Math 111
Calculus II

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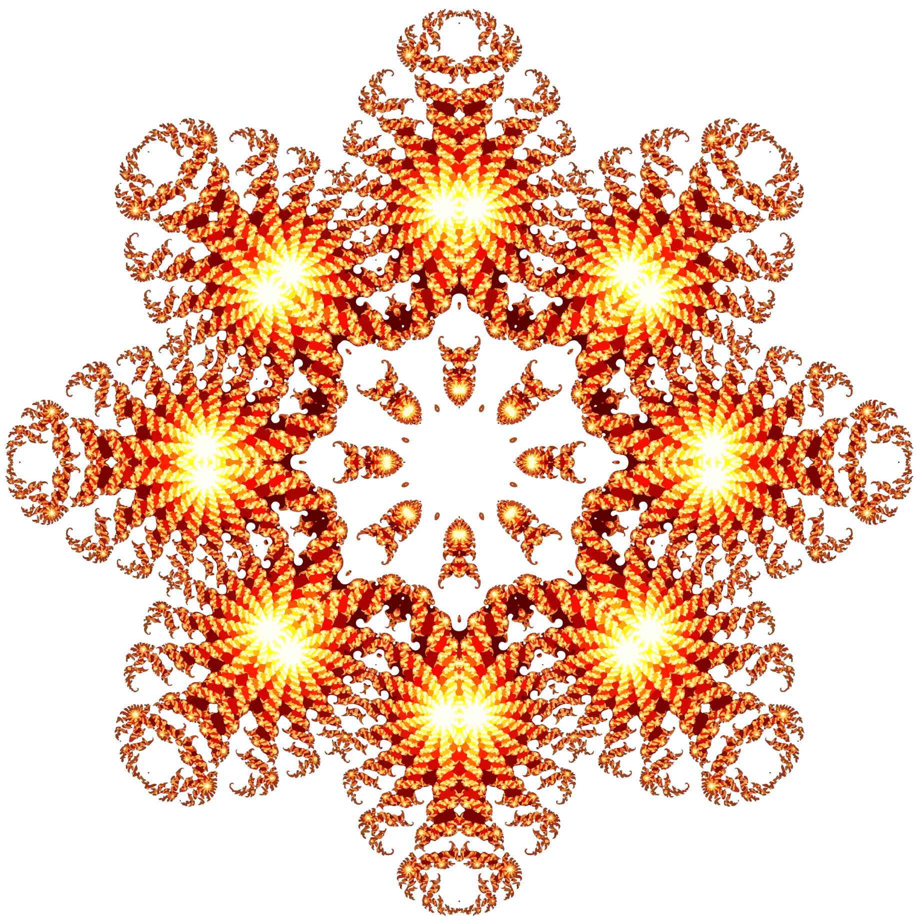

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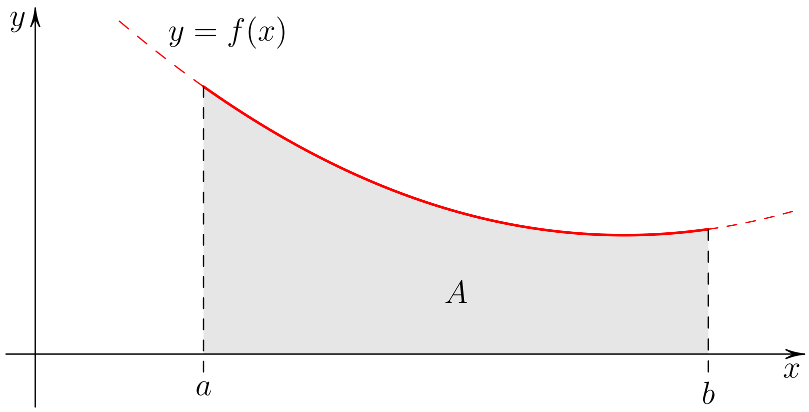
Chapter 1: Integration Review

1.1 The Meaning of the Definite Integral

The **definite integral** of the function $f(x)$ between $x = a$ and $x = b$ is written:

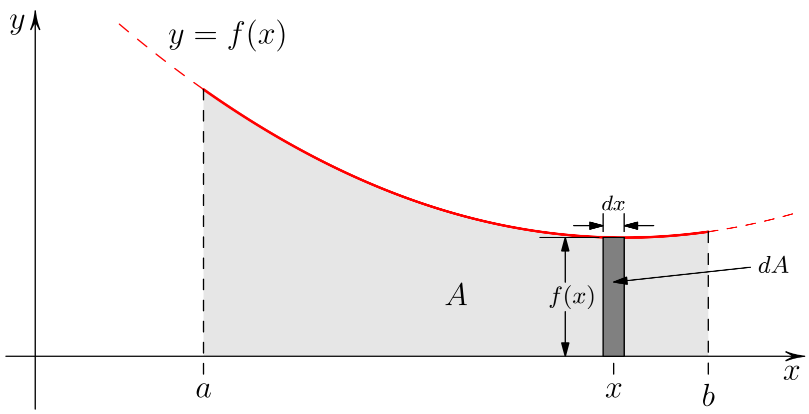
$$\int_a^b f(x)\,dx$$

Geometrically it equals the **area** A between the curve $y = f(x)$ and the x-axis between the vertical lines $x = a$ and $x = b$:



More precisely, assuming $a < b$, the definite integral is the net sum of the **signed** areas between the curve $y = f(x)$ and the x-axis where areas below the x-axis (i.e. where $f(x)$ dips below the x-axis) are counted **negatively**.

The notation used for the definite integral, $\int_a^b f(x)\,dx$, is elegant and intuitive. We are $\int$umming ($\int dA$) the (infinitesimally) small differential rectangular areas $dA = f(x) \cdot dx$ of height $f(x)$ and width dx at each value x between $x = a$ and $x = b$:



We will see soon how viewing integrals as sums of differentials can be used to come up with formulas for calculations aside from just area.

1.2 The Fundamental Theorem of Calculus

As seen in a previous calculus course, the definite integral can be written as a limiting sum (Riemann Sum) of N rectangles of finite width $\Delta x = (b-a)/N$ where we let the number of rectangles (N) go to infinity (and consequently the width $\Delta x \to 0$). This method of evaluating a definite integral is hard or impossible to compute exactly yfor most functions. An easy way to evaluate definite integrals is due to the **Fundamental Theorem of Calculus** which relates the calculation of a definite integral with the evaluation of the **antiderivative** $F(x)$ of $f(x)$:

Theorem 1-1: The Fundamental Theorem of Calculus:

If f is continuous on $[a, b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

for any F an antiderivative of f, i.e. $F'(x) = f(x)$.

Notationally we write $F(b) - F(a)$ with the shorthand $F(x)\big|_a^b$, i.e.

$$F(x)\big|_a^b = F(b) - F(a) \ ,$$

where, unlike the integral sign, the bar is placed *on the right*.

1.3 Indefinite Integrals

Because of the intimate relationship between the antiderivative and the definite integral, we define the **indefinite integral** of $f(x)$ (with no limits a or b) to just be the antiderivative, i.e.

$$\int f(x)\ dx = F(x) + C$$

where $F(x)$ is an antiderivate of $f(x)$ (so $F'(x) = f(x)$) and C is an arbitrary constant. The latter is required since the antiderivative of a function is not unique as $\frac{d}{dx}C = 0$ implies we can always add a constant to an antiderivative to get another antiderivative of the same function.

Using our notation for indefinite integrals and our knowledge of derivatives gives the following.

Table of Indefinite Integrals

1. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \qquad (n \neq -1)$
2. $\displaystyle\int \cos x\,dx = \sin x + C$
3. $\displaystyle\int \sin x\,dx = -\cos x + C$
4. $\displaystyle\int \sec^2 x\,dx = \tan x + C$
5. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$
6. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$
7. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$
8. $\displaystyle\int c f(x)\,dx = c\int f(x)\,dx$
9. $\displaystyle\int [f(x) \pm g(x)]\ dx = \int f(x)\,dx \pm \int g(x)\,dx$

In the last two integration formulae $f(x)$ and $g(x)$ are functions while c is a constant. For indefinite integrals we say, for example, that $\frac{1}{n+1}x^{n+1}+C$ is the **(indefinite) integral** of x^n where x^n is the **integrand**. The **process** of finding the integral is called **integration**. Each of these indefinite integrals may be verified by differentiating the right hand side and verifying that the integrand is the result.

1.4 Integration by Substitution

The last two general integral results allow us to break up an integral of sums or differences into integrals of the individual pieces and to pull out any constant multipliers. Another useful way of solving an integral is to use the **Substitution Rule** which arises by working the differentiation Chain Rule in reverse.

Theorem 1-2: Substitution Rule (Indefinite Integrals): Suppose $u = g(x)$ is a differentiable function whose range of values is an interval I upon which a further function f is continuous, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du\ .$$

where the right hand integral is to be evaluated at $u = g(x)$ after integration.

Here the du appearing on the right side is the differential:

$$du = g'(x)dx$$

which, recall, can be remembered by thinking $\frac{du}{dx} = g'(x)$ and multiplying both sides by dx.

When using the Substitution Rule with definite integrals we can avoid the final back-substitution of $u = g(x)$ of the indefinite case by instead just changing the limits of the integral appropriately to the u-values corresponding to the x-limits:

Theorem 1-3: Substitution Rule (Definite Integrals): Suppose $u = g(x)$ is a differentiable function whose derivative g' is continuous on $[a, b]$ and a further function f is continuous on the range of $u = g(x)$ (evaluated on $[a, b]$), then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du\ .$$

1.5 Integration Examples

Example 1-1

Evaluate the following integrals:

1. $\displaystyle\int \left(x^2 + \frac{1}{x^2} - 3\cos x\right) dx$
2. $\displaystyle\int_0^{\pi/2} \left(\sqrt{x} + \sin x\right) dx$
3. $\displaystyle\int \left(\sqrt[3]{x^2} + \sec x \tan x\right) dx$
4. $\displaystyle\int \frac{(x+2)^2}{\sqrt{x}}\,dx$
5. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx$
6. $\displaystyle\int t^2 \sec(t^3+5)\tan(t^3+5)\,dt$
7. $\displaystyle\int \sqrt{1+\tan\theta}\,\sec^2\theta\,d\theta$
8. $\displaystyle\int_{-1}^{0} x\sqrt{3x+4}\,dx$

Solution:

1. Using basic integration formula and recalling that $\dfrac{1}{x^n} = x^{-n}$ one has:

$$\begin{aligned}\int \left(x^2 + \frac{1}{x^2} - 3\cos x\right) dx &= \int \left(x^2 + x^{-2} - 3\cos x\right) dx \\ &= \frac{1}{3}\, x^3 - x^{-1} - 3\sin x + C = \frac{1}{3}\, x^3 - \frac{1}{x} - 3\sin x + C\end{aligned}$$

2. Recalling that $\sqrt[n]{x} = x^{\frac{1}{n}}$ we evaluate the definite integral to get:

$$\begin{aligned}\int_0^{\pi/2} \left(\sqrt{x} + \sin x\right) dx &= \int_0^{\pi/2} \left(x^{\frac{1}{2}} + \sin x\right) dx = \left[\frac{2}{3} x^{\frac{3}{2}} - \cos x\right|_0^{\frac{\pi}{2}} \\ &= \left[\frac{2}{3}\left(\frac{\pi}{2}\right)^{\frac{3}{2}} - \cos\left(\frac{\pi}{2}\right)\right] - \left[\frac{2}{3}(0)^{\frac{3}{2}} - \cos(0)\right] = \frac{\sqrt{2}\pi^{\frac{3}{2}}}{6} + 1\end{aligned}$$

3. Recalling that and $(x^m)^n = x^{mn}$ gives:

$$\begin{aligned}\int \left(\sqrt[3]{x^2} + \sec x \tan x\right) dx &= \int (x^{2/3} + \sec x \tan x) dx \\ &= \frac{3}{5} x^{5/3} + \sec x + C\end{aligned}$$

4. Expanding the numerator, dividing through by the denominator and simplifying $\dfrac{x^m}{x^n} = x^{m-n}$ before integrating gives:

$$\begin{aligned}\int \frac{(x+2)^2}{\sqrt{x}}\, dx &= \int \frac{x^2 + 4x + 4}{\sqrt{x}} dx \\ &= \int \left(x^{3/2} + 4x^{1/2} + 4x^{-1/2}\right) dx \\ &= \frac{2}{5} x^{5/2} + 4\left(\frac{2}{3}\right) x^{3/2} + 4\,(2)\, x^{1/2} + C \\ &= \frac{2}{5} x^{5/2} + \frac{8}{3} x^{3/2} + 8x^{1/2} + C\end{aligned}$$

5. Using the substitution $u = x^2 + 1$ so $du = 2x\, dx \implies \frac{1}{2}\, du = x\, dx$ one has:

$$\begin{aligned}\int \frac{x}{\sqrt{x^2+1}}\, dx = \int \frac{1}{2} \cdot \frac{1}{\sqrt{u}}\, du &= \frac{1}{2} \int u^{-1/2}\, du \\ &= u^{1/2} + C = \sqrt{x^2+1} + C\end{aligned}$$

6. Using the substitution $u = t^3 + 5$ so $du = 3t^2\, dt \implies \frac{1}{3}\, du = t^2\, dt$ one has:

$$\begin{aligned}\int t^2 \sec(t^3+5) \tan(t^3+5)\, dt &= \frac{1}{3} \int \sec u \tan u\, du \\ &= \frac{1}{3} \sec u + C = \frac{1}{3} \sec(t^3+5) + C\end{aligned}$$

7. Using the substitution $u = 1 + \tan\theta$ so $du = \sec^2\theta\, d\theta$ one has:

$$\begin{aligned}\int \sqrt{1+\tan\theta}\, \sec^2\theta\, d\theta = \int \sqrt{u}\, du &= \int u^{1/2}\, du \\ &= \frac{2}{3}\, u^{2/3} + C = \frac{2}{3}(1+\tan\theta)^{3/2} + C\end{aligned}$$

8. Make the following substitution $u = 3x + 4$ so $du = 3dx \implies \frac{1}{3}\,du = dx$. Solving for x gives $x = \frac{1}{3}(u-4)$ allowing its replacement in the integral. Finally we change to limits in u:

$$x = 0 \implies u = 0 + 4 = 4$$
$$x = -1 \implies u = -3 + 4 = 1$$

$$\begin{aligned}
\int_{-1}^{0} x\sqrt{3x+4}\,dx &= \int_{1}^{4} \frac{1}{3}(u-4)\sqrt{u}\,\frac{1}{3}\,du \\
&= \frac{1}{9}\int_{1}^{4} (u^{3/2} - 4u^{1/2})\,du = \frac{1}{9}\left[\frac{2}{5}u^{5/2} - 4\left(\frac{2}{3}\right)u^{3/2}\right|_{1}^{4} \\
&= \frac{1}{9}\left[\frac{2}{5}(4)^{5/2} - \frac{8}{3}(4)^{3/2}\right] - \frac{1}{9}\left[\frac{2}{5}(1)^{5/2} - \frac{8}{3}(1)^{3/2}\right] \\
&= \frac{1}{9}\left[\frac{64}{5} - \frac{64}{3}\right] - \frac{1}{9}\left[\frac{2}{5} - \frac{8}{3}\right] = \frac{1}{9}\left[\frac{64}{5} - \frac{64}{3} - \frac{2}{5} + \frac{8}{3}\right] \\
&= \frac{1}{9}\left[\frac{62}{5} - \frac{56}{3}\right] = \frac{1}{9}\left[-\frac{94}{15}\right] \\
&= -\frac{94}{135}
\end{aligned}$$

Further Questions:

Evaluate the following integrals:

1. $\displaystyle\int \left(t^2 + \sqrt{t} - \frac{2}{t^2}\right) dt$
2. $\displaystyle\int_0^1 \left(t^2+1\right)^2\, dt$
3. $\displaystyle\int x^2\left(x^3+2\right)^{\frac{1}{3}}\, dx$
4. $\displaystyle\int_0^{\frac{\pi}{4}} (\sec x - \tan x)\sec x\, dx$
5. $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$
6. $\displaystyle\int_1^2 x\sqrt{x-1}\,dx$
7. $\displaystyle\int \sin(5\theta)\, d\theta$
8. $\displaystyle\int_2^3 \frac{3x^2-1}{\left(x^3-x\right)^2}\, dx$
9. $\displaystyle\int t^2\sin\left(1-t^3\right) dt$
10. $\displaystyle\int \frac{x-\sqrt{3x}}{\sqrt{2x}}\, dx$
11. $\displaystyle\int_0^4 (4x+9)^{\frac{3}{2}}\, dx$
12. $\displaystyle\int (\cos\theta + \sin\theta)\left(\cos\theta - \sin\theta\right)^4 d\theta$

Chapter 1 Review Exercises

Answers: Page 257

1-12: Evaluate the given integral.

1. $\displaystyle\int \left(x^3 + \frac{1}{\sqrt{x}} + \frac{5}{x^3}\right) dx$
2. $\displaystyle\int \frac{x^3 + \sqrt{x} + 1}{\sqrt{x}}\, dx$
3. $\displaystyle\int \frac{\tan\sqrt{x}\ \sec\sqrt{x}}{\sqrt{x}}\, dx$
4. $\displaystyle\int_0^1 \left(x^2 - x^3\right)\, dx$
5. $\displaystyle\int_0^{\pi/2} \cos x\,(1 + \sin x)^3\, dx$
6. $\displaystyle\int x^2 \sin\left(x^3 + 2\right)\, dx$
7. $\displaystyle\int \sqrt{2x+5}\, dx$
8. $\displaystyle\int_0^3 x\left(x^2 + 16\right)^{3/2}\, dx$
9. $\displaystyle\int \frac{\sin(3x)}{(2 + \cos(3x))^2}\, dx$
10. $\displaystyle\int_5^{12} \frac{1}{(x-4)^{1/3}}\, dx$
11. $\displaystyle\int \frac{1}{(1+\sqrt{x})^4}\, dx$
12. $\displaystyle\int x\,(x+4)^5\, dx$

Chapter 2: Inverses and Other Functions

2.1 Inverse Functions

Example 2-1

The inverse function of the function $f(x) = x^3$ is $g(x) = x^{\frac{1}{3}}$.

Intuitively $g(x) = x^{\frac{1}{3}}$ is the inverse of is $f(x) = x^3$ because g undoes the action of f. So if f acts on the value 2 so $f(2) = 2^3 = 8$ and we act g on the result, $g(8) = 8^{\frac{1}{3}} = 2$ we are returned to the original value.

One may wonder whether all functions have inverses. The answer is no. A necessary and sufficient condition for a function to have an inverse is that the function be *one-to-one*.

Definition: A function f with domain A and range B is said to be **one-to-one** if whenever $f(x_1) = f(x_2)$ (in B) one has that $x_1 = x_2$ (in A).

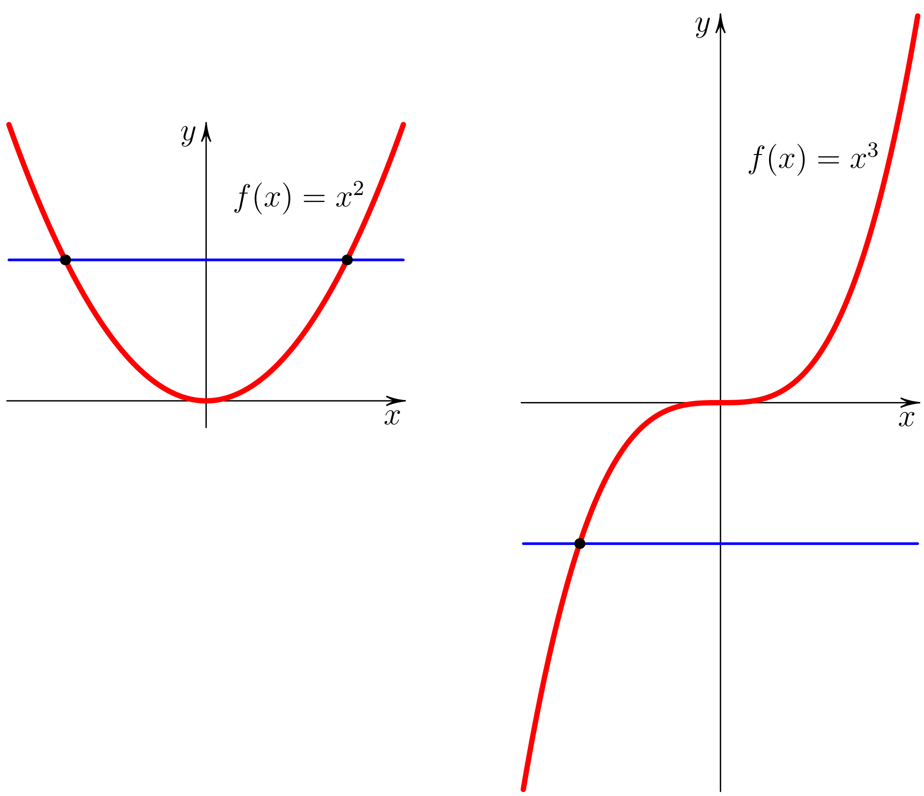
A logically equivalent condition is that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. In words, no two elements in the domain A have the same image in the range B.

2.1.1 Horizontal Line Test

The **Horizontal Line Test** says that a function $f(x)$ will be one-to-one if and only if every horizontal line intersects the graph of $y = f(x)$ at most once.

Example 2-2

The horizontal line test shows that the function $y = x^2$ is not one-to-one while $y = x^3$ is one-to-one.



The following theorem is intuitively true when one considers the Horizontal Line Test.

Theorem 2-1: Suppose a function f has a domain D consisting of an interval. If the function is increasing everywhere or decreasing everywhere on D then f is one-to-one.

Example 2-3

Determine whether the given functions are one-to-one:

1. $f(x) = 2x^3 + 5$
2. $f(x) = \dfrac{x}{x-4}$
3. $f(x) = \dfrac{\sqrt{x+1}}{\sqrt{x}}$

Solution:

1. $f(x) = 2x^3 + 5$

   - Method 1: Using the definition:

$$\begin{aligned} f(x_1) = f(x_2) &\implies 2x_1^3 + 5 = 2x_2^3 + 5 \\ &\implies 2x_1^3 = 2x_2^3 \\ &\implies x_1^3 = x_2^3 \text{ (Take the cubic root of both sides.)} \\ &\implies x_1 = x_2 \end{aligned}$$

   - Method 2: Using the derivative:

$$f'(x) = 6x^2 > 0 \text{ for all } x \text{ in } D = (-\infty, \infty)$$

   Therefore, $f(x)$ is always increasing on an interval which implies that $f(x)$ is one-to-one.

2. $f(x) = \dfrac{x}{x-4}$

   We use the definition to prove $f(x)$ is one-to-one:

$$\begin{aligned} f(x_1) = f(x_2) &\implies \frac{x_1}{x_1 - 4} = \frac{x_2}{x_2 - 4} \\ &\implies x_1(x_2 - 4) = x_2(x_1 - 4) \\ &\implies x_1x_2 - 4x_1 = x_2x_1 - 4x_2 \\ &\implies -4x_1 = -4x_2 \\ &\implies x_1 = x_2 \end{aligned}$$

   Note the derivative of $f(x)$ is negative everywhere it is defined:

$$f'(x) = \frac{(1)(x-4) - x(1)}{(x-4)^2} = \frac{-4}{(x-4)^2} < 0$$

   Therefore $f(x)$ is decreasing for all x in the domain of f. However the domain $D = \mathbb{R} - \{4\} = (-\infty, 4) \cup (4, \infty)$ is broken into two intervals so this is not sufficient to prove $f(x)$ is one-to-one. Consider, as a counter-example, $f(x) = \tan x$ which increases everywhere on its domain but fails hopelessly to be one-to-one as is seen easily by the horizontal line test! Care must be taken to inspect the domain when using the derivative approach. One would need to show that the range of the function on each interval it contained was disjoint (did not overlap) with the range of the function on the other intervals. This occurs in this example where f maps $(-\infty, 4)$ to $(-\infty, 1)$ and maps $(4, \infty)$ to $(1, \infty)$.

3. $f(x) = \dfrac{\sqrt{x+1}}{\sqrt{x}}$

   The square roots require $x \geq -1$ and $x \geq 0$. Furthermore we cannot divide by zero. Therefore the domain of the function is composed of a single open interval, $D = (0, \infty)$. We use the method of the derivative:

$$\begin{aligned} f'(x) &= \frac{\frac{1}{2}(x+1)^{-1/2}\sqrt{x} - \sqrt{x+1}\left(\frac{1}{2}x^{-1/2}\right)}{x} \\ &= \frac{\frac{\sqrt{x}}{\sqrt{x+1}} - \frac{\sqrt{x+1}}{\sqrt{x}}}{2x} \cdot \frac{\sqrt{x}\sqrt{x+1}}{\sqrt{x}\sqrt{x+1}} = \frac{x-(x+1)}{2x^{3/2}\sqrt{x+1}} \\ &= \frac{-1}{2x^{3/2}\sqrt{x+1}} < 0 \text{ for all } x > 0 \end{aligned}$$

   Therefore, $f(x)$ is decreasing on interval $(0, \infty)$ which implies that $f(x)$ is one-to-one.

