000	0.008	0.053	0.144	0.180	0.098	0.017	0.000	0.000	0.000
12	0.000	0.000	0.000	0.000	0.002	0.024	0.096	0.180	0.153	0.044	0.001	0.000	0.000
13	0.000	0.000	0.000	0.000	0.001	0.008	0.052	0.145	0.192	0.095	0.007	0.000	0.000
14	0.000	0.000	0.000	0.000	0.000	0.002	0.022	0.093	0.192	0.164	0.027	0.002	0.000
15	0.000	0.000	0.000	0.000	0.000	0.001	0.007	0.047	0.149	0.218	0.080	0.011	0.000
16	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.017	0.087	0.218	0.180	0.053	0.001
17	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.005	0.036	0.154	0.285	0.179	0.014
18	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.009	0.068	0.285	0.377	0.159
19	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.014	0.135	0.377	0.826

n=20

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.818	0.358	0.122	0.012	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.165	0.377	0.270	0.058	0.007	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.016	0.189	0.285	0.137	0.028	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	0.001	0.060	0.190	0.205	0.072	0.012	0.001	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.013	0.090	0.218	0.130	0.035	0.005	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.002	0.032	0.175	0.179	0.075	0.015	0.001	0.000	0.000	0.000	0.000	0.000
6	0.000	0.000	0.009	0.109	0.192	0.124	0.037	0.005	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.002	0.055	0.164	0.166	0.074	0.015	0.001	0.000	0.000	0.000	0.000
8	0.000	0.000	0.000	0.022	0.114	0.180	0.120	0.035	0.004	0.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.007	0.065	0.160	0.160	0.071	0.012	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.002	0.031	0.117	0.176	0.117	0.031	0.002	0.000	0.000	0.000
11	0.000	0.000	0.000	0.000	0.012	0.071	0.160	0.160	0.065	0.007	0.000	0.000	0.000
12	0.000	0.000	0.000	0.000	0.004	0.035	0.120	0.180	0.114	0.022	0.000	0.000	0.000
13	0.000	0.000	0.000	0.000	0.001	0.015	0.074	0.166	0.164	0.055	0.002	0.000	0.000
14	0.000	0.000	0.000	0.000	0.000	0.005	0.037	0.124	0.192	0.109	0.009	0.000	0.000
15	0.000	0.000	0.000	0.000	0.000	0.001	0.015	0.075	0.179	0.175	0.032	0.002	0.000
16	0.000	0.000	0.000	0.000	0.000	0.000	0.005	0.035	0.130	0.218	0.090	0.013	0.000
17	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.012	0.072	0.205	0.190	0.060	0.001
18	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.028	0.137	0.285	0.189	0.016
19	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.007	0.058	0.270	0.377	0.165
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.012	0.122	0.358	0.818

n=25

	Probability												
x	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	0.778	0.277	0.072	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.196	0.365	0.199	0.024	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.024	0.231	0.266	0.071	0.007	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	0.002	0.093	0.226	0.136	0.024	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.027	0.138	0.187	0.057	0.007	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.006	0.065	0.196	0.103	0.020	0.002	0.000	0.000	0.000	0.000	0.000	0.000
6	0.000	0.001	0.024	0.163	0.147	0.044	0.005	0.000	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.007	0.111	0.171	0.080	0.014	0.001	0.000	0.000	0.000	0.000	0.000
8	0.000	0.000	0.002	0.062	0.165	0.120	0.032	0.003	0.000	0.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.029	0.134	0.151	0.061	0.009	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.012	0.092	0.161	0.097	0.021	0.001	0.000	0.000	0.000	0.000
11	0.000	0.000	0.000	0.004	0.054	0.147	0.133	0.043	0.004	0.000	0.000	0.000	0.000
12	0.000	0.000	0.000	0.001	0.027	0.114	0.155	0.076	0.011	0.000	0.000	0.000	0.000
13	0.000	0.000	0.000	0.000	0.011	0.076	0.155	0.114	0.027	0.001	0.000	0.000	0.000
14	0.000	0.000	0.000	0.000	0.004	0.043	0.133	0.147	0.054	0.004	0.000	0.000	0.000
15	0.000	0.000	0.000	0.000	0.001	0.021	0.097	0.161	0.092	0.012	0.000	0.000	0.000
16	0.000	0.000	0.000	0.000	0.000	0.009	0.061	0.151	0.134	0.029	0.000	0.000	0.000
17	0.000	0.000	0.000	0.000	0.000	0.003	0.032	0.120	0.165	0.062	0.002	0.000	0.000
18	0.000	0.000	0.000	0.000	0.000	0.001	0.014	0.080	0.171	0.111	0.007	0.000	0.000
19	0.000	0.000	0.000	0.000	0.000	0.000	0.005	0.044	0.147	0.163	0.024	0.001	0.000
20	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.020	0.103	0.196	0.065	0.006	0.000
21	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.007	0.057	0.187	0.138	0.027	0.000
22	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.024	0.136	0.226	0.093	0.002
23	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.007	0.071	0.266	0.231	0.024
24	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.024	0.199	0.365	0.196
25	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.004	0.072	0.277	0.778

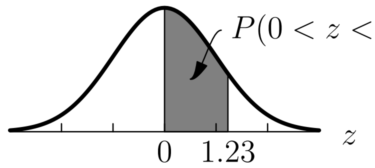
Poisson Distribution

	μ										
x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5
0	0.905	0.819	0.741	0.670	0.607	0.549	0.497	0.449	0.407	0.368	0.223
1	0.090	0.164	0.222	0.268	0.303	0.329	0.348	0.359	0.366	0.368	0.335
2	0.005	0.016	0.033	0.054	0.076	0.099	0.122	0.144	0.165	0.184	0.251
3	0.000	0.001	0.003	0.007	0.013	0.020	0.028	0.038	0.049	0.061	0.126
4	0.000	0.000	0.000	0.001	0.002	0.003	0.005	0.008	0.011	0.015	0.047
5	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.002	0.003	0.014
6	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.004
7	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001
8	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