Further Questions:

Determine whether the given functions are one-to-one:

1. $f(x) = 2x^3 + 5$
2. $f(x) = \dfrac{3-x}{x+1}$

Definition: Suppose f is a one-to-one function defined on domain A with range B. The **inverse function of f** denoted by $\boldsymbol{f^{-1}}$ is defined on domain B with range A and satisfies

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

Here the symbol $\iff$ means "if and only if". "*This* if and only if *that*" itself means that both the following hold

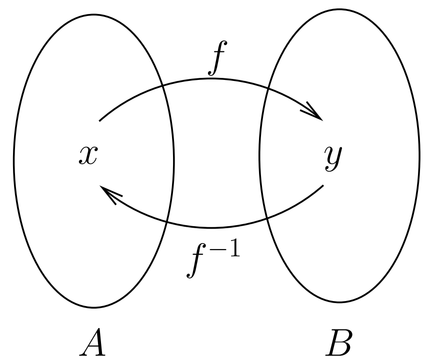
- "If *this* then *that*."
- "If *that* then *this*."

Also observe that the notation f^{-1} makes it clear that f is the function for which this is the inverse.

Notes:

1. $f^{-1} \neq \dfrac{1}{f}$ We call $\frac{1}{f}$ the *reciprocal* of f.

2. The definition says that if f maps x to y, then f^{-1} maps y back to x.



3. The domain of f^{-1} is the range of f while the range of f^{-1} is the domain of f.

4. Reversing the roles of x and y gives

$$f^{-1}(x) = y \iff f(y) = x$$

or equivalently

$$f(y) = x \iff f^{-1}(x) = y$$

This implies that f itself is the inverse function of f^{-1}.

5. The following hold (see last diagram)

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in } A \\ f\left(f^{-1}(y)\right) &= y \quad \text{for every } y \text{ in } B \end{aligned}$$

The first relationship highlights the utility of the inverse function in *solving equations* for if we have, say, $f(x) = 3$ for some one-to-one function f for which we know the inverse $f^{-1}(x)$, it follows, applying f^{-1} to both sides that $f^{-1}(f(x)) = x = f^{-1}(3)$. We are applying inverse functions all the time when we isolate variables in equations.

This explains why "cube-rooting both sides" of $x^3 = 64$ is a safe way to find the solution to this equation while "square-rooting both sides" of $x^2 = 64$ is not. The latter finds only one of the two solutions. (Applying a *function* in this way can only produce one number.)

2.1.2 Finding Inverse Functions

To find the inverse function of a one-to-one function f proceed with the following steps:

1. Write $y = f(x)$
2. Solve the equation for x in terms of y (if possible).
3. Interchange the roles of x and y. The resulting equation is $y = f^{-1}(x)$.

Example 2-4

Find the inverse function of the given function:

1. $f(x) = 2x^3 + 5$
2. $f(x) = \dfrac{x}{x-4}$
3. $f(x) = \dfrac{\sqrt{x+1}}{\sqrt{x}}$

Solution:

These are the same functions from Example 2-3. As they were all shown to be one-to-one we know they are invertible.

1. To find the inverse function of $f(x) = 2x^3 + 5$, we solve $y = f(x)$ for x in terms of y.

$$\begin{aligned} y = f(x) &\implies y = 2x^3 + 5 \implies y - 5 = 2x^3 \\ &\implies \frac{y-5}{2} = x^3 \implies x = \sqrt[3]{\frac{y-5}{2}} \\ &\qquad \text{Exchange } x \text{ and } y \implies y = \sqrt[3]{\frac{x-5}{2}} \\ &\qquad \text{Therefore } f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}} \end{aligned}$$

2. To find the inverse function of $f(x) = \dfrac{x}{x-4}$, we solve $y = f(x)$ for x in terms of y.

$$\begin{aligned} y = f(x) \implies y = \frac{x}{x-4} &\implies y(x-4) = x \implies yx - 4y = x \\ &\implies yx - x = 4y \implies x(y-1) = 4y \\ &\implies x = \frac{4y}{y-1} \\ &\qquad \text{Exchange } x \text{ and } y \implies y = \frac{4x}{x-1} \\ &\qquad \text{Therefore } f^{-1}(x) = \frac{4x}{x-1} \end{aligned}$$

3. To find the inverse function of $f(x) = \dfrac{\sqrt{x+1}}{\sqrt{x}}$, we solve $y = f(x)$ for x in terms of y.

$$\begin{aligned} y = f(x) &\implies y = \frac{\sqrt{x+1}}{\sqrt{x}} = \sqrt{\frac{x+1}{x}} \\ &\implies y^2 = \frac{x+1}{x} \implies xy^2 = x + 1 \implies xy^2 - x = 1 \\ &\implies x(y^2 - 1) = 1 \implies x = \frac{1}{y^2 - 1} \\ &\qquad \text{Exchange } x \text{ and } y \implies y = \frac{1}{x^2 - 1} \\ &\qquad \text{Therefore } f^{-1}(x) = \frac{1}{x^2 - 1} \end{aligned}$$

Further Questions:

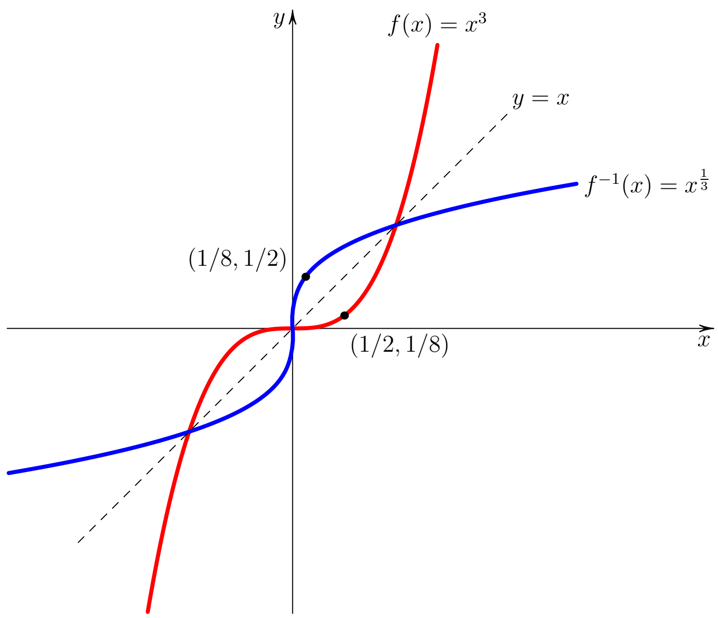
Find the inverse function of the given function:

1. $f(x) = 2x^3 + 5$
2. $f(x) = \dfrac{3-x}{x+1}$
3. $f(x) = x^2 - 9$

Note that if you are able to solve your expression uniquely for x in terms of y in the second step it follows that the function is one-to-one since, given any y value in the range B there can only be a single value x in A which maps to it, namely the value which results from evaluating your solved expression with y.

2.1.3 Graphs of Inverse Functions

The definition of the inverse function implies that if (x, y) lies on the graph of $y = f(x)$ then (y, x) will lie on the graph of f^{-1}. Geometrically this means that the graph of f^{-1} may be obtained by reflecting the graph of f about the line $y = x$.



Graphically a discontinuity in f would imply a discontinuity in f^{-1} and vice versa. We have the following theorem.

Theorem 2-2: Suppose f is a one-to-one continuous function defined on an interval then its inverse f^{-1} is also continuous.

2.1.4 Derivative of an Inverse Function

If we let $g(x)$ be the inverse of f then our earlier relationship $x = f\left(f^{-1}(x)\right) = f(g(x))$. Differentiating the left side with respect to x just gives 1. Differentiating the right side of the equation with respect to x can be done with the Chain Rule. Solving for the derivative $g'(a)$ gives the following result.

Theorem 2-3: Suppose f is a one-to-one differentiable function with inverse $g = f^{-1}$. If $f'(g(a)) \neq 0$ then the inverse function is differentiable at a with

$$g'(a) = \frac{1}{f'(g(a))}$$

More generally the derivative of the inverse function is

$$g'(x) = \frac{1}{f'(g(x))} \ .$$

Example 2-5

For the function $f(x) = x^2 + 1,\ x \geq 0$:

1. Show that f is one-to-one.
2. Calculate $g = f^{-1}$ and find its domain and range.
3. Calculate $g'(5)$ using your result from part 2.
4. Find $g'(5)$ from the formula $g'(x) = \dfrac{1}{f'(g(x))}$.

Solution:

1. Using the definition of one-to-one:

$$\begin{aligned} f(x_1) = f(x_2) &\implies x_1^2 + 1 = x_2^2 + 1 \\ &\implies x_1^2 - x_2^2 = 0 \\ &\implies (x_1 - x_2)(x_1 + x_2) = 0 \\ &\implies x_1 = x_2 \text{ or } x_1 = -x_2 \end{aligned}$$

   Since the domain of f is restricted to $x \geq 0$, $x_1 = -x_2$ is not possible as one of the numbers would necessarily be negative, except in the case of $x_1 = x_2 = 0$. We conclude f is one-to-one.

2. Solve $y = f(x)$ for x:

$$\begin{aligned} y = f(x) &\implies y = x^2 + 1 \implies x^2 = y - 1 \\ &\implies x = \pm\sqrt{y-1} \quad (\leftarrow \text{Reject negative due to domain of } f.) \\ &\implies x = \sqrt{y-1} \\ &\quad \text{Exchange } x \text{ and } y \implies y = \sqrt{x-1} \\ &\quad \text{Therefore } g(x) = f^{-1}(x) = \sqrt{x-1} \end{aligned}$$

   Domain of g requires $x - 1 \geq 0 \implies x \geq 1$. So $D_g = [1, \infty)$. Range of g equals the domain of f and so $R_g = [0, \infty)$.

3. Differentiating using the Chain Rule gives:

$$g'(x) = \frac{1}{2}(x-1)^{-\frac{1}{2}}(1+0) = \frac{1}{2\sqrt{x-1}}$$
$$\text{and so } g'(5) = \frac{1}{2\sqrt{5-1}} = \frac{1}{4}$$

4. Since $f'(x) = 2x$ we have using the inverse derivative formula:

$$g'(5) = \frac{1}{f'(g(5))} = \frac{1}{f'(\sqrt{5-1})} = \frac{1}{f'(2)} = \frac{1}{2(2)} = \frac{1}{4}$$

Further Questions:

For the function $f(x) = \dfrac{1}{x-1}$:

1. Show that f is one-to-one.

2. Calculate $g = f^{-1}$ and find its domain and range.
3. Calculate $g'(2)$ using your result from part 2.
4. Find $g'(2)$ from the formula $g'(x) = \dfrac{1}{f'(g(x))}$.

Example 2-6

Find $(f^{-1})'(a)$ for the given a:

1. $f(x) = x^3 + 4x + 3$, $a = -2$
2. $f(x) = \dfrac{x^3}{x^2+4}$, $a = -\dfrac{1}{5}$

Solution:

1. $f'(x) = 3x^2 + 4 > 0$ on $(-\infty, \infty) \implies f(x)$ has an inverse

$$(f^{-1})'(-2) = \frac{1}{f'(f^{-1}(2))}$$

We need to find $f^{-1}(-2)$. To do so, call the unknown value w, use the properties of the inverse to get an equation for it, and then solve for w as follows:

$$\begin{aligned} f^{-1}(-2) = w &\implies f(w) = -2 \implies w^3 + 4w + 3 = -2 \\ &\implies w = -1 \text{ (By inspection or direct solution.)} \\ f'(f^{-1}(-2)) &= f'(-1) = 3(-1)^2 + 4 = 3 + 4 = 7 \\ \text{Therefore } (f^{-1})'(-2) &= \frac{1}{f'(f^{-1}(-2))} = \frac{1}{7} \end{aligned}$$

2. $f'(x) = \dfrac{3x^2(x^2+4) - x^3(2x)}{(x^2+4)^2} = \dfrac{x^4 + 12x^2}{(x^2+4)^2}$

$$\begin{aligned} (f^{-1})'\left(-\frac{1}{5}\right) &= \frac{1}{f'(f^{-1}(-\frac{1}{5}))} \\ f^{-1}\left(-\frac{1}{5}\right) &= w \implies f(w) = -\frac{1}{5} \implies \frac{w^3}{w^2+4} = -\frac{1}{5} \\ &\implies 5w^3 + w^2 - 4 = 0 \implies w = -1 \\ f'(-1) &= \frac{(-1)^4 + 12(-1)^2}{((-1)^2+4)^2} = \frac{1+12}{5^2} = \frac{13}{25} \\ \text{Therefore } (f^{-1})'\left(-\frac{1}{5}\right) &= \frac{1}{f'(f^{-1}(-\frac{1}{5}))} = \frac{1}{f'(-1)} = \frac{1}{\frac{13}{25}} = \frac{25}{13} \end{aligned}$$

Further Questions:

Find the following derivatives:

1. $\left(f^{-1}\right)'(1)$ if $f(x) = x^3 + x + 1$.
2. $g'(-1)$ if $f(x) = 3x - \cos x$ and $g = f^{-1}$.

2.1.5 Creating Invertible Functions

So far one-to-one (and hence invertible) functions seem uncommon. However this is only because we only considered functions defined on their natural domains, i.e. the set of numbers for which the function may be evaluated. We can choose to define a function with a smaller domain and by suitable restriction we can create a function that is one-to-one and hence invertible.

Example 2-7

Define the function $f(x)$ to have the value $f(x) = x^2$ but only be defined on the domain $A = [0, \infty)$. Since f is increasing everywhere on this interval it is one-to-one and hence has an inverse, $f^{-1}(x) = x^{\frac{1}{2}} = \sqrt{x}$. If we restricted the domain to be $A = (-\infty, 0]$ the inverse would be $f^{-1}(x) = -\sqrt{|x|}$!

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Exercise 2-1

1-4: Show that the given function is one-to-one and find its inverse.

1. $f(x) = 2x^3 + 5$
2. $f(x) = \dfrac{2x}{3x-1}$
3. $f(x) = \sqrt{2x-3}$
4. $f(x) = \left(x^3+2\right)^7$

5-7: In the following problems

(a) Show that f is one-to-one.

(b) Find $f^{-1}(x)$ and state the domain and range of f^{-1}.

5. $f(x) = x^5 + 4$
6. $f(x) = \dfrac{x+3}{2x+1}$
7. $f(x) = \dfrac{\sqrt{x}+1}{\sqrt{x}+2}$

8-10: Find $\left(f^{-1}\right)'(a)$ for the given a.

8. $f(x) = 2x^3 + 5, a = 7$
9. $f(x) = x^5 + \sin x + 2x + 2, a = 2$
10. $f(x) = \sqrt[3]{x} + x + 1, a = 11$

---

11. Suppose f and g are invertible functions. Consider the composite function $h = f \circ g$ defined by $h(x) = f\left(g(x)\right)$.

    (a) Show that h is one-to-one.

    (b) Find $h^{-1}(x)$.

2.2 Exponential Functions

If we write the number 2^5, then this, recall, means $2 \times 2 \times 2 \times 2 \times 2$. We call 2 the *base* and 5 the *exponent*. We have already seen that one way to create a function is to replace the base with a variable. This produces *power functions* like

$$f(x) = x^2 \qquad y = x^{\frac{1}{2}} = \sqrt{x} \qquad y = x^{-1} = \tfrac{1}{x}$$

In general, a power function is of the form $y = x^r$ where r is any real constant.

If, on the other hand, we let the exponent be a variable and the base a constant, like:

$$f(x) = 2^x, \; y = (1/2)^x$$

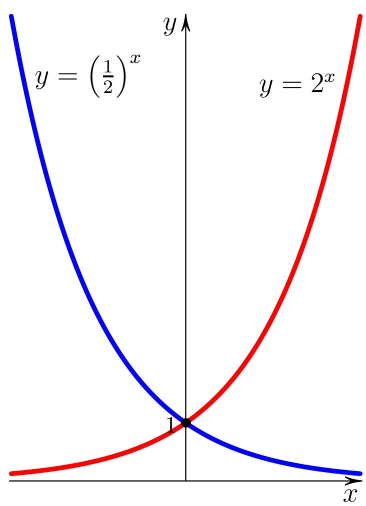
we get *exponential functions*.

Definition: Let $a > 0$. The function

$$f(x) = a^x$$

is an exponential function.

The graph of an exponential function has the following form depending on whether a is greater than or less than 1. Two typical values of a are shown.

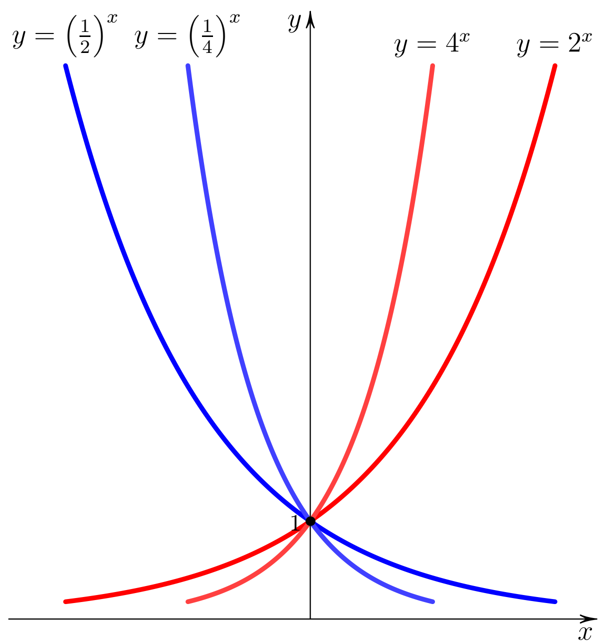


Notes:

1. If $x = 0$ then $a^x = a^0 = 1$. Therefore all exponential functions go through the point $(0, 1)$.
2. If $x = n$, a positive integer then $a^x = a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$.
3. If $x = -n$, n a positive integer, then $a^x = a^{-n} = \frac{1}{a^n}$.
4. If $x = \frac{1}{n}$, n a positive integer, then $a^x = a^{\frac{1}{n}} = \sqrt[n]{a}$. (Hence $a < 0$ is excluded.)
5. If x is rational, $x = \frac{p}{q}$, then $a^x = a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}} = \sqrt[q]{a^p}$.
6. If $a \neq 1$ (and $a > 0$) then $f(x) = a^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$.

7. If $0 < a < 1$ then $f(x) = a^x$ is a decreasing function.
8. If $a > 1$ then $f(x) = a^x$ is an increasing function.
9. If $a, b > 0$ and $x, y \in \mathbb{R}$ then (a) $a^x a^y = a^{x+y}$ (b) $\frac{a^x}{a^y} = a^{x-y}$ (c) $(a^x)^y = a^{xy}$ (d) $(ab)^x = a^x b^x$. These relations are readily apparent when one considers x and y as positive integers.

For two bases greater than one the base which is larger is the steeper curve while for two bases less than one the base which is smaller is steeper.



As depicted in the previous graphs, we have the following limits:

Theorem 2-4: For exponential functions we have the following limits at infinity : If $a > 1$, then $\lim_{x\to-\infty} a^x = 0$ and $\lim_{x\to\infty} a^x = \infty$.
If $0 < a < 1$, then $\lim_{x\to-\infty} a^x = \infty$ and $\lim_{x\to\infty} a^x = 0$.

So the x-axis is a horizontal asymptote for a^x provided $a > 0$, $a \neq 1$.

2.2.1 The Natural Exponential Function

Consider the derivative of $f(x) = a^x$:

$$f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x a^h - a^x}{h} = \left(\lim_{h\to 0} \frac{a^h - 1}{h} \right) a^x$$

where here we are able to pull out the a^x from the limit because a^x does not involve the limit variable h. The result shows the derivative is proportional to the function $f(x) = a^x$ itself with constant of proportionality c given by the evaluation of the limit:

$$c = \lim_{h\to 0} \frac{a^h - 1}{h}$$

Due to the presence of the constant a in the limit, one anticipates correctly that the constant c depends on the choice of base a. Interestingly, one can ask the question if there is some choice of base a for which the constant is $c = 1$. The answer is yes, the base is given by *Euler's Number* :

$$e = 2.71828\ldots$$

for which we have that $c = 1$ in the above limit:

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1 \ .$$

More constructively, as opposed to e being the solution of such a limit equation, it will be shown that e may be written as the following limit:

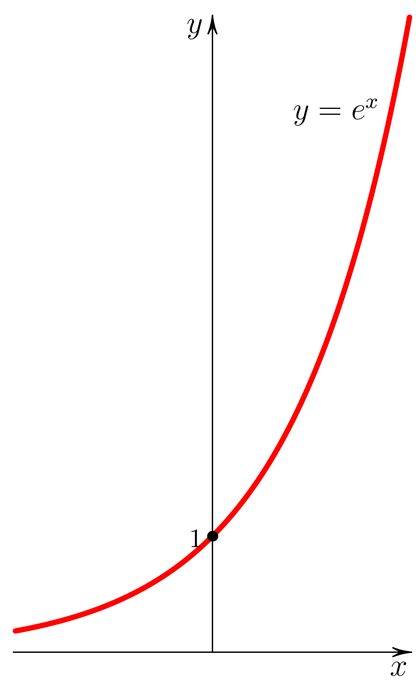
$$e = \lim_{h\to 0} (1+h)^{\frac{1}{h}} \ ,$$

or, setting $h = 1/n$,

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \ .$$

Definition: If $a = e = 2.71828\ldots$, then $f(x) = e^x$ is the *natural exponential function* .

Since $e = 2.71\ldots > 1$ the natural exponential function shares all the aforementioned properties of $f(x) = a^x$ where $a > 1$. (i.e. continuous, increasing function with domain $\mathbb{R}$, range $(0, \infty)$, limits, etc.)



You should identify the natural exponential key $\boxed{e^x}$ on your calculator.

2.2.2 Derivative of e^x

Furthermore from the preceding discussion we have the important result:

Theorem 2-5: The derivative of the natural exponential function is:

$$\frac{d}{dx}e^x = e^x \ .$$

Proof is, as above,

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = \left(\lim_{h\to 0}\frac{e^h-1}{h}\right)e^x = (1)e^x = e^x \ .$$

A corollary of this theorem, applying the Chain Rule to the function e^u with $u = g(x)$ is:

Theorem 2-6: $\frac{d}{dx}e^u = e^u\frac{du}{dx}$ or $\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}g'(x)$

Example 2-8

Differentiate the following functions:

1. $f(x) = e^{x^2+5}$
2. $y = \cos\left(e^{2x}\right)$
3. $f(t) = \left(t^2+5\right)e^{t^3}$
4. $2xy^2 + xe^y = e^{3x}$

Solution:

1. $f(x) = e^{x^2+5}$
   Using the Chain Rule:

$$\begin{aligned} f'(x) &= e^{x^2+5}\left(x^2+5\right)' = e^{x^2+5}\left(2x+0\right) \\ &= 2x\, e^{x^2+5} \end{aligned}$$

2. $y = \cos\left(e^{2x}\right)$

$$\begin{aligned} y' &= -\sin\left(e^{2x}\right)\left(e^{2x}\right)' = -\sin\left(e^{2x}\right)e^{2x}(2) \\ &= -2e^{2x}\sin\left(e^{2x}\right) \end{aligned}$$

3. $f(t) = \left(t^2+5\right)e^{t^3}$
   Using Product and Chain Rules:

$$\begin{aligned} f'(t) &= 2te^{t^3} + \left(t^2+5\right)e^{t^3}(3t^2) \\ &= 2te^{t^3} + \left(3t^4+15t^2\right)e^{t^3} \end{aligned}$$

4. $2xy^2 + xe^y = e^{3x}$
   Using implicit differentiation we have:

$$\begin{aligned} 2y^2 + 4xyy' + e^y + xe^y\,y' &= 3e^{3x} \\ \implies (4xy + xe^y)\,y' &= 3e^{3x} - 2y^2 - e^y \\ \implies y' &= \frac{3e^{3x}-2y^2-e^y}{4xy+xe^y} \end{aligned}$$

Further Questions:

Find $\frac{dy}{dx}$ for the following:

1. $y = e^{2x}$
2. $y = e^{x^3+x}$
3. $y = e^{\sec x} + \sec(e^x)$
4. $y = e^{x^4} \sin\left(x^2+1\right)$
5. $xy + e^x = 2xy^2$

2.2.3 Integral of e^x

Since $\frac{d}{dx} e^x = e^x$, we have the following:

Theorem 2-7: The indefinite integral of e^x is

$$\int e^x \, dx = e^x + C$$

Example 2-9

Evaluate the following integrals:

1. $\displaystyle\int e^{6x} \, dx$
2. $\displaystyle\int e^x \cos(e^x) \, dx$
3. $\displaystyle\int_0^1 x e^{x^2+1} \, dx$
4. $\displaystyle\int \frac{e^{4x}}{\sqrt{2+e^{4x}}} \, dx$

Solution:

1. Using substitution $u = 6x$ so $du = 6\,dx \implies \frac{1}{6} du = dx$ one has:
$$\int e^{6x} \, dx = \int \frac{1}{6} e^u \, du = \frac{1}{6} e^u + C = \frac{1}{6} e^{6x} + C$$
2. Using substitution $u = e^x$ so $du = e^x \, dx$:
$$\int e^x \cos(e^x) \, dx = \int \cos u \, du = \sin u + C = \sin(e^x) + C$$
3. Let $u = x^2 + 1$ so $du = 2x \, dx \implies \frac{1}{2} du = x \, dx$ and change limits:
$$x = 0 \implies u = 1$$
$$x = 1 \implies u = 2$$
$$\int_0^1 x e^{x^2+1} \, dx = \int_1^2 e^u \frac{1}{2} \, du = \frac{1}{2} e^u \Big|_1^2 = \frac{1}{2}\left(e^2 - e^1\right) = \frac{1}{2}\left(e^2 - e\right)$$
4. Let $u = 2 + e^{4x}$ so $du = 4e^{4x} \, dx \implies \frac{1}{4} du = e^{4x} \, dx$:
$$\int \frac{e^{4x}}{2+e^{4x}} \, dx = \frac{1}{4} \int \frac{1}{\sqrt{u}} \, du = \frac{1}{4} \int u^{-1/2} \, du = \frac{1}{4}(2)\, u^{1/2} + C$$
$$= \frac{1}{2}\sqrt{2+e^{4x}} + C$$

Further Questions:

Evaluate the following integrals:

1. $\int e^{2x}\,dx$
2. $\int x^3 e^{x^4+1}\,dx$
3. $\int_0^1 e^{-x}\,dx$
4. $\int \frac{2e^x}{(1+e^x)^2}\,dx$
5. $\int \frac{2+3e^x}{e^x}\,dx$
6. $\int \frac{1-4e^{3x}}{e}\,dx$

2.2.4 Simplifying Exponential Expressions

Using the rules of exponents we are often able to consolidate expressions involving several exponents into an expression involving one exponent.

Example 2-10

The expression $\frac{e^2\sqrt{e^x}}{(2e^x)^3}$ may be simplified as follows:

$$\begin{aligned}
\frac{e^2\sqrt{e^x}}{(2e^x)^3} &= \frac{e^2\,(e^x)^{\frac{1}{2}}}{2^3\,(e^x)^3} \quad \left(\text{since } \sqrt[n]{a} = a^{\frac{1}{n}},\ (ab)^x = a^x b^x\right) \\
&= \frac{e^2 e^{\frac{1}{2}x}}{2^3 e^{3x}} \quad (\text{since } (a^x)^y = a^{xy}) \\
&= \frac{e^{2+\frac{1}{2}x}}{8e^{3x}} \quad (\text{since } a^x a^y = a^{x+y}) \\
&= \frac{1}{8}e^{2+\frac{1}{2}x-3x} \quad \left(\text{since } \frac{a^x}{a^y} = a^{x-y}\right) \\
&= \frac{1}{8}e^{2-\frac{5}{2}x}
\end{aligned}$$

The usefulness in consolidating exponents in this manner is clear when solving equations.

Example 2-11

Solving the equation

$$\frac{e^2\sqrt{e^x}}{(2e^x)^3} = \frac{1}{2}$$

Is equivalent, by using our previous result and multiplying both sides by 8, to

$$e^{2-\frac{5}{2}x} = 4$$

Now if we could apply an *inverse* to the natural exponential function on both sides we could solve for x.

Exercise 2-2

Answers: Page 258

1-3: Evaluate the given limit.

1. $\lim\limits_{x\to-\infty} e^{-x}$

2. $\lim\limits_{x\to\infty} e^{-x^6}$

3. $\lim\limits_{x\to\infty} 5^{e^{-x}}$

4-10: Differentiate the given function.

4. $f(x) = \left(x^4+1\right)e^{-2x}$

5. $g(t) = \dfrac{2t^2+e^{3t}}{t^2e^{3t}}$

6. $y = \dfrac{e^x+1}{e^x+3}$

7. $f(x) = \cos\left(e^{2x}+x\right)$

8. $g(x) = \tan\left(e^x\right) + e^{\tan x}$

9. $y = \sin\sqrt{x^2+e^{2x}}$

10. $f(x) = \dfrac{e^x+e^{-x}}{e^x+3e^{-x}}$

11-12: Find the equation of the tangent line to the curve at the given point.

11. $y = x - e^{-x}, \quad (0,\,-1)$

12. $x - xy + e^y + e^x = 2e, \quad (1,\,1)$

13-20: Evaluate the given integral.

13. $\displaystyle\int \left(e^{2x}+e^{-x}\right)^2\,dx$

14. $\displaystyle\int xe^{x^2}\,dx$

15. $\displaystyle\int e^x\sin\left(e^x\right)\,dx$

16. $\displaystyle\int \frac{\left(e^{2x}+1\right)^2}{e^x}\,dx$

17. $\displaystyle\int_0^1 \frac{e^{3x}}{\left(e^{3x}+2\right)^2}\,dx$

18. $\displaystyle\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

19. $\displaystyle\int \left(1+e^{\tan x}\right)\sec^2 x\,dx$

20. $\displaystyle\int \frac{1}{\left(e^x+e^{-x}\right)^2}\,dx$

2.3 Logarithmic Functions

We finished the last section by suggesting that inverses of exponential functions would be useful for, among other things, solving equations involving exponentials. Since the exponential function $f(x) = a^x$ with constant $a > 0$ and $a \neq 1$ is either everywhere decreasing ($0 < a < 1$) or increasing ($1 < a$) on open interval $\mathbb{R} = (-\infty, \infty)$, the exponential function is one-to-one and hence has an inverse function f^{-1}.

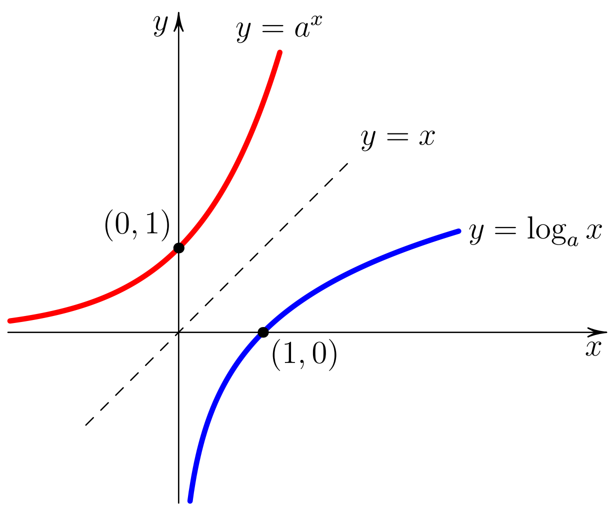
Definition: Given constant $a > 0$, $a \neq 0$, the **logarithmic function of base** $\boldsymbol{a}$, written $\log_a x$ is defined by

$$\log_a x = y \iff a^y = x$$

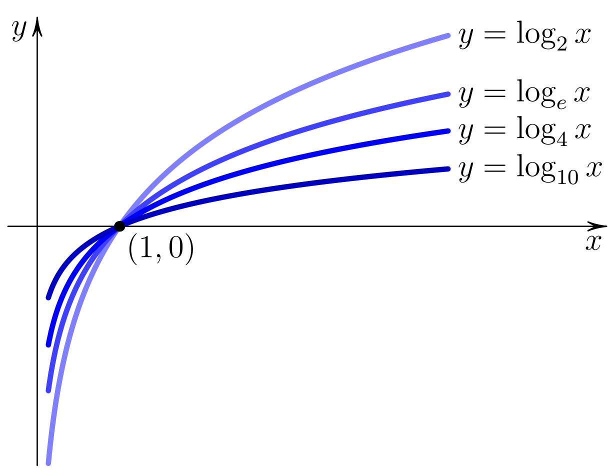
That is, it is the inverse of the exponential function $f(x) = a^x$.

In words, the logarithm of a value x to a base a is the *exponent* to which you must take a to get x.

For the case $\boxed{a > 1}$, which, as you recall, is the case for $a = e = 2.71\ldots$, a representative graph of $y = a^x$ and its inverse $y = \log_a x$ are as follows:



For base $a > 1$ we saw that larger values of a led to steeper $y = a^x$ curves, it follows that larger values of a will make the logarithmic curves more horizontal in this case:



2.3.1 Logarithmic Function Properties

Because of their relationship to exponentials as inverses the following are true for logarithmic functions:

1. $y = \log_a x$ has domain $(0, \infty)$ and range $\mathbb{R}$.
2. $y = \log_a x$ is continuous on its domain.
3. $y = f(x) = \log_a x$ is one-to-one with inverse function $f^{-1}(x) = a^x$.
4. $\log_a(1) = 0$
5. The following limits hold (see graph for $a > 1$ case):
   - If $0 < a < 1$ then $f(x) = \log_a(x)$ is a decreasing function with
     $$\lim_{x \to 0^+} \log_a x = +\infty \qquad \lim_{x \to \infty} \log_a x = -\infty$$
   - If $a > 1$, then $f(x) = \log_a(x)$ is an increasing function with
     $$\lim_{x \to 0^+} \log_a x = -\infty \qquad \lim_{x \to \infty} \log_a x = \infty$$

   Note the y-axis is a vertical asymptote in either case.
6. The following inverse relations hold:
   $$\begin{aligned} \log_a (a^x) &= x \quad \text{for any } x \text{ in } \mathbb{R} \\ a^{\log_a x} &= x \quad \text{for any } x > 0 \end{aligned}$$

The special multiplication, division, and power laws of exponents induce the following important logarithmic results.

Theorem 2-8: For $x > 0$ and $y > 0$ and any real number r the following hold:

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a \left(\frac{x}{y} \right) = \log_a x - \log_a y$
3. $\log_a (x^r) = r \log_a x$

To prove the theorem note that if $x > 0$ and $y > 0$ then $m = \log_a x$ and $n = \log_a y$ exist and, exponentiating both sides, it follows that $x = a^m$ and $y = a^n$. Evaluating the first equation's left hand side we have:

$$\log_a(xy) = \log_a (a^m a^n) = \log_a \left(a^{m+n} \right) = m + n = \log_a x + \log_a y$$

The other conclusions are similarly proven.

These results can be used to simplify logarithmic expressions:

Example 2-12

Simplify the following:

1. $\log_5 50 + \log_5 2 - \log_5 4$
2. $5 \log \left(10^4 \right) - 6 \log \left(\sqrt{10} \right) + 2 \log \left(10^{3/2} \right)$

Solution:

1. $\log_5 50 + \log_5 2 - \log_5 4 = \log_5\left(\frac{(50)(2)}{4}\right) = \log_5(25) = \log_5\left(5^2\right) = 2$
2. Recalling that by convention log means $\log_{10}$ we have:

$$\begin{aligned} 5\log\left(10^4\right) - 6\log\left(\sqrt{10}\right) + 2\log\left(10^{3/2}\right) &= (5)(4)\log(10) - 6\frac{1}{2}\log 10 + 2\frac{3}{2}\log 10 \\ &= 20(1) - 3(1) + 3(1) = 20 - 3 + 3 = 20 \end{aligned}$$

Further Questions:

Simplify the following:

1. $\log_2 4 + \log_2 10 - \log_2 5$
2. $\log_5 3 + \log_5 3^4 + \log_5 1$

2.3.2 The Natural Logarithmic Function

Definition: The logarithmic function with base a equal to $e = 2.71\ldots$ is called the **natural logarithmic function** and is denoted by $\mathbf{\ln x}$. In symbols:

$$\ln x = \log_e x$$

All the properties for a logarithm with base $a > 1$ apply to the natural logarithm. In terms of the notation for natural logarithms and exponentials we have the definition:

$$\ln x = y \iff e^y = x$$

and the properties :

$$\begin{aligned} \ln e &= 1 \\ \ln e^x &= x \quad (x \in \mathbb{R}) \\ e^{\ln x} &= x \quad (x > 0) \\ \ln(xy) &= \ln x + \ln y \quad (x, y > 0) \\ \ln\left(\frac{x}{y}\right) &= \ln x - \ln y \quad (x, y > 0) \\ \ln\left(x^r\right) &= r\ln x \quad (x > 0, r \in \mathbb{R}) \end{aligned}$$

Note the following:

$$\log_a(x + y) \neq \log_a x + \log_a y$$

$$\log_a(x - y) \neq \log_a x - \log_a y$$

In the specific case of natural logarithms ($a = e$):

$$\ln(x + y) \neq \ln x + \ln y$$

$$\ln(x - y) \neq \ln x - \ln y$$

You should identify the natural logarithm key $\boxed{\ln}$ on your calculator. Note that the key $\boxed{\log}$ on the calculator means base 10 logarithm $\log_{10} x$.[1]

Example 2-13

Simplify the following:

1. $\ln\left(\sqrt{e}\right) + 2\ln(3e) - \ln\left(e^5\right)$
2. $\ln\left(\dfrac{1}{e^{3/2}}\right)$

Solution:

1. $\ln\left(\sqrt{e}\right) + 2\ln(3e) - \ln\left(e^5\right) = \ln(e^{\frac{1}{2}}) + 2(\ln 3 + \ln e) - 5 = \dfrac{1}{2} + 2(\ln 3 + 1) - 5 = -\dfrac{5}{2} + 2\ln 3$
2. $\ln\left(\dfrac{1}{e^{3/2}}\right) = \ln 1 - \ln\left(e^{3/2}\right) = 0 - \dfrac{3}{2}\ln e = -\dfrac{3}{2}(1) = -\dfrac{3}{2}$

Further Questions:

Simplify the following:

1. $\ln 5 + 2\ln 3 + \ln 1$
2. $\dfrac{1}{2}\ln(4t) - \ln(t^2+1)$
3. $e^{\ln\left(x^2+1\right)} + 3x^2 - 5$

2.3.3 Solving Exponential and Logarithmic Equations

Solving equations involving logarithmic or exponential functions typically involves using properties of these functions to simplify those expressions involving the variable and then applying the appropriate inverse function to undo the exponential or logarithm. Finally one may solve for the variable.[2]

Example 2-14

We saw that the equation

$$\frac{e^2\sqrt{e^x}}{(2e^x)^3} = \frac{1}{2}$$

could be written, using properties of exponentials, as

$$e^{2-\frac{5}{2}x} = 4\ .$$

Applying ln, the inverse of the exponential e^x, to both sides of the equation, gives

$$2 - \frac{5}{2}x = \ln 4\ .$$

Solving for x gives

$$x = \frac{2}{5}\left(2 - \ln 4\right)\ .$$

[1] However in other areas (some computer applications) the symbol log will often refer to a natural logarithm so one needs to be careful.

[2] More complicated equations may allow themselves to be written as a product of factors equal to zero:

$$(\text{factor}_1)(\text{factor}_2)\cdot\ldots\cdot(\text{factor}_n) = 0$$

where the factors themselves involve logarithms or exponentials. Note that a strictly exponential factor equalling zero will provide no solution as $a^x \neq 0$ for all x. A strictly logarithmic factor equalling zero will be equivalent to the argument of the logarithm equalling 1.

Example 2-15

Solve the following equations for x.

1. $e^{2x+5} = 3$
2. $\ln(\ln x) = 2$
3. $\ln(x-2) = 2 + \ln(x-3)$

Solution:

1. $e^{2x+5} = 3$

   Take the natural logarithm of both sides.

$$\begin{aligned} \ln\left(e^{2x+5}\right) &= \ln 3 \\ &\implies 2x+5 = \ln 3 \\ &\implies 2x = \ln 3 - 5 \\ &\implies x = \frac{1}{2}(\ln 3 - 5) \end{aligned}$$

2. $\ln(\ln x) = 2$

   Exponentiate both sides.

$$e^{\ln(\ln x)} = e^2 \implies \ln x = e^2$$

   Exponentiate both sides again.

$$e^{\ln x} = e^{(e^2)} \implies x = e^{(e^2)}$$

   Note that by convention we can write $e^{(e^2)} = e^{e^2}$ (without parentheses). This is because $(e^e)^2$ would just be written e^{2e} using our rules of exponents. In general, remember parentheses when you ladder exponents if you do not remember the convention.

3. $\ln(x-2) = 2 + \ln(x-3)$

   First simplify the expression.

$$\begin{aligned} \ln(x-2) - \ln(x-3) &= 2 \\ \ln\left(\frac{x-2}{x-3}\right) &= 2 \end{aligned}$$

   Next exponentiate both sides.

$$\begin{aligned} e^{\ln\left(\frac{x-2}{x-3}\right)} = e^2 &\implies \frac{x-2}{x-3} = e^2 \implies x - 2 = e^2x - 3e^2 \\ &\implies x - e^2x = 2 - 3e^2 \implies x(1-e^2) = 2 - 3e^2 \\ &\implies x = \frac{2-3e^2}{1-e^2} \end{aligned}$$

Further Questions:

Solve the following equations for x:

1. $5e^{x-3} = 4$
2. $\ln\left(x^2 - 3\right) = 0$
3. $4e^x e^{-2x} = 6$
4. $\ln(2\ln x - 5) = 0$
5. $e^{x^2-5x+6} = 1$
6. $3e^{2x-4} = 10$
7. $\ln\left(\dfrac{x-2}{x-1}\right) = 1 + \ln\left(\dfrac{x-3}{x-1}\right)$

The following relates logarithms in other bases to the natural logarithm .

Theorem 2-9: For $a > 0$, $a \neq 1$ we have:

$$\log_a x = \frac{\ln x}{\ln a}$$

Proof comes from observing that since $a^{\log_a x} = x$ we can take the natural logarithm of both sides and then use the power rule for the natural logarithm to get

$$(\log_a x)(\ln a) = \ln x \; .$$

Solving for $\log_a x$ gives our result.

This theorem is useful for evaluating an arbitrary base a logarithm on a calculator.

Example 2-16

Write in terms of the natural logarithm (ln):

1. $\log_{12} 5$
2. $\log_2(e^{5x})$
3. $\log_x(x^2 + 2x + 1)$

Solution:

1. $\log_{12} 5 = \dfrac{\ln 5}{\ln 12}$
2. $\log_2(e^{5x}) = \dfrac{\ln(e^{5x})}{\ln 2} = \dfrac{5x}{\ln 2}$
3. $\log_x(x^2 + 2x + 1) = \dfrac{\ln(x^2 + 2x + 1)}{\ln x} = \dfrac{\ln\left((x+1)^2\right)}{\ln x} = \dfrac{2\ln(x+1)}{\ln x}$

Further Questions:

Write in terms of the natural logarithm (ln):

1. $\log_5 7$
2. $\log_{20}\left(x^2 + 1\right)$
3. $\log_{10}\left(e^{2x}\right)$

2.3.4 Derivative of the Natural Logarithmic Function

Theorem 2-10: The derivative of the natural logarithmic function is

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

To prove the theorem note that if $y = \ln x$ then by definition of the logarithm as an inverse we have

$$e^y = x \ .$$

Differentiating this implicit equation with respect to x on both sides gives

$$e^y y' = 1 \ ,$$

and so

$$y' = \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x} \ .$$

A corollary of this result, applying the Chain Rule to the function $\ln u$ with $u = g(x)$ is

Theorem 2-11: $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$ or $\frac{d}{dx}\ln[g(x)] = \frac{g'(x)}{g(x)}$

Example 2-17

Find the derivative of the given functions.

1. $y = \ln(x^3 + e^x)$
2. $f(x) = \ln(\tan x)$
3. $f(t) = e^{2t+\ln t}$

Solution:

In each of the following we use the Chain Rule.

1. $y = \ln(x^3 + e^x)$

$$y' = \frac{1}{x^3 + e^x}\left(x^3 + e^x\right)' = \frac{3x^2 + e^x}{x^3 + e^x}$$

2. $f(x) = \ln(\tan x)$

$$f'(x) = \frac{(\tan x)'}{\tan x} = \frac{\sec^2 x}{\tan x}$$

3. $f(t) = e^{2t+\ln t}$

$$f'(t) = e^{2t+\ln t}\,(2t + \ln t)' = \left(2 + \frac{1}{t}\right)e^{2t+\ln t}$$

Further Questions:

Differentiate the following functions:

1. $y = \ln\left(x^2 - 3x + 1\right)$
2. $y = \ln(x + \ln x)$
3. $y = \ln\left(\frac{x+1}{\sqrt{x+2}}\right)$
4. $y = e^{(2+x\ln x)}$

2.3.5 Derivatives Using Arbitrary Bases

Theorem 2-12: The derivative of the logarithm function to base $a > 0$ $(a \neq 1)$ is

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \qquad \frac{d}{dx}[\log_a g(x)] = \frac{g'(x)}{g(x)\ln a}$$

Proof of the former derivative follows from the identity $\log_a x = \frac{\ln x}{\ln a}$:

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{d}{dx}\left(\frac{1}{\ln a} \cdot \ln x\right) = \frac{1}{\ln a} \cdot \frac{d}{dx}(\ln x) = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}$$

Here note that we used that $\frac{1}{\ln a}$ is constant since a is constant. The latter derivative in the theorem follows from the Chain Rule applied to this former result.

Theorem 2-13: The derivative of an exponential function with base $a > 0$, $a \neq 1$ is

$$\frac{d}{dx}a^x = a^x \ln a \qquad \frac{d}{dx}\left[a^{g(x)}\right] = a^{g(x)}g'(x)\ln a$$

Proof of the former derivative follows by the observation that by our inverse identies the base a may be written $a = e^{\ln a}$ and using the Chain Rule:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{\ln a}\right)^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a}\frac{d}{dx}(x \ln a) = e^{x \ln a}(\ln a) = \left(e^{\ln a}\right)^x(\ln a) = a^x \ln a$$

Once again the latter derivative given in the theorem is just the result arising from using the Chain Rule with the former result.

Example 2-18

Differentiate the following functions.

1. $y = 5^{x^2-3x}$
2. $f(\theta) = \log_3(\sin\theta)$
3. $f(x) = \log_5(x^2+3x) + 10^{\ln x}$

Solution:

1. $y = 5^{x^2-3x}$

$$\begin{aligned} y' &= 5^{x^2-3x}(\ln 5)(x^2-3x)' \\ &= (2x-3)\ln 5 \cdot 5^{x^2-3x} \end{aligned}$$

2. $f(\theta) = \log_3(\sin\theta)$

$$f'(\theta) = \frac{1}{\sin(\theta)\ln 3} \cdot (\sin\theta)' = \frac{\cos\theta}{(\ln 3)\sin\theta}$$

3. $f(x) = \log_5(x^2+3x) + 10^{\ln x}$
   We differentiate term by term to get:

$$\begin{aligned} f'(x) &= \frac{1}{(x^2+3x)\ln 5} \cdot (x^2+3x)' + \cdot 10^{\ln x}\ln(10)(\ln x)' \\ &= \frac{2x+3}{x^2+3x} \cdot \frac{1}{\ln 5} + \frac{1}{x}\ln 10 \cdot 10^{\ln x} \end{aligned}$$

Further Questions:

Differentiate the following functions:

1. $y = \log_{10}\left(3x^2 + e^x\right)$
2. $y = 5^{2e^x+3x}$
3. $y = a^{3x}\log_4 x$
4. $y = 4^{\cos x}$

2.3.6 Logarithmic Differentiation

Using the properties of logarithms makes taking derivatives of logarithms of products, quotients, and powers easy.

Example 2-19

To differentiate $y = \ln[x(x^2+1)(x-3)]$ is easily done if we expand the logarithm first and then differentiate:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\ln[x(x^2+1)(x-3)] \\ &= \frac{d}{dx}\left[\ln x + \ln(x^2+1) + \ln(x-3)\right] \\ &= \frac{1}{x} + \frac{2x}{x^2+1} + \frac{1}{x-3} \end{aligned}$$

Wouldn't it be nice if when working with products, etc., we were always differentiating their logarithm? In *logarithmic differentiation* we take the *logarithm of both sides* of an equation before differentiating.

Example 2-20

To differentiate $y = x\left(2x^3+1\right)^3(x+5)^{\frac{1}{2}}\left(x^2+3x-1\right)^{\frac{1}{3}}$ one could use the (generalized) Product Rule. Instead, try taking the logarithm of both sides of the equation to get:

$$\ln y = \ln x + 3\ln\left(2x^3+1\right) + \frac{1}{2}\ln(x+5) + \frac{1}{3}\ln\left(x^2+3x-1\right)$$

Next differentiate both sides of the equation with respect to x to get:

$$\frac{1}{y}y' = \frac{1}{x} + 3\frac{6x^2}{2x^3+1} + \frac{1}{2}\frac{1}{x+5} + \frac{1}{3}\frac{2x+3}{x^2+3x-1}$$

Multiplying both sides by y and substituting in its value gives our derivative:

$$y' = \left[\frac{1}{x} + \frac{18x^2}{2x^3+1} + \frac{1}{2(x+5)} + \frac{2x+3}{3\left(x^2+3x-1\right)}\right] x\left(2x^3+1\right)^3(x+5)^{\frac{1}{2}}\left(x^2+3x-1\right)^{\frac{1}{3}}$$

Note that implicit differentiation is used to differentiate the $\ln y$ that shows up on the left hand side of the equation with respect to x. This gives the $\frac{1}{y}y'$ which is why we need to multiply both sides by y (for which we have the function).

Steps in Logarithmic Differentiation

1. Take logarithms of both sides of the equation $y = f(x)$.
2. Differentiate with respect to x on both sides, remembering to use implicit differentiation on $\ln y$ to get $\frac{1}{y}y'$.
3. Solve for y' and substitute $f(x)$ for y.