	μ										
x	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0
0	0.135	0.082	0.050	0.030	0.018	0.011	0.007	0.004	0.002	0.002	0.001
1	0.271	0.205	0.149	0.106	0.073	0.050	0.034	0.022	0.015	0.010	0.006
2	0.271	0.257	0.224	0.185	0.147	0.112	0.084	0.062	0.045	0.032	0.022
3	0.180	0.214	0.224	0.216	0.195	0.169	0.140	0.113	0.089	0.069	0.052
4	0.090	0.134	0.168	0.189	0.195	0.190	0.175	0.156	0.134	0.112	0.091
5	0.036	0.067	0.101	0.132	0.156	0.171	0.175	0.171	0.161	0.145	0.128
6	0.012	0.028	0.050	0.077	0.104	0.128	0.146	0.157	0.161	0.157	0.149
7	0.003	0.010	0.022	0.039	0.060	0.082	0.104	0.123	0.138	0.146	0.149
8	0.001	0.003	0.008	0.017	0.030	0.046	0.065	0.085	0.103	0.119	0.130
9	0.000	0.001	0.003	0.007	0.013	0.023	0.036	0.052	0.069	0.086	0.101
10	0.000	0.000	0.001	0.002	0.005	0.010	0.018	0.029	0.041	0.056	0.071
11	0.000	0.000	0.000	0.001	0.002	0.004	0.008	0.014	0.023	0.033	0.045
12	0.000	0.000	0.000	0.000	0.001	0.002	0.003	0.007	0.011	0.018	0.026
13	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.003	0.005	0.009	0.014
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004	0.007
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.003
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001
17	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001
18	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

	μ								
x	7.5	8.0	8.5	9.0	9.5	10.0	12.0	15.0	20.0
0	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.004	0.003	0.002	0.001	0.001	0.000	0.000	0.000	0.000
2	0.016	0.011	0.007	0.005	0.003	0.002	0.000	0.000	0.000
3	0.039	0.029	0.021	0.015	0.011	0.008	0.002	0.000	0.000
4	0.073	0.057	0.044	0.034	0.025	0.019	0.005	0.001	0.000
5	0.109	0.092	0.075	0.061	0.048	0.038	0.013	0.002	0.000
6	0.137	0.122	0.107	0.091	0.076	0.063	0.025	0.005	0.000
7	0.146	0.140	0.129	0.117	0.104	0.090	0.044	0.010	0.001
8	0.137	0.140	0.138	0.132	0.123	0.113	0.066	0.019	0.001
9	0.114	0.124	0.130	0.132	0.130	0.125	0.087	0.032	0.003
10	0.086	0.099	0.110	0.119	0.124	0.125	0.105	0.049	0.006
11	0.059	0.072	0.085	0.097	0.107	0.114	0.114	0.066	0.011
12	0.037	0.048	0.060	0.073	0.084	0.095	0.114	0.083	0.018
13	0.021	0.030	0.040	0.050	0.062	0.073	0.106	0.096	0.027
14	0.011	0.017	0.024	0.032	0.042	0.052	0.090	0.102	0.039
15	0.006	0.009	0.014	0.019	0.027	0.035	0.072	0.102	0.052
16	0.003	0.005	0.007	0.011	0.016	0.022	0.054	0.096	0.065
17	0.001	0.002	0.004	0.006	0.009	0.013	0.038	0.085	0.076
18	0.000	0.001	0.002	0.003	0.005	0.007	0.026	0.071	0.084
19	0.000	0.000	0.001	0.001	0.002	0.004	0.016	0.056	0.089
20	0.000	0.000	0.000	0.001	0.001	0.002	0.010	0.042	0.089
21	0.000	0.000	0.000	0.000	0.000	0.001	0.006	0.030	0.085
22	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.020	0.077
23	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.013	0.067
24	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.008	0.056
25	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.005	0.045
26	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.034
27	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.025
28	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.018
29	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.013
30	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.008
31	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.005
32	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.003
33	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002
34	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001
35	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001
36	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Normal Curve Areas



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998

Probability Formulae

Counting Rules

$${}_nP_r = \frac{n!}{(n-r)!}$$

$$n^r$$

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

Probability Rules

$$P(A) = \frac{n(A)}{n(S)}$$

$$0 \leq P(A) \leq 1$$

$$P(A) + P(\text{-}A) = 1$$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

$$P(A \text{ or } B) = P(A) + P(B)$$

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Probability Distribution Functions

$$P(X) = {}_nC_X \cdot \pi^X \cdot (1-\pi)^{n-X}$$

$$P(X) = \frac{({}_SC_X) \cdot ({}_{N-S}C_{n-X})}{{}_NC_n}$$

$$P(X) = \frac{\mu^X \cdot e^{-\mu}}{X!}$$

Parameters of Probability Distributions

General Formulae for Discrete Distributions

$$\mu = \sum X \cdot P(X)$$

$$\sigma = \sqrt{\sum (X-\mu)^2 \cdot P(X)}$$

$$\sigma = \sqrt{\sum X^2 \cdot P(X) - \left(\sum X \cdot P(X)\right)^2}$$

Shortcuts for the Binomial Distribution

$$\mu = n\pi$$

$$\sigma = \sqrt{n \cdot \pi \cdot (1-\pi)}$$

Standard Scores

$$Z = \frac{X-\mu}{\sigma}$$

$$Z = \frac{X-\bar{X}}{s}$$

$$X = \mu + Z\sigma$$

$$X = \bar{X} + Zs$$

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