Example 2-21

Differentiate the following using logarithmic differentiation:

1. $y = (x+2)^{e^x}$
2. $y = \dfrac{(2x+5)^9\sqrt{x^2+3x}}{e^{\sin x}}$
3. $f(x) = (\sin x + x)^x$

Solution:

1. First take the logarithm of both sides of $y = (x+2)^{e^x}$.

$$\ln y = \ln\left[(x+2)^{e^x}\right]$$
$$\implies \ln y = e^x \ln(x+2)$$

Next differentiate both sides with respect to x.

$$\frac{1}{y}y' = e^x \ln(x+2) + e^x \frac{1}{x+2}(1+0)$$
$$\implies y'(x) = \left[e^x \ln(x+2) + \frac{e^x}{x+2}\right] \underbrace{(x+2)^{e^x}}_{=y}$$

2. Take the logarithm of $y = \dfrac{(2x+5)^9\sqrt{x^2+3x}}{e^{\sin x}}$ to get:

$$\ln y = \ln\left[(2x+5)^9\right] + \ln\left[\left(x^2+3x\right)^{\frac{1}{2}}\right] - \ln\left[e^{\sin x}\right]$$
$$\ln y = 9\ln(2x+5) + \frac{1}{2}\ln\left(x^2+3x\right) - \sin x$$

Differentiate both sides with respect to x.

$$\frac{y'}{y} = 9\frac{2+0}{2x+5} + \frac{1}{2}\frac{2x+3}{x^2+3x} - \cos x$$
$$y' = \left[\frac{18}{2x+5} + \frac{x+3/2}{x^2+3x} - \cos x\right]\frac{(2x+5)^9\sqrt{x^2+3x}}{e^{\sin x}}$$

3. Taking the logarithms of both sides of $f(x) = (\sin x + x)^x$ we have:

$$\ln f(x) = \ln\left[(\sin x + x)^x\right] = x\ln(\sin x + x)$$

Differentiate both sides with respect to x.

$$\frac{f'(x)}{f(x)} = (1)\ln(\sin x + x) + x\frac{\cos x + 1}{\sin x + x}$$
$$\implies f'(x) = \left[\ln(\sin x + x) + x\frac{\cos x + 1}{\sin x + x}\right] f(x)$$
$$\implies f'(x) = \left[\ln(\sin x + x)\,\frac{\cos x + 1}{\sin x + x}\right](\sin x + x)^x$$

Further Questions:

Differentiate the following using logarithmic differentiation:

1. $y = \dfrac{\left(x^3+5\right)\left(x^2-3x\right)^4}{x-2}$
2. $y = \left(x^2+3\right)^{x^3}$
3. $f(x) = \left(e^x+1\right)^{\ln x}$
4. $y = (2x+1)^{\sqrt{x}}$

We note that logarithmic differentiation allows, as shown in the previous example, the ability to differentiate functions of the form $y = [f(x)]^{g(x)}$ which up until now we had no method to differentiate. A function like $y = x^x$ for instance is neither a power function (x^a) nor an exponential function (a^x). Neither of the rules for differentiating those give the correct answer of $\frac{d}{dx}x^x = x^x \ln x + x^x$ obtained by logarithmic differentiation (show this!). One approach, equivalent to logarithmic differentiation, is to rewrite the expression as $x^x = (e^{\ln x})^x = e^{x \ln x}$, so the base is now constant, and then use the rule for exponential derivatives along with the Chain Rule:

$$\frac{d}{dx}x^x = \frac{d}{dx}e^{x\ln x} = e^{x\ln x}\left[(1)\ln x + x\frac{1}{x}\right] = x^x(\ln x + 1)$$

where we returned $e^{x \ln x}$ to x^x in the last step. More generally one can write $[f(x)]^{g(x)} = e^{g(x)\ln[f(x)]}$ and differentiate that directly as well. We prefer the logarithmic differentiation approach above as it avoids pushing the base into the exponent and returning again when you are done.

The approach of pushing the base into the exponent may be useful when the function one wishes to differentiate involves a sum of expressions only one of which requires logarithmic differentiation. In that case one would need to evaluate the derivative of that single term on the side with logarithmic differentiation and insert the result in the final derivative. Pushing the base function into the exponent, on the other hand, can be done inline with the rest of the derivatives. For example if $f(x) = e^e + e^x + x^e + x^x$ then

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(e^e + e^x + x^e + x^x\right) = \frac{d}{dx}\left(e^e + e^x + x^e + e^{x\ln x}\right) \\ &= 0 + e^x + (e-1)x^{e-1} + x^x(\ln x + 1) \end{aligned}$$

where we differentiated the last term as above. If we wanted to use logarithmic differentiation we would need to assign $y = x^x$, perform the logarithmic differentiation on the side to find the derivative y', and then insert that answer into the derivative $f'(x)$.

2.3.7 Integral of $\frac{1}{x}$ and a^x

The Power Rule for integration is

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} \quad (n \neq -1)$$

The answer for the indefinite integral clearly indicates that it cannot work for $n = -1$ as one would be dividing by zero. Since $\frac{d}{dx}\ln x = \frac{1}{x}$ however, we now have an antiderivative for $x^{-1} = \frac{1}{x}$, namely $\ln x$. This will only work for values of $x > 0$ since the domain of $\ln x$ is only positive numbers. However a second antiderivative of $\frac{1}{x}$ that will work when $x < 0$ is $\ln(-x)$, since $\frac{d}{dx}\ln(-x) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$ by the Chain Rule. We can combine the results using absolute value bars in the following theorem.

Theorem 2-14: The indefinite integral of x^{-1} is

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

A useful corollary of this result is that one can now integrate the tangent function. Using the substitution $u = \cos x$ (so $du = -\sin x\, dx$) one has

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C = \ln\left(|\cos x|^{-1}\right) + C = \ln|\sec x| + C$$

Since $\frac{d}{dx}a^x = a^x \ln a$ it follows, dividing both sides of the equation by the constant $\ln a$, that $\frac{d}{dx}\frac{a^x}{\ln a} = a^x$ and so we also have the result:

Theorem 2-15: $\displaystyle\int a^x\, dx = \frac{a^x}{\ln a} + C$

Example 2-22

Evaluate the following integrals:

1. $\displaystyle\int \frac{2}{x\ln x}\, dx$
2. $\displaystyle\int x^2 10^{x^3+1}\, dx$
3. $\displaystyle\int \frac{4x-1}{4x^2-2x+10}\, dx$
4. $\displaystyle\int \frac{dx}{(3x-2)\ln(9x-6)}$

Solution:

1. Let $u = \ln x$ so $du = \frac{1}{x}\, dx$.

$$\int \frac{2}{x\ln x}\, dx = 2\int \frac{1}{u}\, du = 2\ln|u| + C = 2\ln|\ln x| + C$$

2. Let $u = x^3 + 1$, so $du = 3x^2\, dx \implies \frac{1}{3}du = x^2\, dx$

$$\int x^2 10^{x^3+1}\, dx = \frac{1}{3}\int 10^u\, du = \frac{1}{3}\cdot\frac{10^u}{\ln 10} + C = \frac{10^{x^3+1}}{3\ln 10} + C$$

3. Let $u = 4x^2 - 2x + 10$ so $du = (8x-2)dx \implies \frac{1}{2}du = (4x-1)dx$

$$\int \frac{4x-1}{4x^2-2x+10}\, dx = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + C = \ln|4x^2-2x+10| + C$$

4. Let $u = 9x - 6$ so $du = 9dx \implies \frac{1}{9}du = dx$. Also solve for the remaining factor in terms of u to get $u = 3(3x-2) \implies 3x - 2 = \frac{u}{3}$. Then

$$\int \frac{dx}{(3x-2)\ln(9x-6)} = \frac{1}{9}\int \frac{1}{\frac{u}{3}\ln u}du = \frac{1}{3}\int \frac{1}{u\ln u}du$$

Next let $w = \ln u$ so $dw = \frac{1}{u}du$. The integral becomes

$$= \frac{1}{3}\int \frac{1}{w}dw = \frac{1}{3}\ln|w| + C = \frac{1}{3}\ln|\ln u| + C = \frac{1}{3}\ln|\ln(9x-6)| + C$$

Further Questions:

Evaluate the following integrals:

1. $\displaystyle\int \frac{3}{2x}\,dx$
2. $\displaystyle\int \frac{x^2}{x^3+5}\,dx$
3. $\displaystyle\int \frac{\ln x}{x}\,dx$
4. $\displaystyle\int_1^4 \frac{10^{\sqrt{x}}}{\sqrt{x}}\,dx$
5. $\displaystyle\int e^x 5^{e^x}\,dx$
6. $\displaystyle\int \frac{\sec x\tan x}{3+5\sec x}\,dx$
7. $\displaystyle\int \cot x\,dx$

Answers: Page 258

Exercise 2-3

1-3: Use the properties of logarithms to expand the given expression.

1. $\log_5\left(\dfrac{x+1}{2x+3}\right)^4$
2. $\ln\dfrac{(x+1)^2\sqrt{x+4}}{\sqrt[3]{x+2}}$
3. $\ln\dfrac{e^{2x}(2x+1)^3}{\sqrt{e^x+1}}$

4-6: Express the given quantity as a single logarithm.

4. $\ln(2x) - 3\ln(e^x+2) - \dfrac{1}{2}\ln(x+4)$
5. $3\log_{10}\left(x^2+1\right) + \dfrac{1}{3}\log_{10}(x+4) - \dfrac{1}{4}\log_{10}\left(3x^2+5\right))$
6. $3\ln x + 2\log_3\left(x^3+2\right) - \dfrac{1}{2}\ln(3x+1)$

7-11: Solve for x.

7. $\ln\left(x^2+3\right) = 4$
8. $e^{2x} - 5e^x + 6 = 0$
9. $\ln(2x+1) + \ln x = \ln\left(x^2+2\right)$
10. $\ln(e^x-2) + \ln e^x = \ln 8$
11. $e^x 2^x = 5$
    (Hint: Convert $2^x = \left(e^{\ln 2}\right)^x = e^{x\ln 2}$)

12-15: For the following functions find

(a) The domain and the range of f, and

(b) $f^{-1}(x)$ and its domain.

12. $f(x) = \sqrt{2 + 3e^x}$

13. $f(x) = \ln(3x + 2)$

14. $f(x) = \dfrac{e^x - 2}{e^x + 3}$

15. $f(x) = \dfrac{\ln x + 1}{\ln x + 2}$

16-22: Differentiate the given function.

16. $f(x) = \ln\left(x^2 + e^x\right)$

17. $y = \log_4(\sin x + 3x)$

18. $g(t) = 10^{t+2}\ln(\ln t + 5)$

19. $f(x) = \ln\left[x^2\sqrt{x^2+3}\left(e^{3x}+1\right)\right]$

20. $y = \dfrac{e^{2x}\sqrt{x^2+5}}{\sqrt[3]{x+1}}$

21. $f(x) = \left(x^2 + e^x\right)^{\ln x}$

22. $y = x^{\sin x}$

23-31: Evaluate the given integral.

23. $\displaystyle\int \frac{e^{-2x}}{\sqrt{e^{-2x}+3}}\,dx$

24. $\displaystyle\int \frac{e^x}{e^x+5}\,dx$

25. $\displaystyle\int_0^2 \frac{1}{t+9}\,dt$

26. $\displaystyle\int \frac{2}{x\,[2+\ln x]^3}\,dx$

27. $\displaystyle\int (2x+1)\;\; 5^{x^2+x}\,dx$

28. $\displaystyle\int_1^4 \frac{10^{\sqrt{x}}}{\sqrt{x}}\,dx$

29. $\displaystyle\int \frac{\sin 2x}{3+\cos 2x}$

30. $\displaystyle\int \frac{3}{x\,(\ln x)^4}\,dx$

31. $\displaystyle\int \frac{\log_4 x}{x}\,dx$

---

32. (a) Show the (remarkable) result that $a^{\ln x} = x^{\ln a}$.

(b) Using the identity from (a) find the integral $\displaystyle\int 5^{\ln x}\,dx$.

(c) Integrate $\displaystyle\int 5^{\ln x}\,dx$ using the substitution $u = \ln x$.

(d) Is $f(x) = 5^{\ln x}$ a power function, an exponential function, or neither?

2.4 Exponential Growth and Decay

Many quantities, such as the number of cells being cultured in a lab dish or the number of radioactive nucleii of a particular isotope in a radioactive sample remaining undecayed, have a population $y(t)$ that satisfies the differential equation

$$\boxed{\frac{dy}{dt} = ky}\,.$$

Here the constant k, called the **relative growth rate**, characterizes the population under consideration. It will be positive ($k > 0$) if the population $y(t)$ is increasing in time and negative ($k < 0$) if it is decreasing. The preceding equation is called the **law of natural growth** or **law of natural decay** respectively. The constant k is called **relative** since if we solve for it, $k = \frac{1}{y}\frac{dy}{dt}$, we see the rate dy/dt is constant only relative to the population size y at an given time.

The differential $dy = \frac{dy}{dt}dt$ satisfies

$$dy = ky\,dt\,.$$

Over a fixed time interval Δt we have the analogous relation

$$\Delta y = ky\Delta t\,,$$

where y, by the Mean Value Theorem, is evaluated at some time t in the interval. Assuming Δt is small enough this can be effectively any time t in the interval as y will be approximately constant over such an interval. The change Δy in the population $y(t)$ over a fixed small time interval Δt is therefore proportional to the population itself

$$\Delta y \propto y\,.$$

which is expected for a population whose growth (or loss) depends on the current size of the population. Additionally the relation shows the change will also be approximately proportional to the length of the time interval Δt considered,

$$\Delta y \propto \Delta t\,,$$

assuming again that Δt is sufficiently small, a result that is also reasonable.

To understand how y changes in time we need to find the function $y(t)$ that satisfies (solves) the differential equation

$$\frac{dy}{dt} = ky\,.$$

If the right hand side of the equation just involved t explicitly, like $\frac{dy}{dt} = t^2$, the answer would just be the antiderivative $y(t) = \int t^2 = \frac{1}{3}t^3 + C$. Our differential equation is not of this form, however, as it has the dependent variable y on the right hand side. Solving such a differential equation such as ours can be done by the process of **separation of variables** . Inspired by the Leibniz notation, one formally proceeds by isolating, if possible, a function of the dependent variable y and its differential dy on one side of the equation and a function of the independent variable t and its differential dt on the other to get

$$\frac{dy}{y} = k\,dt\,.$$

One then integrates both sides:

$$\begin{aligned} \int \frac{dy}{y} &= \int k\,dt \\ \Rightarrow\ \ln y &= kt + D\,, \end{aligned}$$

where we have combined the integration constants C_1 and C_2 arising from both sides of the integral into $D = C_2 - C_1$. Finally we can solve for y by taking the natural exponential of both sides:

$$e^{\ln y} = e^{kt+D}$$
$$\Rightarrow \; y = e^{kt}e^{D}$$

Calling a new (positive) constant $C = e^D$ we have the final solution of the differential equation

$$y(t) = Ce^{kt}\,.$$

Despite the lack of rigour in our separation of variable approach, one may readily confirm that $y(t) = Ce^{kt}$ does satisfy the original differential equation as required.

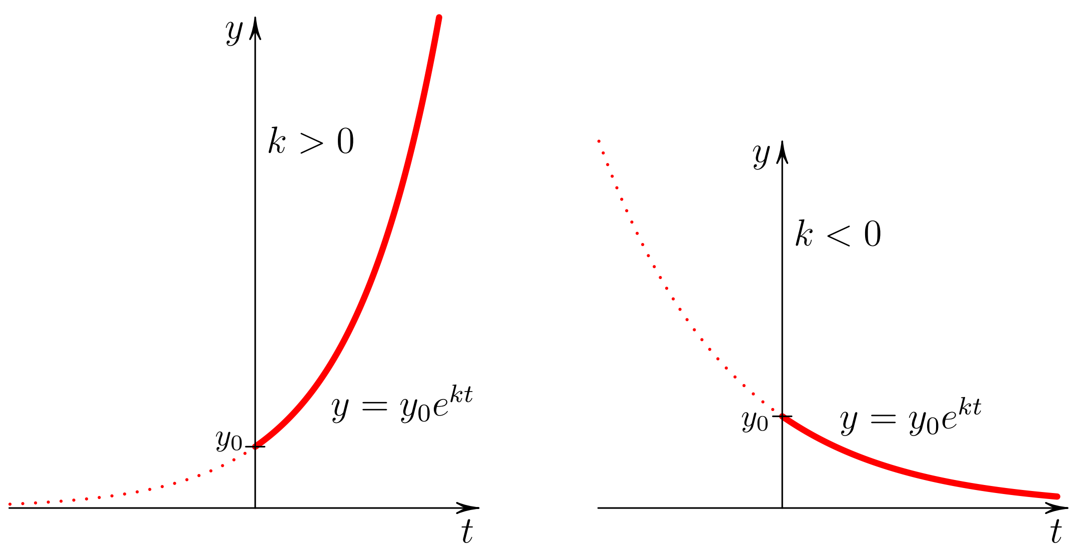
The constant of integration, C can be determined by providing an additional piece of information regarding the system. If, for instance, one knows the initial size of the population is $y(0) = y_0$, then the solution of the resulting initial value problem gives

$$y_0 = y(0) = Ce^{k(0)} = Ce^0 = C(1) = C\,.$$

Placing this value for $C = y_0$ back in $y(t)$ gives

$$\boxed{y(t) = y_0e^{kt}}\,.$$

As such the population at arbitrary time, assuming it is undergoing exponential growth or decay, is characterized completely by the growth constant k and its initial size y_0. The graphs of the cases where $k > 0$ (growth) and $k < 0$ (decay) are shown below.



Example 2-23

A quantity $P(t)$ measuring the number of bacteria growing in a controlled laboratory setting increases exponentially. Suppose it doubles in 8 hours.

1. Find the relative growth rate.
2. After how many hours has P increased by 50%?

Solution:

1. $P(8 \text{ hr}) = 2P_0$ where P_0 is the initial population.

$$\begin{aligned} \implies P_0 e^{k(8\,\text{hr})} = 2P_0 \implies e^{(8\,\text{hr})k} &= 2 \\ \ln\left(e^{(8\,\text{hr})k}\right) &= \ln 2 \\ (8\,\text{hr})k &= \ln 2 \\ k &= \frac{\ln 2}{8}\ (1/\text{hr}) \end{aligned}$$

   Therefore $P(t) = P_0 e^{\frac{\ln 2}{8\,\text{hr}} t}$.

2. An increase of 50% means that at the unknown time t the population has grown to $P(t) = P_0 + (0.5)P_0 = 1.5P_0$.

$$\begin{aligned} \implies P_0 e^{\frac{\ln 2}{8\,\text{hr}} t} = 1.5P_0 \implies e^{\frac{\ln 2}{8\,\text{hr}} t} &= 1.5 \\ \frac{\ln 2}{8\,\text{hr}} t &= \ln 1.5 \\ t &= \frac{8 \ln 1.5}{\ln 2}\ \text{hr} \approx 4.68\,\text{hr} \end{aligned}$$

Further Questions:

Fox squirrels introduced into a city see their population increase from 50 to 12000 in 4 years. Assuming the growth was exponential over this time period,

1. Find the relative growth rate k.
2. When will the squirrel population exceed 1 million?
3. Is the latter likely? Explain.

When $k < 0$ we have a decay formula and the amount y decreases over time. Then the positive constant $\lambda = |k| = -k$, called the **decay constant** , may be introduced and our formula becomes

$$y(t) = y_0 e^{-\lambda t}.$$

Rather than the decay constant, one often uses the **half-life** constant T for a radioactive sample. It is defined to be the time required for half of the initial decaying substance to disappear (i.e. decay into a new form), and so $y(T) = \frac{1}{2} y_0$. This can be used to determine the decay constant λ.

Example 2-24

The mass of a certain radioactive material is given by $m(t) = m_0 e^{kt}$. If the half-life of this material is 1600 years, find the decay constant $\lambda = -k$.

Solution:

$$m(t) = m_0 e^{kt} \implies m(1600\,\text{yr}) = \frac{1}{2}m_0 \implies m_0 e^{k(1600\,\text{yr})} = \frac{1}{2}m_0 \implies e^{k(1600\,\text{yr})} = \frac{1}{2}$$

$$\implies \ln\left(e^{k(1600\,\text{yr})}\right) = \ln\left(\frac{1}{2}\right) = \ln 1 - \ln 2 = 0 - \ln 2 = -\ln 2$$

$$\implies k(1600\,\text{yr}) = -\ln 2 \implies k = -\frac{\ln 2}{1600}\,(1/\text{yr})$$

$$\implies \lambda = \frac{\ln 2}{1600}\,(1/\text{yr}) \approx 0.000433\,(1/\text{yr})$$

Further Questions:

Cobalt-60 is a radioactive isotope used in early radiotherapy and other applications. Sixty is the **mass number** of the nucleus, the number of nucleons (protons and neutrons) it contains. A sample of Cobalt-60 undergoes exponential decay with a half-life of 5.2714 years.

1. Find the decay constant $\lambda = -k$ of Cobalt-60.
2. How long would it take for a sample containing 40 grams of the isotope to decay to a sample containing only 10 grams of it?

Finally we note that there are many examples of quantities besides population counts which satisfy the differential equation $\frac{dy}{dt} = ky$ with solution $y = y_0 e^{kt}$. As an example, the voltage across a discharging capacitor in an electronic circuit containing only a resistor and a capacitor (an RC circuit) undergoes exponential decay from an initial voltage V_0.

Answers:
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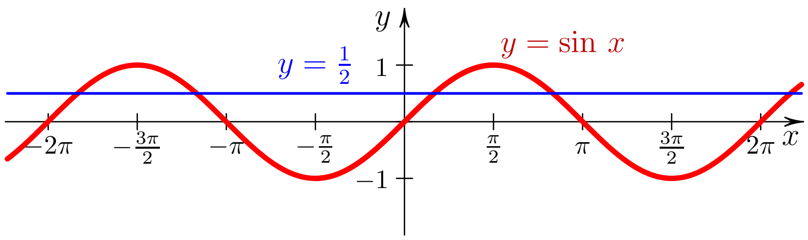
Exercise 2-4

1. A certain type of bacteria grows exponentially and doubles every 5 hours.
   (a) Find the growth constant.
   (b) How many bacteria will there be after 20 hours assuming the initial population is 200 bacteria?
2. The initial deer population in a forest was 25 deers. After one year the number of deers had increased to 75. If the number of deers grows exponentially, how many deers will there be at the end of two years?
3. In 2016, the population of a small island was estimated to be 30,000 people with an annual rate of increase of 2.5%.
   (a) Find the growth constant.
   (b) Estimate the population of the island in 2025.
4. Magnesium-27 decays exponentially and has a half-life of 9.45 minutes.
   (a) Find the decay constant for Magnesium-27.
   (b) When will an initiall mass of 20 mg be reduced to 5 mg?
5. A patient is given a 400 mg dose of medicine that degrades by 20% every 3 hours. What is the remaining drug concentration after a day?
6. A certain radiactive isotope decays exponentially according to the formula $m = m_0 e^{-0.3t}$ where m is the mass (in grams) of the isotope at the end of t days and m_0 is the initial mass. Find the half-life of this isotope.

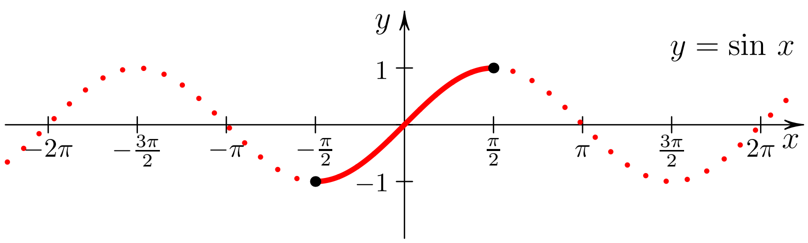
2.5 Inverse Trigonometric Functions

2.5.1 Inverse Sine

The sine function $y = \sin x$ on its natural domain $(-\infty, \infty)$ is not a one-to-one function. It clearly fails the horizontal line test as the intersection with the line $y = \frac{1}{2}$ clearly shows:

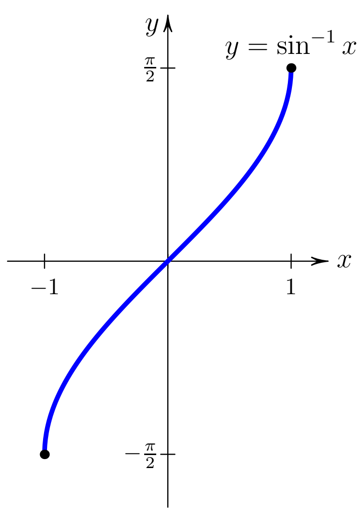


However the function $y = \sin x$ on domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is a one-to-one function:



Definition: The inverse function of $y = \sin x$; $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is called the **inverse sine function** or **arcsine function** and is denoted by $\boldsymbol{y = \sin^{-1} x}$ or $\boldsymbol{y = \arcsin x}$. It satisfies

$$y = \sin^{-1} x \iff x = \sin y \quad \left(-\frac{\pi}{2} \le y \le \frac{\pi}{2}\right)$$



The domain of inverse sine is $[-1, 1]$ and range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The usual inverse identities apply:

$$\begin{aligned} \sin^{-1}(\sin x) &= x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \sin\left(\sin^{-1} x\right) &= x \quad \text{for} \quad -1 \le x \le 1 \end{aligned}$$

Example 2-25

Evaluate the following:

1. $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$
2. $\sec\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$

Solution:

1. Since inverse trigonometric functions return angles, call $\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$. Then by definition of the inverse we have:

$$\sin\theta = \frac{1}{\sqrt{2}} \text{ with } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\implies \theta = \frac{\pi}{4}$$

The latter result is by inspection of a $45° - 45° - 90°$ triangle with side lengths 1, 1, $\sqrt{2}$ and recalling sine is the opposite over the hypotenuse.

2. Evaluate the inverse trigonometric function as in part 1 and then insert it to find

$$\sec\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] = \sec\left(\frac{\pi}{2}\right) = \frac{1}{\cos\left(\frac{\pi}{2}\right)} = \frac{1}{1/\sqrt{2}} = \sqrt{2},$$

where here we used the $45° - 45° - 90°$ triangle again to evaluate the cosine.

Further Questions:

Evaluate the following:

1. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
2. $\tan\left[\sin^{-1}\left(\frac{1}{2}\right)\right]$
3. $\sin\left(2\sin^{-1}x\right)$

Theorem 2-16: The derivative of inverse sine is

$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}} \;,$$

where $-1 < x < 1$.

To prove the theorem note that if $y = \sin^{-1}x$ then by definition of the inverse:

$$\sin y = x$$

Implicit differentiation of both sides with respect to x yields

$$(\cos y)\, y' = 1$$

and so $\frac{dy}{dx} = \frac{1}{\cos y}$. By the trigonometric identity $\cos^2 y + \sin^2 y = 1$ it follows that

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

where here the positive solution was taken since $-\frac{\pi}{2} < y < \frac{\pi}{2}$ implies $\cos y > 0$. Inserting this in the formula for $\frac{dy}{dx}$ gives the result.

Note that the Chain Rule result is, as expected:

$$\frac{d}{dx}\left(\sin^{-1} g(x)\right) = \frac{g'(x)}{\sqrt{1-g^2(x)}} .$$

Example 2-26

Differentiate the following function:

$$y = \sin^{-1}(e^x + 2)$$

Solution:

By the Chain Rule:

$$y' = \left(\sin^{-1}(e^x+2)\right)' = \frac{1}{\sqrt{1-(e^x+2)^2}}(e^x+0) = \frac{e^x}{\sqrt{1-(e^x+2)^2}}$$

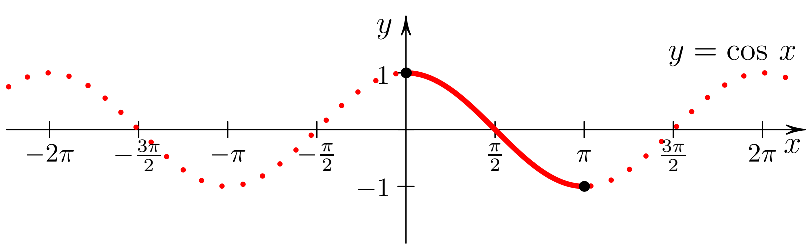
Further Questions:

Differentiate the following functions:

1. $y = \sin^{-1}(\ln x + 3)$
2. $y = e^{\sin^{-1} x} + \sin^{-1}(e^x)$

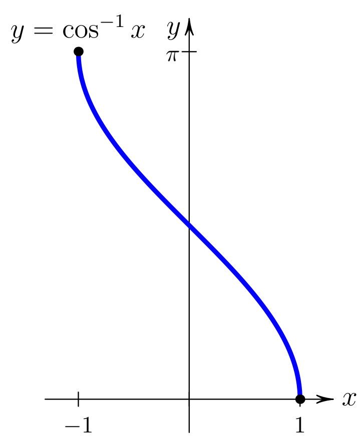
2.5.2 Inverse Cosine

The function $y = \cos x$ on domain $[0, \pi]$ is a one-to-one function:



Definition: The inverse function of $y = \cos x$; $[0, \pi]$ is called the **inverse cosine function** or **arccosine function** and is denoted by $\boldsymbol{y = \cos^{-1} x}$ or $\boldsymbol{y = \arccos x}$. It satisfies

$$y = \cos^{-1} x \iff x = \cos y \quad (0 \le y \le \pi)$$



The domain of inverse cosine is $[-1, 1]$ and range is $[0, \pi]$. The inverse identities are:

$$\begin{aligned} \cos^{-1}(\cos x) &= x \quad \text{for} \quad 0 \leq x \leq \pi \\ \cos\left(\cos^{-1} x\right) &= x \quad \text{for} \quad -1 \leq x \leq 1 \end{aligned}$$

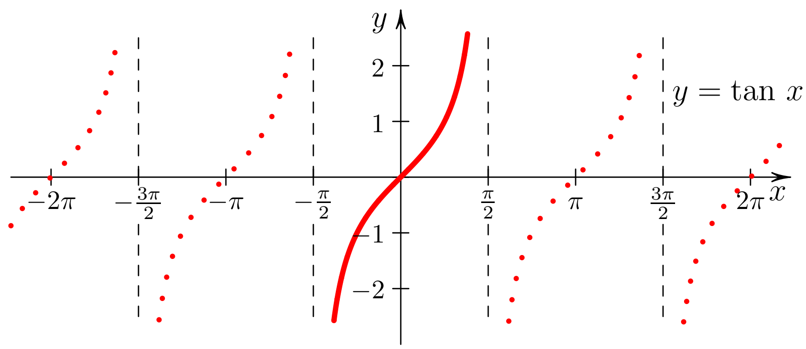
Theorem 2-17: The derivative of inverse cosine is

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}} \ ,$$

where $-1 < x < 1$.

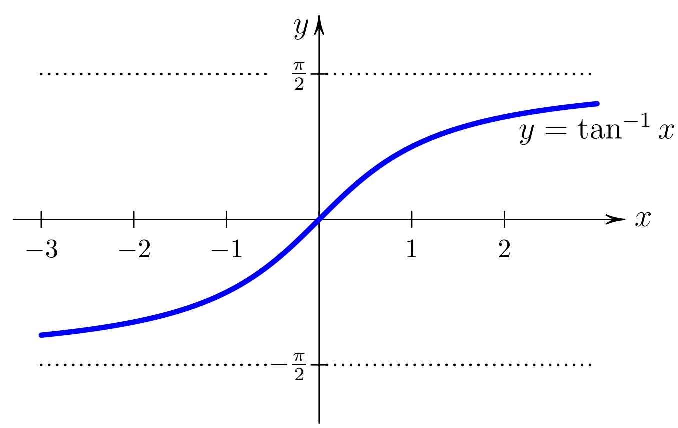
2.5.3 Inverse Tangent

The function $y = \tan x$ on domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is a one-to-one function:



Definition: The inverse function of $y = \tan x$; $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is called the **inverse tangent function** and is denoted by $\boldsymbol{y = \tan^{-1} x}$ or $\boldsymbol{y = \arctan x}$. It satisfies

$$y = \tan^{-1} x \iff x = \tan y \quad \left(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right)$$



The domain of inverse tangent is $(-\infty, \infty)$ and range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The usual inverse identities apply:

$$\begin{aligned} \tan^{-1}(\tan x) &= x \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \tan\left(\tan^{-1} x\right) &= x \quad \text{for} \quad x \in \mathbb{R} \end{aligned}$$

as well as the limits:

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2} \qquad\qquad \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

So $y = \pm\frac{\pi}{2}$ are horizontal asymptotes of the function.

Theorem 2-18: The derivative of inverse tangent is

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} .$$

2.5.4 Other Trigonometric Inverses

Similarly one defines $y = \csc^{-1} x$, $y = \sec^{-1} x$, and $y = \cot^{-1} x$.[3]

Notes:

- Since trigonometric functions are functions of angles, inverse trigonometric functions return angles. All our angles above are in radians. On your calculator you must have it set to radian mode to get these inverse trigonometric function results. If you have your calculator set to degree mode you will get your answers in degrees.
- The -1 in $\sin^{-1} x$ means inverse *not* taking to the power of -1 (reciprocal) like the 2 in $\sin^2 x$ means. If you mean take to the power of -1, i.e. $\frac{1}{\sin x}$ then you must write $(\sin x)^{-1}$ or simply use the reciprocal trigonometric function $\csc x$.
- It is because none of the trig functions are one-to-one and hence not invertible on their natural domains that solving a trigonometric equation $\text{trig}(x) = \#$ requires more than just "applying the inverse" to both sides (unlike, say comparable logarithmic or exponential equations). So to solve $\sin x = \frac{1}{2}$ the result $x = \sin^{-1}(1/2) = \pi/6$ is only one of many solutions. (See the intersections between $y = \sin x$ and $y = 1/2$ in our initial graph in this section.)

A complete table of the inverse trigonometric derivatives is as follows :

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

Here the derivatives exist on the domains of the inverse trigonometric function except at those values where the expression is undefined. For the Chain Rule formulae simply replace x by $g(x)$ in each formula and multiply the result by $g'(x)$.

[3] Note that the convention for the inverse secant and inverse cosecant functions used here is that the domain of secant (cosecant), and hence the range of inverse secant (cosecant) is $[0, \pi/2) \cup [\pi, 3\pi/2)$ ($(0, \pi/2] \cup (\pi, 3\pi/2]$). One can also choose the more intuitive interval $[0, \pi/2) \cup (\pi/2, \pi]$ for secant and $[-\pi/2, 0) \cup (0, \pi/2]$ for cosecant but then absolute value bars are required about the x outside the radical in the derivative formulae.

Example 2-27

Differentiate the following functions:

1. $f(x) = \cos^{-1}(\ln x)$
2. $y = \dfrac{\tan^{-1}(x^2+3)}{x}$
3. $\tan^{-1} y + \sin^{-1} x = e^y$
4. $y = \sec^{-1}(x^2)$

Solution:

1. $f'(x) = \dfrac{d}{dx}\cos^{-1}(\ln x) = \dfrac{-1}{\sqrt{1-(\ln x)^2}}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x\sqrt{1-(\ln x)^2}}$
2. We can use the Product Rule:

$$\begin{aligned} y' &= \left(\tan^{-1}(x^2+3)\cdot\left(\frac{1}{x}\right)\right)' = \frac{2x+0}{1+(x^2+3)^2}\left(\frac{1}{x}\right) + \tan^{-1}(x^2+3)\left(-\frac{1}{x^2}\right) \\ &= \frac{2}{1+(x^2+3)^2} - \frac{\tan^{-1}(x^2+3)}{x^2} \end{aligned}$$

3. Using implicit differentiation differentiate both sides of $\tan^{-1} y + \sin^{-1} x = e^y$ to get:

$$\begin{aligned} &\Longrightarrow \frac{1}{1+y^2}(y') + \frac{1}{\sqrt{1-x^2}} = e^y y' \\ &\Longrightarrow y'\left(\frac{1}{1+y^2} - e^y\right) = -\frac{1}{\sqrt{1-x^2}} \\ &\Longrightarrow y' = \frac{-\frac{1}{\sqrt{1-x^2}}}{\frac{1}{1+y^2} - e^y} \quad \left(\leftarrow \text{Multiply by } 1 = \frac{1+y^2}{1+y^2}\right) \\ &\Longrightarrow y' = -\frac{1+y^2}{\sqrt{1-x^2}(1-e^y(1+y^2))} \end{aligned}$$

4. $y' = \dfrac{d}{dx}\sec^{-1}(x^2) = \dfrac{1}{x^2\sqrt{(x^2)^2-1}}(2x) = \dfrac{2}{x\sqrt{x^4-1}}$

Further Questions:

Differentiate the following functions:

1. $y = \sin^{-1}(2x-1)$
2. $y = \tan^{-1}\left(\dfrac{x}{3}\right) + \ln\sqrt{\dfrac{x-3}{x+3}}$
3. $y = x\cos^{-1} x - \sqrt{1-x^2}$
4. $y = \sin^{-1}\left(\tan^{-1} x\right)$
5. $y = \tan^{-1}(\ln x)\, e^{x^2+3}$
6. $y = \cos^{-1}\left(e^{2x}-5\right)$
7. $y = \tan^{-1}\left(x^2+3\right) - \tan\left(\cos^{-1} x + 1\right)$
8. $f(t) = \sec^{-1}\left(e^t + \ln t\right)$
9. $\sin^{-1} y = x^2 + y^2 + e^y$

The derivatives of the inverse trigonometric functions give the following results:

Theorem 2-19: We have the following indefinite integrals:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C, \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C, \qquad \int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$$

If one considers the more general integral

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx\ ,$$

where $a > 0$ is constant this can be solved by first noting that

$$\sqrt{a^2-x^2} = \sqrt{a^2\left(1-\frac{x^2}{a^2}\right)} = \sqrt{a^2}\sqrt{1-\frac{x^2}{a^2}} = |a|\sqrt{1-\frac{x^2}{a^2}} = a\sqrt{1-\frac{x^2}{a^2}}$$

and then using the substitution $u = \frac{x}{a}$ (so $du = \frac{dx}{a}$) to get:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a\sqrt{1-\frac{x^2}{a^2}}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C = \sin^{-1}\frac{x}{a} + C\ .$$

Similar generalization can be done to the inverse tangent integral. We thus have:

Theorem 2-20: For constant $a > 0$,

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$$

Note on Remembering Formulas

Integral formulas appearing in Theorem 2-20 with literal constants (i.e. a) are what one finds typically in integral tables. The placement of the a's need not be memorized as they can be inferred by *dimensional analysis*. Suppose x has units of length. Then from the form of the integrand a would also have units of length since a^2 is added to or subtracted from x^2. The resulting integrals involve inverse trigonometric functions whose arguments must be dimensionless since the result of any trigonometric function is a dimensionless ratio. To achieve that we divide x in our original integrals of Theorem 2-19 by a. Next to remember whether we need a $\frac{1}{a}$ in front or not we consider the integrand. For the first one we have dx in the numerator which will have the same length dimension as x since it is just an (infinitessimal) interval of x. In the denominator we have the square root of a length-squared which is just a length. As such for the first integral we are summing a dimensionless quantity since it is a length divided by a length and the result must be dimensionless. The resulting arcsine function, which returns a dimensionless angle, therefore cannot have an a out front as that would introduce a length dimension. The latter two integrals, however, are sums of quantities of dimension length/length^2 = 1/length. To achieve that dimension in our result we must multiply the dimensionless inverse trigonometric function by $\frac{1}{a}$. Once again one notes how the differential dx in our notation makes this work. Thank-you Leibniz!

Example 2-28

Evaluate the given integrals:

1. $\displaystyle\int \frac{1}{\sqrt{4-t^2}}\,dt$

2. $\displaystyle\int_0^2 \frac{dx}{4+x^2}\,dx$

Solution:

1. To get the leading 1 we factor out 4:

$$\int \frac{1}{\sqrt{4-t^2}}\,dt = \int \frac{1}{\sqrt{4\left(1-\left(\frac{t}{2}\right)^2\right)}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{t}{2}\right)^2}}\,dt$$

Then let $u = \frac{t}{2}$ so $du = \frac{1}{2}dt \implies dt = 2du$. The integral is then:

$$\begin{aligned} &= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\cdot 2\,du = \sin^{-1} u + C \\ &= \sin^{-1}\left(\frac{t}{2}\right) + C \end{aligned}$$

2. Again we factor out 4 and to get the desired 1:

$$\int_0^2 \frac{dx}{4+x^2}\,dx = \int_0^2 \frac{1}{4}\cdot\frac{1}{1+(\frac{x}{2})^2}\,dx$$

Substitute: $u = \frac{x}{2}$ so $du = \frac{1}{2}dx \implies 2du = dx$.
Change limits: $x = 0 \implies u = 0$, $x = 2 \implies u = 1$
The integral becomes:

$$\begin{aligned} &= \int_0^1 \frac{1}{4}\cdot\frac{1}{1+u^2}\cdot 2\,du = \frac{1}{2}\tan^{-1}u\Big|_{u=0}^{u=1} \\ &= \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(0)\right] = \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8} \end{aligned}$$

Further Questions:

Evaluate the following integrals:

1. $\displaystyle\int \frac{3}{\sqrt{4-2x^2}}\,dx$

2. $\displaystyle\int \frac{\tan^{-1} x}{1+x^2}\,dx$

3. $\displaystyle\int \frac{4}{t\left[9+(\ln t)^2\right]}\,dt$

4. $\displaystyle\int \frac{e^x}{\sqrt{1-8e^{2x}}}\,dx$

5. $\displaystyle\int \frac{3x+4}{2x^2+3}\,dx$

6. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx$

Exercise 2-5

Answers: Page 261

1-4: Find the exact value of the following expressions.

1. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$
2. $\tan^{-1}\left(\sin\frac{\pi}{2}\right)$
3. $\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)$
4. $\tan\left(\cos^{-1}x\right)$

5-11: Differentiate the given functions.

5. $f(x) = \sin^{-1}(e^x + 2)$
6. $y = \cos^{-1}(\ln x + 5)$
7. $g(x) = \tan^{-1}(\sin x)$
8. $y = \left(\sin^{-1}x\right)^{\ln x}$
9. $\tan^{-1}y + e^y + xy = 5$
10. $\sin^{-1}x + \sin^{-1}y = x^2 + y^2$
11. $f(x) = \sec^{-1}(2x)$

12-20: Evaluate the given integral.

12. $\int \frac{e^x}{3 + 2e^{2x}}\,dx$
13. $\int \frac{2}{x\left[5 + (\ln x)^2\right]}\,dx$
14. $\int \frac{3x}{\sqrt{1 - x^4}}\,dx$
15. $\int \frac{2x + 3}{x^2 + 5}\,dx$
16. $\int \frac{\sin^{-1}x}{\sqrt{1 - x^2}}\,dx$
17. $\int \frac{\sec^2 x}{\sqrt{1 - \tan^2 x}}\,dx$
18. $\int \frac{1}{\sqrt{x}(2 + x)}\,dx$
19. $\int \frac{1}{\sqrt{e^{2x} - 1}}\,dx$
20. $\int_0^{\sqrt{2}/2} \frac{2x}{\sqrt{1 - x^4}}\,dx$

2.6 L'Hôpital's Rule

We have already evaluated limits that are indeterminate forms of the type $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Example 2-29

Evaluate the following limits:

1. $\lim_{x\to 2} \frac{x^2+3x-10}{2x^2-8}$
2. $\lim_{x\to\infty} \frac{7x^2+3x+1}{9x^3+4}$

Solution:

1. Substitution of 2 in the rational function confirms a $\frac{0}{0}$ indeterminate form. The polynomials evaluating to zero at $x = 2$ suggest the presence of a factor of $(x-2)$ is the cause. Factoring confirms this and allows evaluation of the limit:

$$\lim_{x\to 2} \frac{x^2+3x-10}{2x^2-8} = \lim_{x\to 2} \frac{(x-2)(x+5)}{2(x+2)(x-2)} = \lim_{x\to 2} \frac{x+5}{2(x+2)} = \frac{7}{2(4)} = \frac{7}{8}$$

2. As $x \to \infty$ the polynomials in the rational function both get large and we have an $\frac{\infty}{\infty}$ indeterminate form. Factoring out the dominant term in each of the numerator and the denominator, in this case the highest power of x, allows the limit to be evaluated:

$$\lim_{x\to\infty} \frac{7x^2+3x+1}{9x^3+4} = \lim_{x\to\infty} \frac{x^2}{x^3} \cdot \frac{7+\frac{3}{x}+\frac{1}{x^2}}{9+\frac{4}{x^3}} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{7+\frac{3}{x}+\frac{1}{x^2}}{9+\frac{4}{x^3}} = 0 \cdot \frac{7+0+0}{9+0} = 0$$

Further Questions:

Evaluate the following limits:

1. $\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2}$
2. $\lim_{x\to\infty} \frac{3x^2-5x+2}{4x^2+3x-10}$

In general:

- If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ is called the **indeterminate form of type $\frac{0}{0}$**.
- If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ is called the **indeterminate form of type $\frac{\infty}{\infty}$** .

Our techniques used above will not work for evaluating all limits of this type:

Example 2-30

The limit $\lim_{x\to 0} \frac{2^x-1}{x}$ is an indeterminate form of type $\frac{0}{0}$ while $\lim_{x\to\infty} \frac{\ln x}{x}$ is of type $\frac{\infty}{\infty}$. Neither limit may be resolved using the methods of the previous example.

Theorem 2-21: If f and g are differentiable functions with $g'(x) \neq 0$ on an open interval containing the value a and $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, (i.e. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ or $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$) then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} ,$$

provided the limit on the right hand side either exists or is $\pm\infty$. This is **L'Hôpital's Rule**.

Example 2-31

Evaluate the previous limits using L'Hôpital's Rule:

1. $\lim_{x \to 0} \frac{2^x - 1}{x} = \lim_{x \to 0} \frac{2^x \ln 2 - 0}{1} = 2^0 \ln 2 = (1)(\ln 2) = \ln 2$
2. $\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \lim_{x \to \infty} \frac{1}{x} = 0$

Note that when applying L'Hôpital's Rule one is **not** using the Quotient Rule! The derivatives in the numerator and denominator are taken separately.

Example 2-32

Evaluate the given limits:

1. $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$
2. $\lim_{x \to 1} \frac{e^{2x} - e^2}{\ln x}$
3. $\lim_{x \to \infty} \frac{x^2 + 1}{e^x}$

Solution:

1. The functions $f(x) = x^3 - 8$ and $g(x) = x - 2$ are differentiable and

$$f(2) = 2^3 - 8 = 8 - 8 = 0$$
$$g(2) = 2 - 2 = 0 \,.$$

Thus the quotient is of $\frac{0}{0}$ indeterminate form. Therefore we can apply L'Hôpital's Rule:

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} \underset{\text{LH}}{=} \lim_{x \to 2} \frac{3x^2}{1} = 3(2)^2 = 12$$

2. The function $f(x) = e^{2x} - e^2$ and $g(x) = \ln x$ are differentiable and

$$f(1) = e^2 - e^2 = 0$$
$$g(1) = \ln 1 = 0 \,.$$

Thus the quotient is of $\frac{0}{0}$ indeterminate form. Therefore, we can apply L'Hôpital's Rule:

$$\lim_{x \to 1} \frac{e^{2x} - e^2}{\ln x} \underset{\text{LH}}{=} \lim_{x \to 1} \frac{2e^{2x}}{\frac{1}{x}} = \lim_{x \to 1} 2xe^{2x} = 2(1)e^{2(1)} = 2e^2$$

3. The function $f(x) = x^2 + 1$ and $g(x) = e^x$ are differentiable and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(x^2+1\right) = \infty$$
$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \left(e^x\right) = \infty$$

Thus the quotient is of $\frac{\infty}{\infty}$ indeterminate form. Therefore we can apply L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{x^2+1}{e^x} \underset{\text{LH}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \underset{\text{LH}}{=} \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$$

Here we applied L'Hôpital's Rule a second time because $\dfrac{2x}{e^x}$ was also of $\frac{\infty}{\infty}$ determinate form.

Further Questions:

Evaluate the following limits:

1. $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x}$
2. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$
3. $\displaystyle\lim_{x\to 0} \frac{\cos x + 2x - 1}{3x}$
4. $\displaystyle\lim_{x\to 2} \frac{x^2 + 3x + 5}{x^2 - 4}$
5. $\displaystyle\lim_{x\to \infty} \frac{\ln x}{\sqrt{x}}$
6. $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$
7. $\displaystyle\lim_{x\to \infty} \frac{e^x}{\ln x}$
8. $\displaystyle\lim_{x\to 0} \frac{\cos x}{x^2 - 1}$
9. $\displaystyle\lim_{x\to 5} \frac{\sqrt{x-1} - 2}{x^2 - 25}$
10. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x - \tan x}$
11. $\displaystyle\lim_{x\to 0} \frac{3^x - 1}{x}$
12. $\displaystyle\lim_{x\to 0} \frac{4e^{2x} - 4}{e^x - 1}$

2.6.1 Indeterminate Forms of type $0 \cdot \infty$ and $\infty - \infty$

Consider the following indeterminate forms:

- If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$ then $\lim_{x\to a} f(x)g(x)$ is called the **indeterminate form of type $\mathbf{0 \cdot \infty}$** .
- If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a}[f(x) - g(x)]$ is called the **indeterminate form of type $\mathbf{\infty - \infty}$** .

To solve an indeterminate form of type $0 \cdot \infty$, write the product $f \cdot g$ as either

$$f \cdot g = \frac{f}{1/g} \quad \text{or} \quad f \cdot g = \frac{g}{1/f}$$

This will convert the indeterminate form into a form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ which can then potentially evaluated using L'Hôpital's Rule.

For indeterminate forms of type $\infty - \infty$ try to convert the difference into a quotient (by using a common denominator or factoring out common terms or rationalization) to once again reduce the limit to type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 2-33

Evaluate the given limits:

1. $\lim\limits_{x \to \frac{\pi}{2}} \sec x \sin(2x)$
2. $\lim\limits_{x \to 2} \left[\dfrac{4}{x^2 - 4} - \dfrac{1}{x - 2} \right]$

Solution:

1. As $x \to \dfrac{\pi}{2}$ we have $\sec x \to \infty$, and $\sin(2x) \to 0$. Thus the product is of $0 \cdot \infty$ indeterminate form. We write it as a $\dfrac{0}{0}$ form noting that cosine is the reciprocal of secant.

$$\lim_{x \to \frac{\pi}{2}} \sec x \sin(2x) = \lim_{x \to \frac{\pi}{2}} \frac{\sin(2x)}{\cos x} \underset{\text{LH}}{=} \lim_{x \to \frac{\pi}{2}} \frac{2\cos(2x)}{-\sin x}$$

$$= -\frac{2\cos\left(2\frac{\pi}{2}\right)}{\sin\frac{\pi}{2}} = -\frac{2\cos\pi}{(1)} = -2(-1) = 2$$

2. As $x \to 2^+$ one has $\frac{4}{x^2-4} \to \infty$ and $\frac{1}{x-2} \to \infty$. As $x \to 2^-$ one has $\frac{4}{x^2-4} \to -\infty$ and $\frac{1}{x-2} \to -\infty$. Thus, the difference is of $\infty - \infty$ (or $-\infty + \infty$) indeterminate form. Convert the difference into a quotient using a common denominator.

$$\lim_{x \to 2} \left[\frac{4}{x^2 - 4} - \frac{1}{x - 2} \right] = \lim_{x \to 2} \left[\frac{4}{(x+2)(x-2)} - \frac{1}{x - 2} \right] = \lim_{x \to 2} \frac{4 - (x+2)}{x^2 - 4}$$

$$= \lim_{x \to 2} \frac{2 - x}{x^2 - 4} \quad \left(\frac{0}{0} \text{ form}\right) \underset{\text{LH}}{=} \lim_{x \to 2} \frac{-1}{2x} = \frac{-1}{2(2)} = -\frac{1}{4}$$

Further Questions:

Evaluate the following limits:

1. $\lim\limits_{x \to 0^+} x^2 \ln x$
2. $\lim\limits_{x \to \frac{\pi}{2}} (2x - \pi) \sec x$
3. $\lim\limits_{x \to 0} \left(\dfrac{1}{e^x - 1} - \dfrac{1}{x} \right)$
4. $\lim\limits_{x \to 1} \left(\dfrac{1}{x^2 - 1} - \dfrac{1}{x - 1} \right)$
5. $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + x} - x \right)$
6. $\lim\limits_{x \to \infty} (x - \ln x)$

2.6.2 Exponential Indeterminate Forms

Several indeterminate forms arise from $\lim\limits_{x\to a}[f(x)]^{g(x)}$.

- If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$ then **indeterminate form of type $\mathbf{0^0}$** .
- If $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = 0$ then **indeterminate form of type $\mathbf{\infty^0}$** .
- If $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = \pm\infty$ then **indeterminate form of type $\mathbf{1^\infty}$** .

Each of these can be evaluated either by taking the natural logarithm:

$$y = [f(x)]^{g(x)} \;\Rightarrow\; \ln y = g(x)\ln[f(x)] \;,$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x)\ln[f(x)]} \,.$$

In either case an indeterminate form of type $0\cdot\infty$ will result. This resulting indeterminacy is the reason these three exponential forms are themselves indeterminate. We will use the former approach, as we did with logarithmic differentiation, to evaluate these limits.

Example 2-34

As a practical example prove our limit formula for e by evaluating $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}$, a limit of indeterminate form 1^∞.

Let $y = (1+x)^{\frac{1}{x}}$. Taking the natural logarithm of both sides results in

$$\ln y = \frac{1}{x}\ln(1+x)$$

Taking the limit as $x \to 0$ of the righthand side gives an indeterminate form of type $\frac{0}{0}$ readily evaluated using L'Hôpital's Rule:

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = \lim_{x\to 0}\frac{1}{1+x} = \frac{1}{1+0} = 1$$

So

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{\left(\lim\limits_{x\to 0}\ln y\right)} = e^1 = e$$

This was the limit stated for e given before.

Example 2-35

Evaluate the following limits:

1. $\lim\limits_{x\to 0}(\sin x)^x$
2. $\lim\limits_{x\to\infty}(\ln x)^{\frac{1}{x}}$
3. $\lim\limits_{x\to 1}(2x-1)^{\frac{1}{\ln x}}$

Solution:

1. As $x \to 0, (\sin x)^x \to 0^0$ indeterminate form.
   Let $y = (\sin x)^x \implies \ln y = \ln\left[(\sin x)^x\right] = x\ln(\sin x)$

$$\begin{aligned}\text{Then } \lim_{x\to 0}\ln y &= \lim_{x\to 0} x\ln(\sin x) \quad (0\cdot\infty \text{ form})\\ &= \lim_{x\to 0}\frac{\ln(\sin x)}{x^{-1}} \quad \left(\frac{\infty}{\infty}\text{ form}\right)\\ &\underset{\text{LH}}{=} \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = \lim_{x\to 0}\frac{-x^2\cos x}{\sin x} \quad \left(\frac{0}{0}\text{ form}\right)\\ &\underset{\text{LH}}{=} \lim_{x\to 0}\frac{-2x\cos x + x^2\sin x}{\cos x} = \frac{0}{1} = 0\end{aligned}$$

   Finally exponentiate the result $\lim_{x\to 0}\ln y = 0$ to get the desired limit:

$$\lim_{x\to 0}(\sin x)^x = \lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{\lim_{x\to 0}\ln y} = e^0 = 1$$

2. As $x \to \infty, (\ln x)^{\frac{1}{x}} \to \infty^0$ indeterminate form.
   Let $y = (\ln x)^{\frac{1}{x}} \implies \ln y = \ln\left[(\ln x)^{\frac{1}{x}}\right] = \frac{1}{x}\ln(\ln x) = \frac{\ln(\ln x)}{x}$

$$\begin{aligned}\text{Therefore } \lim_{x\to\infty}\ln y = \lim_{x\to\infty} &= \frac{\ln(\ln x)}{x} \quad \left(\frac{\infty}{\infty}\text{ form}\right)\\ &\underset{\text{LH}}{=} \lim_{x\to\infty}\frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x\ln x} = \frac{1}{\infty} = 0\end{aligned}$$

   Exponentiating we have that $\lim_{x\to\infty}\ln y = 0$ implies:

$$\lim_{x\to\infty}(\ln x)^{\frac{1}{x}} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty}\ln y} = e^0 = 1$$

3. As $x \to 1, (2x-1)^{\frac{1}{\ln x}} \to 1^\infty$ indeterminate form.
   Let $y = (2x-1)^{\frac{1}{\ln x}} \implies \ln y = \ln\left[(2x-1)^{\frac{1}{\ln x}}\right] = \frac{\ln(2x-1)}{\ln x}$

$$\begin{aligned}\text{Therefore } \lim_{x\to 1}\ln y &= \lim_{x\to 1}\frac{\ln(2x-1)}{\ln x} \quad \left(\frac{0}{0}\text{ form}\right)\\ &\underset{\text{LH}}{=} \lim_{x\to 1}\frac{\frac{2}{2x-1}}{\frac{1}{x}} = \lim_{x\to 1}\frac{2x}{2x-1}\\ &= \frac{2(1)}{2(1)-1} = \frac{2}{2-1} = \frac{2}{1} = 2\end{aligned}$$

   Then $\lim_{x\to 1}\ln y = 2$ implies:

$$\lim_{x\to 1}(2x-1)^{\frac{1}{\ln x}} = \lim_{x\to 1} y = \lim_{x\to 1} e^{\ln y} = e^{\lim_{x\to 1}\ln y} = e^2$$

Further Questions:

Evaluate the following limits:

1. $\lim_{x\to\infty}(1+e^x)^{e^{-x}}$
2. $\lim_{x\to 0^+}(e^x-1)^x$
3. $\lim_{x\to\infty}\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x$
4. $\lim_{x\to 1^-}(1-x)^{\ln x}$
5. $\lim_{x\to\frac{\pi}{2}^-}(\tan x)^{\cos x}$

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Exercise 2-6

1-16: Evaluate the given limit if it exists. If it does not exist but has an infinite trend ($+\infty$ or $-\infty$) indicate the trend.

1. $\lim_{x\to 2}\frac{2x^2-x-6}{x^2+x-6}$
2. $\lim_{x\to 0}\frac{\sin 3x}{5x}$
3. $\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{2\cos x}$
4. $\lim_{x\to 0}\frac{\tan^{-1}x}{5x}$
5. $\lim_{x\to\infty}\frac{\ln x}{x^2}$
6. $\lim_{x\to\infty}\frac{3e^x+\ln x}{e^x+x}$
7. $\lim_{x\to\infty}(x-2)e^{-x^3}$
8. $\lim_{x\to\infty}e^{-2x}\ln x$
9. $\lim_{x\to 0}\left(\frac{2}{e^x-1}-\frac{3}{x}\right)$
10. $\lim_{x\to-\infty}x\left(\frac{\pi}{2}+\tan^{-1}x\right)$
11. $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{x^2-1}\right)$
12. $\lim_{x\to\infty}[\ln x-\ln(2x+3)]$
13. $\lim_{x\to\infty}\left(1-\frac{5}{x}\right)^{2x}$
14. $\lim_{x\to 0}(e^{2x}-1)^{3x}$
15. $\lim_{x\to\infty}(1+\ln x)^{e^{-x}}$
16. $\lim_{x\to 0}(\sin 2x)^x$

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Chapter 2 Review Exercises

1. For the function $f(x) = 3 + \dfrac{1}{x^3}$:
   (a) Show the function is one-to-one.
   (b) Find the inverse of the function, $g(x) = f^{-1}(x)$.
   (c) Calculate the derivative $g'(11)$.

2-13: Find the indicated derivative.

2. $f(x) = \dfrac{2\ln x + 5}{e^{3x} + 4}, f'(x)$
3. $f(t) = \ln\left(t^4 + e^{2t} + 1\right), f'(t)$
4. $g(x) = \ln \sqrt[3]{\dfrac{x+1}{2x+4}}, g'(x)$
5. $F(y) = \sqrt{e^{4y} + e^{-4y}} + 2e^y, F'(0)$
6. $f(x) = x^2 e^{-x^2}, f''(x)$
7. $y = (2x+3)^{4x}, y'$
8. $f(t) = 10^{e^t}, f'(t)$
9. $y = (\ln x)^{\cos x}, y'$
10. $\ln x + \ln y = e^{xy}, y'$
11. $f(x) = \tan^{-1}\left(e^x + \ln x\right), f'(x)$
12. $g(t) = \sin^{-1}(\tan t + 3), g'(t)$
13. $h(x) = \log\left(\cos^{-1} x + 1\right), h'(x)$

14-23: Evaluate the given integral.

14. $\displaystyle\int \frac{e^{4\sqrt{x}}}{\sqrt{x}}\, dx$
15. $\displaystyle\int \frac{4x^2}{2x+1}\, dx$
16. $\displaystyle\int 4^x e^{2x}\, dx$
17. $\displaystyle\int 4^x\left(4 + \cos 4^x\right)\, dx$
18. $\displaystyle\int_0^1 \frac{e^{2x}}{1+e^{4x}}\, dx$
19. $\displaystyle\int \frac{\cos x}{\sqrt{3 - \sin^2 x}}\, dx$
20. $\displaystyle\int \frac{\tan(\ln x)}{x}\, dx$
21. $\displaystyle\int \frac{x+2}{\sqrt{16 - x^2}}\, dx$
22. $\displaystyle\int \frac{2}{3x\sqrt{x^4 - 1}}\, dx$
23. $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\, dx$

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24. Find the points on the graph of $y = \tan^{-1}(2x)$ at which the tangent line is parallel to the line $x - 2y - 8 = 0$.

25-31: Evaluate the given limit.

25. $\lim_{x\to\infty} \dfrac{x^3+4x+2}{\ln(x+2)}$

26. $\lim_{x\to 0} \dfrac{\sin^{-1}(2x)}{\tan^{-1}(2x)}$

27. $\lim_{x\to 1} (\ln x)^{x^2-1}$

28. $\lim_{x\to 0} \left(\dfrac{1}{e^x-1} - \dfrac{1}{e^x-e^{-x}} \right)$

29. $\lim_{x\to\infty} (e^x+x)^{1/x}$

30. $\lim_{x\to 1} (1+2\ln x)^{\frac{1}{x-1}}$

31. $\lim_{x\to\infty} [\ln(5x+2) - \ln(2x+5)]$

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32. A certain bacteria culture doubles in size every 5 minutes. If there are 1000 bacteria initially present, how many will there be in 30 minutes?

33. A drug used for sedation decays exponentially. It is observed that the amount decays by 25% after each hour. A patient receiving this drug should not drive until there is only 30 mg left in their system. If the initial dose is 400 mg, how long will it be until it is safe to drive?

Chapter 3: Integration Methods

3.1 Integration by Parts

Just as the Chain Rule for differentiation leads to the useful Method of Substitution for solving integrals, so too does the Product Rule result in a useful method for solving integrals. Starting with the Product Rule

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)\ ,$$

one can integrate both sides of the equation to get:

$$\int \frac{d}{dx}\left[f(x)g(x)\right] dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

An antiderivative of the derivative of a function is just the function itself so the left-hand side becomes $f(x)g(x)+C$. The constant C may be absorbed into the indefinite integrals on the right and so one has:

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx\ .$$

Reordering the terms gives

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx\ .$$

The formula suggests that a useful strategy for evaluating an integral is to consider an integrand as a product of two terms, one of which may be differentiated ($f(x)$) and one which may be integrated ($g'(x)$) to produce a new integral that is perhaps more easy to evaluate than the original. It is customary to define $u = f(x)$ and $v = g(x)$. One then has the corresponding differentials $du = f'(x)dx$ and $dv = g'(x)dx$. The formula becomes:

$$\int u\,dv = uv - \int v\,du$$

This is the **Integration by Parts** formula.

Example 3-1

Integrate $\int x^2 \ln x\,dx$.

The fact that $\ln x$ is easily differentiated and x^2 easily integrated suggests we reorder the terms and identify u and dv as follows:

$$\int \underbrace{\ln x}_{=u}\,\underbrace{x^2\,dx}_{=dv}$$

Then $u = \ln x$ implies (differentiating) that $du = \frac{1}{x}\,dx$. The differential $dv = x^2\,dx$ is integrated to give $v = \frac{1}{3}x^3$. The Integration by Parts formula $\int u\,dv = uv - \int v\,du$ implies:

$$\begin{aligned}
\int (\ln x)\cdot\left(x^2\,dx\right) &= (\ln x)\cdot\left(\frac{1}{3}x^3\right) - \int \left(\frac{1}{3}x^3\right)\cdot\left(\frac{1}{x}\,dx\right) \\
&= \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2\,dx \\
&= \frac{1}{3}x^3\ln x - \frac{1}{3}\cdot\frac{1}{3}x^3 + C \\
&= \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C
\end{aligned}$$

In general, to apply Integration by Parts select u and dv so that

1. The product $u\,dv$ is equal to the original integrand.
2. dv can be integrated.
3. The new integral $\int v\,du$ is easier than the original integral.
4. For integrals involving $x^p e^{ax}$ try $u = x^p$, $dv = e^{ax}\,dx$.
5. For integrals involving $x^p(\ln x)^q$ try $u = (\ln x)^q$, $dv = x^p\,dx$.

As with assigning u to $\ln x$ in the last suggestion, we often will assign u to an inverse function because its derivative is known. Integration by Parts can then (hopefully) convert the expression into something that can be integrated.

An integral may have several ways it can be broken into u and dv that can be tried when applying Integration by Parts. Only one of these, or indeed none of these, may actually work to allow integration of the function.

Example 3-2

Evaluate the following integrals:

1. $\displaystyle\int (x^2+x)e^{-x}dx$ 2. $\displaystyle\int e^{2x}\sin x\,dx$ 3. $\displaystyle\int_0^1 \sin^{-1} x\,dx$

Solution:

1. Let $u = x^2 + x$ so $du = (2x+1)dx$ and $dv = e^{-x}dx$. Integrating the latter (using the substitution $w = -x$ if needed) gives $v = -e^{-x}$. Integration by Parts gives:

$$\begin{aligned}\int (x^2+x)e^{-x}dx &= (x^2+x)\left(-e^{-x}\right) - \int \left(-e^{-x}\right)(2x+1)dx \\ &= -(x^2+x)e^{-x} + \int (2x+1)e^{-x}dx\end{aligned}$$

Use Integration by Parts again. Let $u = (2x+1)$ so $du = 2dx$ and $dv = e^{-x}$ so $v = -e^{-x}$ again to get:

$$\begin{aligned}\int (x^2+x)e^{-x}dx &= -(x^2+x)e^{-x} + (2x+1)\left(-e^{-x}\right) - \int \left(-e^{-x}\right)2\,dx \\ &= -(x^2+x)e^{-x} - (2x+1)e^{-x} + 2\int e^{-x}dx \\ &= -(x^2+x)e^{-x} - (2x+1)e^{-x} - 2e^{-x} + C \\ &= (-x^2-3x-3)e^{-x} + C\end{aligned}$$

2. Let $u = e^{2x}$ so $du = 2e^{2x}dx$ and $dv = \sin x\,dx$ so (integrating) $v = -\cos x$. Using Integration by Parts gives:

$$\int e^{2x}\sin x\,dx = e^{2x}(-\cos x) - \int(-\cos x)e^{2x}\,dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx$$

Use Integration by Parts on the final integral with $u = e^{2x}$ so $du = 2e^{2x}$ and $dv = \cos x\,dx$ so $v = \sin x$ to get

$$\begin{aligned}\int e^{2x} \sin x\,dx &= -e^{2x}\cos x + 2\left[e^{2x}\sin x - \int (\sin x)\left(2e^{2x}\right)\,dx\right]\\ &= -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx\end{aligned}$$

This is a wrap-around integral in that we have produced the same integral with which we have started. Solve for the integral as we would any variable:

$$\begin{aligned}\int e^{2x}\sin x\,dx + 4\int e^{2x}\sin x\,dx &= -e^{2x}\cos x + 2e^{2x}\sin x\\ \implies 5\int e^{2x}\sin x\,dx &= -e^{2x}\cos x + 2e^{2x}\sin x\\ \implies \int e^{2x}\sin x\,dx &= -\frac{1}{5}e^{2x}\cos x + \frac{2}{5}e^{2x}\sin x + C\end{aligned}$$

Here a constant of integration must be included, as with any indefinite integral, to provide the most general solution.

3. Let $u = \sin^{-1} x$ so $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$ and $dv = dx$ so $v = x$:

$$\int_0^1 \sin^{-1} x\,dx = x\sin^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

Note for Integration by Parts how the limits carry over to the new expression. They do not change as they would for a substitution. To evaluate the integral on the right-hand side we do however require a substitution. Let $w = 1 - x^2$ so $dw = -2x\,dx \implies \frac{1}{2}\,dw = -x\,dx$. The new limits are $x = 1 \implies w = 1 - 1^2 = 0$ and $x = 0 \implies w = 1 - 0^2 = 1$ and we have:

$$\begin{aligned}\int \sin^{-1}x\,dx &= x\sin^{-1}x\Big|_0^1 + \int_1^0 \frac{1}{\sqrt{w}}\frac{dw}{2}\\ &= x\sin^{-1}x\Big|_0^1 + \sqrt{w}\Big|_1^0\\ &= (1)\sin^{-1}(1) - (0)\sin^{-1}(0) + \sqrt{0} - \sqrt{1} = \frac{\pi}{2} - 1\end{aligned}$$

Alternatively, to use the same limits all the way through, evaluate the indefinite integral on the side to get the antiderivative:

$$-\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{w}}\frac{dw}{2} = \sqrt{w} + C = \sqrt{1-x^2} + C$$

The original integral becomes:

$$\begin{aligned}\int_0^1 \sin^{-1}x\,dx &= x\sin^{-1}x\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \left[x\sin^{-1}x + \sqrt{1-x^2}\right|_0^1\\ &= \left[(1)\sin^{-1}(1) + \sqrt{1-1^2}\right] - \left[(0)\sin^{-1}(0) - \sqrt{1-0^2}\right] = \frac{\pi}{2} - 1\end{aligned}$$

as before.

Further Questions:

Evaluate the following integrals:

1. $\displaystyle\int \ln x\, dx$
2. $\displaystyle\int xe^x\, dx$
3. $\displaystyle\int x^2e^{-x}\, dx$
4. $\displaystyle\int e^x \cos x\, dx$
5. $\displaystyle\int_0^1 \tan^{-1} x\, dx$
6. $\displaystyle\int x^3(\ln x)^2\, dx$
7. $\displaystyle\int \sin(\ln x)\, dx$
8. $\displaystyle\int \theta \sec^2 \theta\, d\theta$
9. $\displaystyle\int x^5e^{-x^3}\, dx$
10. $\displaystyle\int x \sin(x^2)\, dx$
11. $\displaystyle\int \cos^2 x\, dx$

Exercise 3-1

Answers: Page 263

1-10: Evaluate the given integral.

1. $\displaystyle\int x^2\, e^{-5x}\, dx$
2. $\displaystyle\int x^2\, \cos^{-1} x\, dx$
3. $\displaystyle\int \sqrt{t}\, e^{2\sqrt{t}}\, dt$
4. $\displaystyle\int x^{10} \ln x\, dx$
5. $\displaystyle\int x^5\, \sin x^3\, dx$
6. $\displaystyle\int e^{2x}\, \sin 4x\, dx$
7. $\displaystyle\int_{\pi/4}^{\pi/3} x\, \sec^2 x\, dx$
8. $\displaystyle\int \sin(3 \ln x)\, dx$
9. $\displaystyle\int x^2\, 5^x\, dx$
10. $\displaystyle\int_0^{\sqrt{3}} \tan^{-1} x\, dx$

3.2 Trigonometric Integrals

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

1. For an odd power of sine ($m = 2k+1$), save one sine factor and express the remaining sine factors in terms of cosine using the identity $\sin^2 x = 1 - \cos^2 x$:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int \left(\sin^2 x\right)^k \cos^n x \sin x \, dx = \int \left(1 - \cos^2 x\right)^k \cos^n x \sin x \, dx$$

Then substitute $u = \cos x$.

2. For an odd power of cosine ($n = 2k+1$), save one cosine factor and express the remaining cosine factors in terms of sine using the identity $\cos^2 x = 1 - \sin^2 x$:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \left(\cos^2 x\right)^k \cos x \, dx = \int \sin^m x \left(1 - \sin^2 x\right)^k \cos x \, dx$$

Then substitute $u = \sin x$.

3. If the powers of both sine and cosine are even, use the trigonometric identities:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

These may need to be used repeatedly. The identity $\sin x \cos x = \frac{1}{2} \sin 2x$ may also be useful.

Either 1 or 2 can be used if the powers of sine and cosine are both odd.

Example 3-3

Evaluate the following integrals:

1. $\displaystyle\int \sin^3 x \cos^6 x \, dx$

2. $\displaystyle\int \tan^2 x \cos^5 x \, dx$

Solution:

1. Take one factor of $\sin x$ out of the odd power of sine to become part of the differential and write the remaining powers in terms of cosine:

$$\int \sin^3 x \cos^6 x \, dx = \int \sin^2 x \cos^6 x \sin x \, dx = \int (1 - \cos^2 x) \cos^6 x \sin x \, dx$$

Then substitute $u = \cos x$ so $du = -\sin x \, dx \implies -du = \sin x \, dx$. The integral becomes:

$$\begin{aligned} &= \int (1 - u^2) u^6 (-du) = \int (u^8 - u^6) du \\ &= \frac{1}{9} u^9 - \frac{1}{7} u^7 + C = \frac{1}{9} \cos^9 x - \frac{1}{7} \cos^7 x + C \end{aligned}$$

2. All trigonometric expressions can be written in terms of sine and cosine. We try that here.

$$\int \tan^2 x \cos^5 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} \cos^5 x \, dx = \int \sin^2 x \cos^3 x \, dx$$

Next take out $\cos x$ from the odd power of cosine to be part of the differential:

$$= \int \sin^2 x \cos^2 x \cos x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$$

Finally let $u = \sin x$ so $du = \cos x \, dx$. The integral becomes

$$= \int u^2 (1 - u^2) \, du = \int (u^2 - u^4) \, du = \frac{1}{3} u^3 - \frac{1}{5} u^5 + C = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$$

Further Questions:

Evaluate the following integrals:

1. $\int \sin^4 x \cos^3 x \, dx$
2. $\int \sin^3 x \, dx$
3. $\int \cot^5 x \sin^2 x \, dx$
4. $\int \sin^2 x \, dx$
5. $\int \sin^4 x \, dx$
6. $\int \cos^2 x \sin^2 x \, dx$

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

1. For an odd power of tangent ($m = 2k + 1$), save a factor of $\sec x \tan x$ and express the remaining factors of tangent in terms of $\sec x$ using the identity $\tan^2 x = \sec^2 x - 1$:

   $$\int \tan^{2k+1} x \sec^n x \, dx = \int \left(\tan^2 x\right)^k \sec^{n-1} x \sec x \tan x \, dx = \int \left(\sec^2 x - 1\right)^k \sec^{n-1} x \sec x \tan x \, dx$$

   Then substitute $u = \sec x$.

2. For an even power of secant ($n = 2k$), save a factor of $\sec^2 x$ and express the remaining secant factors in terms of $\tan x$ using the identity $\sec^2 x = 1 + \tan^2 x$:

   $$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \left(\sec^2 x\right)^{k-1} \sec^2 x \, dx = \int \tan^m x \left(1 + \tan^2 x\right)^{k-1} \sec^2 x \, dx$$

   Then substitute $u = \tan x$.

3. If m is even and $n = 0$ (i.e. no factors of secant), convert a single factor of $\tan^2 x$ using $\tan^2 x = \sec^2 x - 1$. The first term will then be integrable and the procedure may be repeated on the second integral of now lower power.

Strategy for Evaluating $\int \cot^m x \csc^n x \, dx$

1. For an odd power of cotangent ($m = 2k + 1$), save a factor of $\csc x \cot x$ and express the remaining factors of cotangent in terms of $\csc x$ using the identity $\cot^2 x = \csc^2 x - 1$:

   $$\int \cot^{2k+1} x \csc^n x \, dx = \int \left(\cot^2 x\right)^k \csc^{n-1} x \csc x \cot x \, dx = \int \left(\csc^2 x - 1\right)^k \csc^{n-1} x \csc x \cot x \, dx$$

   Then substitute $u = \csc x$.

2. For an even power of cosecant ($n = 2k$), save a factor of $\csc^2 x$ and express the remaining factors of cosecant in terms of $\cot x$ using the identity $\csc^2 x = 1 + \cot^2 x$:

$$\int \cot^m x \csc^{2k} x\, dx = \int \cot^m x \left(\csc^2 x\right)^{k-1} \csc^2 x\, dx = \int \cot^m x \left(1 + \cot^2 x\right)^{k-1} \csc^2 x\, dx$$

   Then substitute $u = \cot x$.

3. If m is even and $n = 0$ (i.e. no factors of cosecant), convert a single factor of $\cot^2 x$ using $\cot^2 x = \csc^2 x - 1$. The first term will then be integrable and the procedure may be repeated on the second integral of now lower power.

Note: This strategy is identical for that of tangents and secants with the identification $\tan \Rightarrow \cot$ and $\sec \Rightarrow \csc$.

Example 3-4

Evaluate the following integrals:

1. $\displaystyle\int \tan^5 x \sec^4 x\, dx$
2. $\displaystyle\int \cot^4 x \csc^4 x\, dx$
3. $\displaystyle\int_0^{\pi/4} \tan x \sec^5 x\, dx$

Solution:

1. Imagining a tangent substitution we pull out a $\sec^2 x$ to be part of the differential. The remaining even power of secant can be converted to tangent as needed.

$$\int \tan^5 x \sec^4 x\, dx = \int \tan^5 x \sec^2 x \sec^2 x\, dx = \int \tan^5 x (1 + \tan^2 x) \sec^2 x\, dx$$

$$\text{Let } u = \tan x \text{ so } du = \sec^2 x :$$

$$= \int u^5 (1 + u^2)\, du = \int (u^5 + u^7)\, du$$

$$= \frac{1}{6} u^6 + \frac{1}{8} u^8 + C = \frac{1}{6} \tan^6 x + \frac{1}{8} \tan^8 x + C$$

   It is to be noted here that a secant substitution will also work here as pulling out $\sec x \tan x$ to be part of the differential leaves an even power of tangent which can be converted to secant. The final answer, in terms of secant, will differ at most by a constant from the answer above.

2. Envisioning a cotangent substitution we try pulling out $\csc^2 x$ to be part of the differential. The remaining even power of cosecant can be converted to cotangent:

$$\int \cot^4 x \csc^4 x\, dx = \int \cot^4 x \csc^2 x \csc^2 x\, dx = \int \cot^4 x (1 + \cot^2 x) \csc^2 x\, dx$$

$$\text{Let } u = \cot x \text{ so } du = -\csc^2 x\, dx:$$

$$= \int u^4 (1 + u^2)(-du) = \int (-u^4 - u^6)\, du$$

$$= -\frac{1}{5} u^5 - \frac{1}{7} u^7 + C = -\frac{1}{5} \cot^5 x - \frac{1}{7} \cot^7 x + C$$

   Note that removing $\csc x \cot x$ for the differential for a $\csc x$ substitution will not work here as the remaining power of cotangent would be odd.

3. Anticipating a secant substitution we separate $\sec x \tan x$ for the differential:

$$\int_0^{\pi/4} \tan x \sec^5 x\, dx = \int_0^{\pi/4} \sec^4 x \sec x \tan x\, dx$$

Let $u = \sec x$ so $du = \sec x \tan x\, dx$. The limits become:

$$x = \pi/4 \implies u = \sec(\pi/4) = \frac{1}{\cos(\pi/4)} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$$

$$x = 0 \implies u = \sec(0) = \frac{1}{\cos(0)} = \frac{1}{1} = 1$$

$$= \int_1^{\sqrt{2}} u^4\, du = \frac{1}{5}u^5\Big|_1^{\sqrt{2}} = \frac{1}{5}\left[(\sqrt{2})^5 - 1^5\right] = \frac{4\sqrt{2}-1}{5}$$

Further Questions:

Evaluate the following integrals:

1. $\int \tan^3 x \sec^3 x\, dx$
2. $\int \tan^2 x \sec^4 x\, dx$
3. $\int \tan^3 x\, dx$
4. $\int \sec x\, dx$
5. $\int \sec^3 x\, dx$
6. $\int \tan^4 x\, dx$
7. $\int \cot^3 x \csc^4 x\, dx$
8. $\int \cot^3 x \csc^3 x\, dx$
9. $\int \csc x\, dx$
10. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc^4 x\, dx$

Strategy for Evaluating $\int \sin mx \cos nx\, dx$, $\int \sin mx \sin nx\, dx$, $\int \cos mx \cos nx\, dx$

Apply the corresponding trigonometric identity:

- $\sin a \cos b = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$
- $\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$
- $\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$

with $a = mx$ and $b = nx$.

Example 3-5

Evaluate the following integrals:

1. $\int \sin 3x \cos 4x\, dx$
2. $\int_0^{\frac{\pi}{2}} \sin 2x \sin 3x\, dx$

Solution:

1. Letting $a = 3x$ and $b = 4x$ in the appropriate trigonometric identity gives:

$$\begin{aligned}\int \sin 3x \cos 4x \, dx &= \frac{1}{2}\int [\sin(3x-4x)+\sin(3x+4x)] \; dx \\ &= \frac{1}{2}\int [\sin(-x)+\sin(7x)] \; dx = \frac{1}{2}\int (-\sin x+\sin 7x) \; dx \\ &= \frac{1}{2}\left(\cos x - \frac{1}{7}\cos 7x\right) + C\end{aligned}$$

   Note here we used that sine is an odd function to write $\sin(-x) = -\sin x$ before integrating. We then integrated term by term doing the substitution $u = 7x$ in the second integral.

2. Letting $a = 2x$ and $b = 3x$ in the appropriate trigonometric identity gives:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin 2x \sin 3x \, dx &= \frac{1}{2}\int_0^{\frac{\pi}{2}} [\cos(2x-3x)-\cos(2x+3x)] \; dx \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} [\cos(-x)-\cos(5x)] \; dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos x-\cos 5x) \; dx \\ &= \frac{1}{2}\left[\sin x - \frac{1}{5}\sin 5x\right|_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\sin\frac{\pi}{2} - \frac{1}{5}\sin\frac{5\pi}{2}\right] - \frac{1}{2}\left[\sin 0 - \frac{1}{5}\sin 0\right] \\ &= \frac{1}{2}\left[1-\frac{1}{5}(1)\right] = \frac{1}{2}\cdot\frac{5-1}{5} = \frac{1}{2}\cdot\frac{4}{5} = \frac{2}{5}\end{aligned}$$

Further Questions:

Evaluate the following integrals:

1. $\displaystyle\int \sin 4x \cos 5x \, dx$
2. $\displaystyle\int \sin 2x \sin 6x \, dx$
3. $\displaystyle\int_0^{\frac{\pi}{4}} \cos 2x \cos 4x \, dx$
4. $\displaystyle\int \sin 2x \sin 6x \cos 2x \, dx$

A Note on Trigonometric Identities

Note that the various trigonometric identities require in this section follow readily from the three basic identities:

a) $\sin^2 x + \cos^2 x = 1$
b) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
c) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

The half-angle identities follow from c) setting $y = x$ and then replacing alternately $\sin^2 x$ or $\cos^2 x$ using a). Dividing a) by $\cos^2 x$ gives the identity involving tangent and secant, while dividing a) by $\sin^2 x$ gives the identity involving cotangent and cosecant. The last three identities on this page follow by solving for the various products using the $+$ and $-$ equations from the appropriate angle addition formula b) or c).

Exercise 3-2

Answers: Page 264

1-10: Evaluate the given integral.

1. $\displaystyle\int \cos^{10} x \, \sin^5 x \, dx$
2. $\displaystyle\int \sin^4 x \, \cos^3 x \, dx$
3. $\displaystyle\int \sin^4 x \, dx$
4. $\displaystyle\int \tan^4 x \, \sec^4 x \, dx$
5. $\displaystyle\int \tan^5 x \, \sec^5 x \, dx$
6. $\displaystyle\int \cot^2 x \, \csc^4 x \, dx$
7. $\displaystyle\int \tan^4 x \, dx$
8. $\displaystyle\int \sin 9x \, \cos 7x \, dx$
9. $\displaystyle\int \sin 3x \, \sin 5x \, dx$
10. $\displaystyle\int \cos 4x \, \cos 5x \, dx$

3.3 Trigonometric Substitution

Some integrals, typically involving roots, may be resolved by using the Substitution Method where the old variable is defined in terms of a new variable via a trigonometric function.

Example 3-6

Find the indefinite integral $\int \sqrt{4-x^2}\,dx$. We consider the substitution $\theta(x)$ defined via

$$x = 2\sin\theta$$

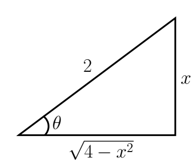
(and so $dx = 2\cos\theta\,d\theta$). Unlike our usual application of the substitution method here we have defined θ implicitly. To make $\theta(x)$ unique as required we add the additional constraint $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. (Equivalently we recognize that the explicit substitution which has been done is just $\theta = \sin^{-1}\left(\frac{x}{2}\right)$ which, recall, is defined with this range.) The integral becomes

$$\begin{aligned}
\int \sqrt{4-x^2}\,dx &= \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta \\
&= \int \sqrt{4}\sqrt{1-\sin^2\theta}\cdot 2\cos\theta\,d\theta \\
&= 4\int \cos\theta\cos\theta\,d\theta = 4\int \cos^2\theta\,d\theta \\
&= 4\int \frac{1}{2}\left(1+\cos 2\theta\right)\,d\theta = 2\int d\theta + 2\int \cos 2\theta\,d\theta \\
&= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C \\
&= 2\sin^{-1}\left(\frac{x}{2}\right) + 2\left(\frac{x}{2}\right)\frac{1}{2}\sqrt{4-x^2} + C \\
&= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2} + C
\end{aligned}$$

Note that when we solved the identity $1-\sin^2\theta = \cos^2\theta$ for $\cos\theta$ we used that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ to get $\cos\theta = \sqrt{1-\sin^2\theta}$ since $\cos\theta$ is indeed positive on the interval. This choice of positive sign was also used in our final step where again $\cos\theta$ was represented by a positive value:

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2/4} = \sqrt{(4-x^2)/4} = \sqrt{4-x^2}\sqrt{1/4} = \frac{1}{2}\sqrt{4-x^2}$$

The restriction of the domain for θ here means that other trigonometric functions appearing in the expression may always be evaluated using an acute triangle. Here, for example, since $\sin\theta = x/2$, we can draw a right triangle with angle θ and length x opposite and hypotenuse of 2 to work out $\cos\theta = \frac{\sqrt{4-x^2}}{2}$.



This method is called **Trigonometric Substitution**. More generally if an integrand contains one of $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$ (where $a > 0$ is constant) then the radical sign can be removed via the appropriate substitution:

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta \quad \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta \quad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta \quad \left(0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2}\right)$	$\sec^2\theta - 1 = \tan^2\theta$

Note that all the ranges of θ have been chosen so that θ will equal the relevant inverse trigonometric function with argument x/a.

Example 3-7

Prove the Archimedian result that the area of a circle of radius R is $A = \pi R^2$.

The area of a semi-circle of radius R is the area under the curve $y = \sqrt{R^2 - x^2}$ between $x = -R$ and $x = R$ and so the area of a circle is

$$A = 2\int_{-R}^{R} \sqrt{R^2 - x^2}\,dx$$

Using substitution $x = R\sin\theta$, with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ gives $dx = R\cos\theta\,d\theta$. So $\theta = \sin^{-1}(x/R)$ and the limits become, for $x = R$, $\theta = \sin^{-1}(R/R) = \sin^{-1}(1) = \pi/2$ and for $x = -R$, $\theta = \sin^{-1}(-R/R) = \sin^{-1}(-1) = -\pi/2$. The solution of the integral follows, similar to the last example,

$$\begin{aligned}
A = 2\int_{-R}^{R} \sqrt{R^2 - x^2}\,dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{R^2 - R^2\sin^2\theta}\cdot R\cos\theta\,d\theta \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{R^2}\sqrt{1 - \sin^2\theta}\cdot R\cos\theta\,d\theta \\
&= 2R^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\cos\theta\,d\theta = 2R^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta \\
&= 2R^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\,d\theta = R^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta \\
&= R^2\left[\theta + \frac{1}{2}\sin 2\theta\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= R^2\left\{\left[\frac{\pi}{2} + \frac{1}{2}\sin\pi\right] - \left[-\frac{\pi}{2} + \frac{1}{2}\sin(-\pi)\right]\right\} \\
&= \pi R^2
\end{aligned}$$

Example 3-8

Evaluate the following integrals:

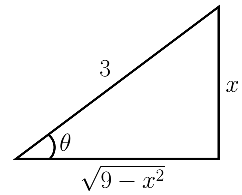
1. $\displaystyle\int \frac{1}{(9-x^2)^{\frac{3}{2}}}\,dx$
2. $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx$
3. $\displaystyle\int \frac{1}{\sqrt{x^2-4x-1}}\,dx$

Solution:

1. Recognizing the form $\sqrt{a^2 - x^2}$, let $x = 3\sin\theta$ so $dx = 3\cos\theta\,d\theta$.

$$\begin{aligned}
\int \frac{1}{(9-x^2)^{\frac{3}{2}}}\,dx = \int \frac{1}{(9-9\sin^2\theta)^{\frac{3}{2}}}3\cos\theta\,d\theta &= \int \frac{3\cos\theta}{\left[9(1-\sin^2\theta)\right]^{\frac{3}{2}}}\,d\theta \\
= \int \frac{3\cos\theta}{\left[9\cos^2\theta\right]^{\frac{3}{2}}}\,d\theta = \int \frac{3\cos\theta}{27\cos^3\theta}\,d\theta &= \frac{1}{9}\int \frac{1}{\cos^2\theta}\,d\theta \\
= \frac{1}{9}\int \sec^2\theta\,d\theta = \frac{1}{9}\tan\theta + C &= \frac{1}{9}\frac{x}{\sqrt{9-x^2}} + C
\end{aligned}$$

In the final step $\tan\theta$ is returned to x by noting that, by the substitution $\sin\theta = \frac{x}{3}$, so a triangle can be drawn with opposite side length of x and hypotenuse length of 3. The remaining side is solved for using the Pythagorean Theorem and then $\tan\theta$ evaluated as the opposite side length over the adjacent. Any other trigonometric function of θ may be evaluated in this manner. If θ is required note that $\theta = \sin^{-1}\left(\frac{x}{3}\right)$.



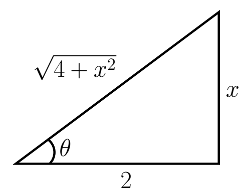
2. Recognizing the form $\sqrt{a^2+x^2}$, let $x = 2\tan\theta$ so $dx = 2\sec^2\theta\, d\theta$.

$$\begin{aligned}\int \frac{x^3}{\sqrt{4+x^2}}\,dx &= \frac{8\tan^3\theta}{\sqrt{4+4\tan^2\theta}}2\sec^2\theta\,d\theta = \int \frac{16\tan^3\theta\sec^2\theta}{\sqrt{4(1+\tan^2\theta)}}\,d\theta \\ &= \int \frac{16\tan^3\theta\sec^2\theta}{\sqrt{4\sec^2\theta}}\,d\theta = \int \frac{16\tan^3\theta\sec^2\theta}{\sqrt{4\sec^2\theta}}\,d\theta = \int \frac{16\tan^3\theta\sec^2\theta}{2\sec\theta}\,d\theta \\ &= 8\int \tan^3\theta\sec\theta\,d\theta = 8\int \tan^2\theta\sec\theta\tan\theta\,d\theta = 8\int(\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta\end{aligned}$$

In the last step we recognized a trigonometric integral and reorganized it anticipating the substitution $u = \sec\theta$ so $du = \sec\theta\tan\theta\,d\theta$. The integral becomes:

$$\begin{aligned}&= 8\int(u^2-1)\,du = 8\left[\frac{1}{3}u^3 - u\right] + C = \frac{8}{3}\sec^3\theta - 8\sec\theta + C \\ &= \frac{8}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^3 - 8\frac{\sqrt{4+x^2}}{2} + C = \frac{1}{3}(4+x^2)^{\frac{3}{2}} - 4\sqrt{4+x^2} + C\end{aligned}$$

Here $\sec\theta$ is returned to x by noting that, by the substitution $\tan\theta = \frac{x}{2}$, so a triangle can be drawn with opposite side of length x and adjacent side of length 2. The remaining side is solved for using the Pythagorean Theorem and then $\sec\theta = 1/\cos\theta$ is evaluated as the hypotenuse side length over that of the adjacent.



3. For this integral we must first remove the linear (x) term by *completing the square*. The polynomial $x^2 - 4x - 1$ has $b = -4$ so dividing that by 2, adding it to x, and squaring gives:

$$(x + (-4/2))^2 = (x-2)^2 = x^2 - 4x + 4 \implies x^2 - 4x = (x-2)^2 - 4 \implies x^2 - 4x - 1 = (x-2)^2 - 5\,.$$

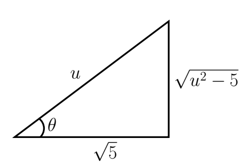
The integral becomes

$$\int \frac{1}{\sqrt{x^2-4x-1}}\,dx = \int \frac{1}{\sqrt{(x-2)^2-5}}\,dx = \int \frac{1}{\sqrt{u^2-5}}\,du$$

where here we did the substitution $u = x - 2$, so $du = dx$, to get the integral into a form ready for a trigonometric substitution. Recognizing the form $\sqrt{u^2-a^2}$. Let $u = \sqrt{5}\sec\theta$ so $du = \sqrt{5}\sec\theta\tan\theta\,d\theta$. The integral becomes:

$$\begin{aligned}&= \int \frac{1}{\sqrt{5\sec^2\theta - 5}}\sqrt{5}\sec\theta\tan\theta\,d\theta = \int \frac{1}{\sqrt{5(\sec^2\theta-1)}}\sqrt{5}\sec\theta\tan\theta\,d\theta \\ &= \int \frac{\sqrt{5}\sec\theta\tan\theta}{\sqrt{5\tan^2\theta}}\,d\theta = \int \frac{\sqrt{5}\sec\theta\tan\theta}{\sqrt{5}\tan\theta}\,d\theta = \int \sec\theta\,d\theta \quad (\leftarrow \text{ Use the known result.}) \\ &= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{u}{\sqrt{5}} + \frac{\sqrt{u^2-5}}{\sqrt{5}}\right| + C = \ln\left|\frac{x-2}{\sqrt{5}} + \frac{\sqrt{x^2-4x-1}}{\sqrt{5}}\right| + C\end{aligned}$$

To return to u note that the substitution implies $\sec\theta = \frac{u}{\sqrt{5}}$. To find $\tan\theta$ a triangle can be drawn, since secant is the reciprocal of cosine, with adjacent side of length $\sqrt{5}$ and hypotenuse of length u. The remaining side is solved for using the Pythagorean Theorem and then the result follows by evaluating the opposite side length over the adjacent side length. Finally we return to x using $u = x - 2$.



Further Questions:

Evaluate the following integrals:

1. $\displaystyle\int_1^2 \frac{1}{x^2\sqrt{16-x^2}}\,dx$
2. $\displaystyle\int \frac{\sqrt{x^2-9}}{x^4}\,dx$
3. $\displaystyle\int \frac{1}{\left(x^2+2x+2\right)^2}\,dx$
4. $\displaystyle\int \frac{2x-3}{x^2-4x+8}\,dx$
5. $\displaystyle\int \frac{1}{x^3\sqrt{x^2-25}}\,dx$
6. $\displaystyle\int \frac{x^2}{(2-9x^2)^{\frac{3}{2}}}\,dx$
7. $\displaystyle\int \frac{1}{(5-4x-x^2)^{\frac{5}{2}}}\,dx$
8. $\displaystyle\int \frac{\sqrt{x-4}}{x}\,dx$

Exercise 3-3

Answers: Page 264

1-10: Evaluate the given integral.

1. $\displaystyle\int \sqrt{16-x^2}\,dx$
2. $\displaystyle\int \frac{\sqrt{x^2+9}}{x}\,dx$
3. $\displaystyle\int \frac{1}{x^4\sqrt{x^2-1}}\,dx$
4. $\displaystyle\int \frac{1}{\left(16-u^2\right)^{5/2}}\,dx$
5. $\displaystyle\int \frac{1}{\sqrt{4x^2-9}}\,dx$
6. $\displaystyle\int \frac{x^2}{\left(x^2+4\right)^{3/2}}\,dx$
7. $\displaystyle\int \frac{x^2}{\sqrt{5-3x^2}}\,dx$
8. $\displaystyle\int \frac{1}{x^2-4x+6}\,dx$
9. $\displaystyle\int \frac{x}{\left(x^2-6x+15\right)^{5/2}}\,dx$
10. $\displaystyle\int \frac{3x+2}{\left(4-4x-x^2\right)^{3/2}}\,dx$

3.4 Partial Fraction Decomposition

The rational function $\dfrac{x+2}{x^3-x^2}$ can be shown to be equal to $-\dfrac{3}{x}-\dfrac{2}{x^2}+\dfrac{3}{x-1}$. Therefore the integral of the former rational function is:

$$\int \frac{x+2}{x^3-x^2}\,dx = \int \left(-\frac{3}{x}-\frac{2}{x^2}+\frac{3}{x-1}\right)dx = -3\ln|x| + \frac{2}{x} + 3\ln|x-1| + C$$

This example suggests that determining a technique to decompose a rational function in this way would provide a method for its integration.

A polynomial $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, $a_n \neq 0$ is said to have degree n. A function $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials is a rational function.

The rational number $\frac{7}{4}$ is called improper because the numerator is larger than the denominator. Through division of 4 into 7 one can write $\frac{7}{4}$ as $1\frac{3}{4}$ where the fractional part, $\frac{3}{4}$ is a proper fraction. Analagous definitions are made for rational functions.

Definition: A rational function $f(x) = \frac{P(x)}{Q(x)}$ is called **proper** if the degree of P is less than the degree of Q. Otherwise $f(x)$ is called **improper** if $\deg(P) \geq \deg(Q)$.

Note:

1. If $f(x) = P(x)/Q(x)$ is proper then it is possible to express it as a sum of simpler fractional functions called **partial fractions** which are integrable.
2. If $f(x)$ is improper, then use long division to divide P by Q until a remainder $R(x)$ is obtained such that $\deg R < \deg Q$. Then

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

   where $S(x)$ and $R(x)$ are polynomials. $S(x)$ is then integrable as it is a polynomial while the proper rational function $R(x)/Q(x)$ can in turn be integrated by the method of partial fractions thereby making $f(x)$ integrable.

Example 3-9

For the following rational functions determine if they are proper or improper. For those that are improper write them as a polynomial plus a proper rational function.

1. $\dfrac{x^3+2}{(x^2+4)^2}$
2. $\dfrac{3x^4+2x^2+x+7}{x^2+2}$

Solution:

1. The numerator $P(x) = x^3 + 2$ so $\deg P = 3$. Expanding the denominator gives

$$Q(x) = (x^2+4)^2 = x^4 + 8x^2 + 16$$

   showing that $\deg Q = 4$ so $\deg P < \deg Q$ and the rational function is proper.

2. Since $P(x) = 3x^4 + 2x^2 + x + 7$ and $Q(x) = x^2 + 2$ we have $4 = \deg P \geq \deg Q = 2$ so the rational function $P(x)/Q(x)$ is improper. Polynomial long division of $P(x)$ by $Q(x)$ gives

$$\begin{array}{r|l} & 3x^2 \qquad\quad - 4 \\ \hline x^2+2 & 3x^4 + 2x^2 + x \;\; + 7 \\ & -3x^4 - 6x^2 \\ \hline & \qquad -4x^2 + x \;\; + 7 \\ & \qquad\;\; 4x^2 \qquad + 8 \\ \hline & \qquad\qquad\quad x + 15 \end{array}$$

and it follows that

$$\frac{3x^4 + 2x^2 + x + 7}{x^2 + 2} = \underbrace{3x^2 - 4}_{S(x)} + \underbrace{\frac{x + 15}{x^2 + 2}}_{R(x)/Q(x)} .$$

Further Questions:

For the following rational functions determine if they are proper or improper. For those that are improper write them as a polynomial plus a proper rational function.

1. $f(x) = \dfrac{x+1}{x^3 - 3x^2 + 2}$
2. $f(x) = \dfrac{x^2+1}{x^2 + 3x}$
3. $f(x) = \dfrac{x^4 + 5x^2 + 1}{x^2 + 2}$

Definition: Let $g(x) = ax^2 + bx + c$ be a quadratic function with real coefficients. If $b^2 - 4ac \geq 0$ then $g(x)$ is called **reducible** because it can be written as a product of linear factors with real coefficients. If $b^2 - 4ac < 0$ then $g(x)$ is called **irreducible** because it cannot be written as a product of linear factors with real coefficients.

Example 3-10

1. The function $g(x) = x^2 + 5x + 6$ has $b^2 - 4ac = 25 - 24 = 1 > 0$ and so is reducible. It clearly factors as $g(x) = (x + 2)(x + 3)$.
2. The function $g(x) = 2x^2 + 4x + 5$ has $b^2 - 4ac = 16 - 40 = -24 < 0$ and is irreducible.

Note: It can be shown, as a consequence of the *Fundamental Theorem of Algebra*, that any polynomial $Q(x)$ with real coefficients can be factored as a product of linear factors of the form $(ax + b)$ and/or quadratic irreducible factors of the form $ax^2 + bx + c$, where a, b, and c are real numbers.

Theorem 3-1: If $P(x)$ and $Q(x)$ are polynomials and $\deg P < \deg Q$ the it follows that

$$\frac{P(x)}{Q(x)} = F_1 + F_2 + \ldots + F_n$$

where each F_i has one of the forms

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax + B}{(ax^2 + bx + c)^j}$$

for some nonnegative integers i and j. The sum $F_1 + F_2 + \ldots + F_n$ is called the **partial fraction decomposition** of $\frac{P(x)}{Q(x)}$ and each F_i is called a **partial fraction**. The denominator polynomials are real linear functions and irreducible quadratics respectively.

Steps for finding Partial Fraction Decomposition

To decompose $f(x) = \frac{P(x)}{Q(x)}$ into partial fractions do the following:

1. If $\deg P \geq \deg Q$ then use long division to get
$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$
2. Express $Q(x)$ as a product of linear and/or quadratic irreducible factors.
3. Express the proper rational function ($P(x)/Q(x)$ or $R(x)/Q(x)$) as a sum of partial fractions of the form
$$\frac{A}{(ax+b)^i} \quad \text{and/or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$
4. Evaluate the constants.

Once the partial fraction decomposition has been accomplished the necessary integration may be completed.

Upon factoring $Q(x)$ there are four cases that are logically possible.

Case I: $Q(x)$ contains a nonrepeated linear factor.

If $Q(x)$ has a nonrepeated linear factor $ax + b$ then the partial fraction decomposition will have the following term due to that factor:
$$\frac{A}{ax+b}$$
where A is constant.

For example, suppose that
$$Q(x) = (a_1x+b_1)(a_2x+b_2)\ldots(a_kx+b_k)$$
where no linear factor is repeated. Then there exist constants A_1, A_2, $\ldots A_k$ such that
$$\frac{P(x)}{Q(x)}\left(\text{or } \frac{R(x)}{Q(x)}\right) = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \ldots + \frac{A_k}{a_kx+b_k}$$

Example 3-11

Evaluate the following integrals:

1. $\displaystyle\int \frac{x+9}{x^2-3x-10}\,dx$
2. $\displaystyle\int \frac{x^2+3x+4}{x(x+1)(x-2)}\,dx$

Solution:

1. The denominator of the rational function factors as $x^2 - 3x - 10 = (x-5)(x+2)$, two linear distinct factors, so the partial fraction decomposition will have the form:
$$\frac{x+9}{(x-5)(x+2)} = \frac{A}{x-5} + \frac{B}{x+2} \implies \frac{x+9}{(x-5)(x+2)} = \frac{A}{x-5}\cdot\frac{x+2}{x+2} + \frac{B}{x+2}\cdot\frac{x-5}{x-5}$$
$$\implies \frac{x+9}{(x-5)(x+2)} = \frac{A(x+2)+B(x-5)}{(x-5)(x+2)}$$

Here we combined the two fractions on the right by getting a common denominator. Equating the numerators we have:

$$x + 9 = A(x + 2) + B(x - 5)$$

which must be true for all x. (Given the rational function is not defined at $x = 5$ or $x = -2$ one might argue these values should be excluded, but in this case the limits as $x \to -2$ and $x \to 5$ on both sides of the equation would need to be equal, and, as these are polynomials, this amounts to the functions themselves being equal at these two values.) Two methods can be used to find A and B.

- Method 1: Choose x values and solve for constants.
  Since the numerator equation is true for all x we can choose any x and solve for the constants. The form of the equation suggests using $x = -2$ and $x = 5$ since then only one constant remains:

$$\begin{aligned} x = -2 &\implies -2 + 9 = A(-2 + 2) + B(-2 - 5) \\ &\implies 7 = -7B \implies B = -1 \\ x = 5 &\implies 5 + 9 = A(5 + 2) + B(5 - 5) \\ &\implies 14 = 7A \implies A = 2 \end{aligned}$$

- Method 2: Equate polynomial coefficients.
  Since $x + 9 = A(x + 2) + B(x - 5)$ is true for all x the polynomials on both sides must be equal. Expanding the right-hand side and collecting like powers of x gives:

$$1x + 9 = (A + B)x + (2A - 5B)$$

  Equating coefficients of like powers of x implies:

$$\begin{aligned} x^0 &: 9 = 2A - 5B \\ x^1 &: 1 = A + B \end{aligned}$$

  This is a linear system of two equations in two unknowns for which many strategies can be used for solution. Multiplying the second equation by 2 gives $2 = 2A + 2B$. Subtracting that on both sides from the first equation gives

$$9 - 2 = 2A - 5B - (2A + 2B) \implies 7 = -7B \implies B = -1$$

  Inserting $B = -1$ in the first equation gives for A:

$$9 = 2A - 5(-1) \implies 4 = 2A \implies A = 2$$

Having $A = 2$ and $B = -1$ our decomposition is then $\dfrac{x + 9}{x^2 - 3x - 10} = \dfrac{2}{x - 5} + \dfrac{-1}{x + 2}$ and the integral is easily solved:

$$\int \frac{x + 9}{x^2 - 3x - 10}\, dx = \int \left(\frac{2}{x - 5} - \frac{1}{x + 2} \right) dx = 2 \ln|x - 5| - \ln|x + 2| + C\,,$$

where here we integrated term by term with substitution $u = x - 5$ and $u = x + 2$ respectively.

2. Here we have three linear factors and the partial fraction decomposition will have the form:

$$\frac{x^2+3x+4}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

Get a common denominator on the right-hand side and equate numerators:

$$\begin{aligned} \implies \frac{x^2+3x+4}{x(x+1)(x-2)} &= \frac{A(x+1)(x-2)+Bx(x-2)+Cx(x+1)}{x(x+1)(x-2)} \\ \implies x^2+3x+4 &= A(x+1)(x-2)+Bx(x-2)+Cx(x+1) \end{aligned}$$

Evaluate the constants A, B, and C by evaluating the numerator equation at different values of x:

$$\begin{aligned} &x=0 \implies 4=-2A \implies A=-2 \\ &x=-1 \implies 1-3+4=B(-1)(-3) \implies 2=3B \implies B=\frac{2}{3} \\ &x=2 \implies 4+6+4=C(2)(3) \implies 14=16C \implies C=\frac{14}{6} \implies C=\frac{7}{3} \end{aligned}$$

Therefore:

$$\begin{aligned} \int \frac{x^2+3x+4}{x(x+1)(x-2)}\,dx &= \int \left[\frac{-2}{x} + \frac{2}{3}\cdot\frac{1}{x+1} + \frac{7}{3}\cdot\frac{1}{x-2}\right] dx \\ &= -2\ln|x| + \frac{2}{3}\ln|x+1| + \frac{7}{3}\ln|x-2| + C \end{aligned}$$

Further Questions:

Evaluate the following integrals:

1. $\displaystyle\int \frac{1}{x^2+2x-3}\,dx$
2. $\displaystyle\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx$
3. $\displaystyle\int \frac{4x^2+3x+1}{x^2-1}\,dx$

Case II: $Q(x)$ contains a repeated linear factor.

If $Q(x)$ has a factor $(ax+b)^r$ then the partial fraction decomposition will have the following terms due to that factor:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_r}{(ax+b)^r}$$

where A_1, A_2, $\ldots A_r$ are constants. Use distinct constants (i.e. A, B, C, etc.) for each such factor.

Example 3-12

The (proper) rational function $\dfrac{x^4+1}{x(3x+2)^3(2x-1)^2}$ decomposes into

$$\frac{x^4+1}{x(3x+2)^3(2x-1)^2} = \frac{A}{x} + \frac{B_1}{3x+2} + \frac{B_2}{(3x+2)^2} + \frac{B_3}{(3x+2)^3} + \frac{C_1}{2x-1} + \frac{C_2}{(2x-1)^2}$$

where the constants A, B_1, B_2, B_3, C_1, and C_2 would then have to be determined.

Example 3-13

Evaluate the integral:

$$\int \frac{x^2+4}{x^2(2x+1)}\,dx$$

Solution:

Since the denominator has linear, repeated factors, the partial fraction decomposition will have the following form:

$$\begin{aligned}
\frac{x^2+4}{x^2(2x+1)} &= \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B}{2x+1} \\
\implies \frac{x^2+4}{x^2(2x+1)} &= \frac{A_1x(2x+1) + A_2(2x+1) + Bx^2}{x^2(2x+1)} \\
\implies x^2+4 &= A_1x(2x+1) + A_2(2x+1) + Bx^2
\end{aligned}$$

Evaluating at $x = 0$ and $2x+1 = 0 \implies x = -1/2$ are obvious choices for finding two of our constants. Due to the repeated x factor we need to choose another arbitrary value to find the remaining constant. We can use the previous results as shown below to find the last constant.

$$\begin{aligned}
x = 0 &\implies 0 + 4 = A_1(0) + A_2(0+1) + B(0) \\
&\implies 4 = A_2 \implies A_2 = 4 \\
x = -\frac{1}{2} &\implies \frac{1}{4} + 4 = A_1(0) + A_2(0) + B\frac{1}{4} \\
&\implies \frac{17}{4} = \frac{1}{4}B \implies B = 17 \\
x = 1 &\implies 1 + 4 = A_1(1)(3) + A_2(3) + B \\
&\implies 5 = 3A_1 + 3(4) + 17 \\
&\implies 5 = 3A_1 + 29 \implies 3A_1 = -24 \implies A_1 = -8
\end{aligned}$$

Note here we could have also used our second method of expanding the right-hand side of our numerator equation and equating polynomial coefficients to get a linear system of equations:

$$x^2 + 4 = (2A_1 + B)x^2 + (A_1 + 2A_2)x + A_2 \implies \begin{cases} A_2 &= 4 \\ A_1 + 2A_2 &= 0 \\ 2A_1 + B &= 1 \end{cases}$$

and solved to get $A_1 = -8, A_2 = 4, B = 17$ as before. Therefore

$$\begin{aligned}
\int \frac{x^2+4}{x^2(2x+1)}\,dx &= \int \left[\frac{-8}{x} + \frac{4}{x^2} + \frac{17}{2x+1}\right] dx \\
&= -8\ln|x| - \frac{4}{x} + \frac{17}{2}\ln|2x+1| + C
\end{aligned}$$

where we integrated term by term and used the substitution $u = 2x+1$, so $du = 2\,dx$, in the last integral.

Further Questions:

Evaluate the following integrals:

1. $\displaystyle\int \frac{x^3 - 4x - 1}{x(x-1)^3}\,dx$
2. $\displaystyle\int \frac{3x^2 + 5x - 10}{x^2(3x-5)}\,dx$

Case III: $Q(x)$ contains a nonrepeated irreducible quadratic factor.

If $Q(x)$ has a nonrepeated irreducible factor $ax^2 + bx + c$ (so $b^2 - 4ac < 0$), then the partial fraction decomposition will have the following term due to that factor:

$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants.

Example 3-14

Evaluate the integral:

$$\int \frac{2x^4 + 7x^2 + 6}{x^3 + 3x}\, dx$$

Solution:

Since the degree of the numerator (4) is greater than or equal to the degree of the denominator (3), this is an improper rational function. Performing polynomial long division one has

$$\begin{array}{r|l} & 2x \\ x^3 + 3x & 2x^4 + 7x^2 + 6 \\ & -2x^4 - 6x^2 \\ \hline & x^2 + 6 \end{array}$$

and the integral can be written as:

$$\int \frac{2x^4 + 7x^2 + 6}{x^3 + 3x}\, dx = \int \left(2x + \frac{x^2 + 6}{x^3 + 3x}\right) dx$$

The denominator can be factorized as $x(x^2 + 3)$. Here $x^2 + 3 = 1x^2 + 0x + 3$ is an irreducible quadratic since $b^2 - 4ac = -12 < 0$. Therefore, the partial fraction decomposition takes the form:

$$\frac{x^2 + 6}{x(x^2 + 3)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3} \implies x^2 + 6 = A(x^2 + 3) + Bx^2 + Cx$$

Because the factor of x is repeated and the irreducible quadratic vanishes for no real value of x the second method of finding constants is preferred. Expanding the polynomial gives:

$$\implies x^2 + 6 = (A + B)x^2 + Cx + 3A \implies \begin{cases} A + B &= 1 \\ C &= 0 \\ 3A &= 6 \end{cases}$$

The solution is straightforward:

$$\begin{aligned} 3A = 6 &\implies A = 2 \\ C &= 0 \\ A + B = 1 &\implies B = 1 - A \implies B = 1 - 2 \implies B = -1 \end{aligned}$$

Therefore

$$\int \frac{2x^4 + 7x^2 + 6}{x^3 + 3x}\, dx = \int \left(2x + \frac{2}{x} - \frac{x}{x^2 + 3}\right) dx = x^2 + 2\ln|x| - \frac{1}{2}\ln|x^2 + 3| + C,$$

where substitution $u = x^2 + 3$, so $du = 2x\, dx$, was used on the final term of the integral.

Further Question:

Evaluate the following integral:

$$\int \frac{x^3 - 4x^2 + 2}{(x^2 + 1)(x^2 + 2)}\, dx$$

Case IV: $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has an irreducible factor $\left(ax^2+bx+c\right)^r$ then the partial fraction decomposition will have the following terms due to that factor:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\ldots+\frac{A_rx+B_r}{\left(ax^2+bx+c\right)^r}$$

where A_1, A_2, ..., A_r, and B_1, B_2, ..., B_r are constants.

Example 3-15

Evaluate the integral:

$$\int\frac{2x+8}{x(x^2+4)^2}\,dx$$

Solution:

The repeated quadratic x^2+4 is irreducible since $0^2-4(1)(4)=-16<0$. The partial fraction decomposition will therefore have the following form:

$$\begin{aligned}
&\frac{2x+8}{x(x^2+4)^2}=\frac{A}{x}+\frac{B_1x+C_1}{x^2+4}+\frac{B_2x+C_2}{(x^2+4)^2}\\
&\implies 2x+8=A(x^2+4)^2+(B_1x+C_1)x(x^2+4)+B_2x^2+C_2x\\
&\implies 2x+8=A(x^4+8x^2+16)+B_1x^4+4B_1x^2+C_1x^3+4C_1x+B_2x^2+C_2x\\
&\implies 2x+8=(A+B_1)x^4+C_1x^3+(8A+4B_1+B_2)x^2+(4C_1+C_2)x+16A
\end{aligned}$$

Solving the system of equations generated by equating coefficients of x^n gives:

$$\begin{aligned}
16A=8 &\implies A=\frac{1}{2}\\
A+B_1=0 &\implies B_1=-A\implies B_1=-\frac{1}{2}\\
8A+4B_1+B_2=0 &\implies 4-2+B_2=0\implies B_2=-2\\
C_1=0&\\
4C_1+C_2=2 &\implies 4(0)+C_2=0\implies C_2=2
\end{aligned}$$

Therefore

$$\begin{aligned}
\int\frac{2x+8}{x(x^2+4)^2}\,dx&=\int\left[\frac{1}{2}\cdot\frac{1}{x}-\frac{1}{2}\cdot\frac{x}{x^2+4}-\frac{2x}{(x^2+4)^2}+\frac{2}{(x^2+4)^2}\right]dx\\
&=\frac{1}{2}\ln|x|-\frac{1}{2}\int\frac{x}{x^2+4}-\int\frac{2x}{(x^2+4)^2}\,dx+\int\frac{2}{(x^2+4)^2}\,dx
\end{aligned}$$

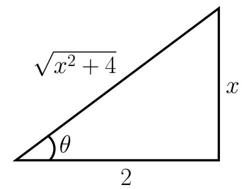
Note that the last partial fraction has been broken into two pieces anticipating the two different integration techniques required. To evaluate the second and third integrals use the substitution $u=x^2+4$ so $du=2x\,dx$:

$$\begin{aligned}
-\frac{1}{2}\int\frac{x}{x^2+4}\,dx&=-\frac{1}{4}\int\frac{1}{u}\,du=-\frac{1}{4}\ln|u|+C=-\frac{1}{4}\ln|x^2+4|+C\\
-\int\frac{2x}{(x^2+4)^2}\,dx&=-\int\frac{1}{u^2}\,du=\frac{1}{u}+C=\frac{1}{x^2+4}+C
\end{aligned}$$

To evaluate $\int \frac{2}{(x^2+4)^2}\,dx$ use the trigonometric substitution $x = 2\tan\theta$ so $dx = 2\sec^2\theta\,d\theta$:

$$\begin{aligned}\int \frac{2}{(x^2+4)^2}\,dx &= 2\int \frac{1}{(4\tan^2\theta+4)^2}\cdot 2\sec^2\theta\,d\theta = 4\int \frac{\sec^2\theta}{16\sec^4\theta}\,d\theta = \frac{1}{4}\int \frac{1}{\sec^2\theta}\,d\theta \\ &= \frac{1}{4}\int \cos^2\theta\,d\theta = \frac{1}{8}\int (1+\cos 2\theta)\,d\theta \\ &= \frac{1}{8}\theta + \frac{1}{16}\sin 2\theta + C = \frac{1}{8}\theta + \frac{1}{8}\sin\theta\cos\theta + C\end{aligned}$$

Note the identity for $\sin 2\theta$ is used to get functions of only the angle θ. Return to x by noting that, by the substitution $\tan\theta = \frac{x}{2}$, a triangle can be drawn with opposite side length of x and adjacent length of 2. The remaining side is solved for using the Pythagorean Theorem and then $\sin\theta$ and $\cos\theta$ are evaluated using their definitions and the triangle. In this integral θ itself is required and we note that $\theta = \tan^{-1}\left(\frac{x}{2}\right)$ by the original substitution. The integral becomes:



$$\begin{aligned}&= \frac{1}{8}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{8}\cdot\frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}} + C \\ &= \frac{1}{8}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{4}\cdot\frac{x}{x^2+4} + C\end{aligned}$$

Putting all the components together gives for the original integral:

$$\begin{aligned}\int \frac{2x+8}{x(x^2+4)^2}\,dx &= \frac{1}{2}\ln|x| - \frac{1}{4}\ln|x^2+4| + \frac{1}{x^2+4} + \frac{1}{8}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{4}\cdot\frac{x}{x^2+4} + C \\ &= \frac{1}{2}\ln|x| - \frac{1}{4}\ln|x^2+4| + \frac{1}{4}\cdot\frac{x+4}{x^2+4} + \frac{1}{8}\tan^{-1}\left(\frac{x}{2}\right) + C\end{aligned}$$

Further Question:

Evaluate the following integral:

$$\int \frac{2x^6+5x^4+2x^2+1}{x\left(x^2+1\right)^2}\,dx$$

Note the following:

- Use distinct constants (i.e. A, B, C, etc.) in each partial fraction.
- Factors in $Q(x)$ are considered the same if they differ only by a multiplicative constant. For example $(3x-1)$ and $(x-1/3)$ are repeated linear factors.

Having looked at all the cases we can write down the partial fraction decomposition of arbitrary rational functions.

Example 3-16

After factoring the denominator $Q(x)$ suppose a (proper) rational function equals

$$\frac{x^2+5x+1}{x\,(x-1)\,(3x+2)^3\,(x^2+2x+4)\,(x^2+9)^2}\ .$$

Then it will have the following partial fraction decomposition

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C_1}{3x+2} + \frac{C_2}{(3x+2)^2} + \frac{C_3}{(3x+2)^3} + \frac{Dx+E}{x^2+2x+4} + \frac{F_1x+G_1}{x^2+9} + \frac{F_2x+G_2}{(x^2+9)^2}\ .$$

Here we had two unrepeated linear factors, x and $(x-1)$, a repeated linear factor $(3x+2)^3$, an unrepeated irreducible quadratic factor (x^2+2x+4), and a repeated irreducible quadratic factor $(x^2+9)^2$.

We would now proceed to solve for the constants (A, B, C_1, C_2, C_3, D, E, F_1, G_1, F_2, and G_2) by combining the partial fractions and equating numerators as discussed above. Once this was done the original rational function could be integrated by integrating the partial fraction decomposition term by term.

Further Questions:

Write down the form of the partial fraction decomposition of the following rational functions. Do not evaluate the constants.

1. $\dfrac{x^2-x-21}{2x^3-x^2+8x-4}$
2. $\dfrac{x^3+2}{(x^2+4)^2}$
3. $\dfrac{x^6+5x^3+x-1}{x^4+5x^2+4}$
4. $\dfrac{x+1}{(x^2-4)^2\,(x^2+3)}$

Rationalizing Substitutions

Some nonrational functions can be changed into rational functions by means of a substitution.

Example 3-17

Evaluate the integral:

$$\int \frac{1}{e^{2x}-1}\,dx$$

Solution:

Noting $e^{2x}=(e^x)^2$ we try the rationalizing substitution $u=e^x$ so $du=e^x dx \implies \dfrac{1}{u}du = dx$:

$$\int \frac{1}{e^{2x}-1}\,dx = \int \frac{1}{u}\cdot\frac{1}{u^2-1}\,du = \int \frac{1}{u(u+1)(u-1)}\,du$$

The partial fraction decomposition takes the form:

$$\frac{1}{u(u+1)(u-1)} = \frac{A}{u} + \frac{B}{u+1} + \frac{C}{u-1}$$
$$\implies 1 = A(u+1)(u-1) + Bu(u-1) + Cu(u+1)$$

Evaluate coefficients by choosing u as follows:

$$u = 0 \implies 1 = -A \implies A = -1$$
$$u = 1 \implies 1 = C(1)(1+1) \implies C = \frac{1}{2}$$
$$u = -1 \implies 1 = B(-1)(-1-1) \implies B = \frac{1}{2}$$

Therefore:

$$\int \frac{1}{e^{2x}-1}\,dx = \int \frac{1}{u(u+1)(u-1)}\,du = \int \left(-\frac{1}{u} + \frac{1}{2}\cdot\frac{1}{u+1} + \frac{1}{2}\cdot\frac{1}{u-1}\right) du$$
$$= -\ln|u| + \frac{1}{2}\ln|u+1| + \frac{1}{2}\ln|u-1| + C$$
$$= -\ln|e^x| + \frac{1}{2}\ln|e^x+1| + \frac{1}{2}\ln|e^x-1| + C$$
$$= -x + \frac{1}{2}\ln|e^x+1| + \frac{1}{2}\ln|e^x-1| + C$$

Further Question:

Evaluate the following integral:

$$\int \frac{4\sqrt{x}}{x-2}\,dx$$

Answers: Page 265

Exercise 3-4

1-10: Evaluate the given integral.

1. $\displaystyle\int \frac{x}{x^2-5x+6}\,dx$
2. $\displaystyle\int \frac{x^3+1}{x^2-9}\,dx$
3. $\displaystyle\int \frac{3x-1}{(x-1)^2(x+2)}\,dx$
4. $\displaystyle\int \frac{2x+4}{(x+1)(x^2+1)}\,dx$
5. $\displaystyle\int \frac{x-9}{x(x^2+3)^2}\,dx$
6. $\displaystyle\int \frac{x^4+x^2+1}{(x-1)^3}\,dx$
7. $\displaystyle\int \frac{10}{x^2(x^2+5)}\,dx$
8. $\displaystyle\int \frac{x^6+x^3-6}{x^4+3x^2}\,dx$
9. $\displaystyle\int \frac{x^2+2x+5}{x^4+4x^2+3}\,dx$
10. $\displaystyle\int \frac{2x^2-4x+4}{x^3-x^2+x-1}\,dx$

3.5 General Strategies for Integration

Unlike differentiation which is largely a deterministic application of rules, integration is an art, with many indefinite integrals not even having an antiderivative that may be written in terms of known functions.

The following basic strategies have been seen

1. Basic Formulas of Integration
2. Substitution
3. Integration by Parts
4. Trigonometric Integrals
5. Trigonometric Substitution
6. Partial Fraction Decomposition
7. Rationalizing Substitution

One or more of these strategies along with using functional identities to rewrite the integrand may need to be applied to evaluate an integral.

Example 3-18

Evaluate the following integrals:

1. $\displaystyle\int \frac{1}{x^2+6x+16}\,dx$ 2. $\displaystyle\int \frac{1}{x^{\frac{3}{2}}-4x^{\frac{1}{2}}}\,dx$ 3. $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}+2e^x-5}}\,dx$

Solution:

1. We observe that $6^2-4(1)(16)=-28<0$ indicates that $x^2+6x+16$ is an irreducible quadratic so this rational function cannot be decomposed further. To integrate it, first complete the square to remove the x term.

$$\int \frac{1}{x^2+6x+16}\,dx = \int \frac{1}{(x+3)^2-9+16}\,dx = \int \frac{1}{(x+3)^2+7}\,dx$$

Next substitute $u=x+3$ so $du=dx$ and the integral becomes:

$$= \int \frac{1}{u^2+7}\,du = \frac{1}{7}\int \frac{1}{\frac{u^2}{7}+1}\,du = \frac{1}{7}\int \frac{1}{(\frac{u}{\sqrt{7}})^2+1}\,du$$

Let $w=\dfrac{u}{\sqrt{7}}$ so $dw=\dfrac{du}{\sqrt{7}} \implies du=\sqrt{7}dw$ to get our integrable form:

$$= \frac{\sqrt{7}}{7}\int \frac{1}{w^2+1}dw = \frac{1}{\sqrt{7}}\tan^{-1}(w)+C = \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{u}{\sqrt{7}}\right)+C = \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x+3}{\sqrt{7}}\right)+C$$

Alternatively we could have used the general integral $\int \frac{1}{u^2+a^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a})$ with $a=\sqrt{7}$ to avoid the w substitution.

2. Perform the rationalizing substitution $u = x^{\frac{1}{2}}$ so $du = \frac{1}{2}x^{-\frac{1}{2}}\,dx \implies 2u\,du = dx$:

$$\int \frac{1}{x^{\frac{3}{2}} - 4x^{\frac{1}{2}}}\,dx = \int \frac{1}{u^3 - 4u} \cdot 2u\,du = \int \frac{2}{u^2 - 4}\,du = \int \frac{2}{(u+2)(u-2)}\,du$$

The partial fraction decomposition is

$$\frac{2}{(u+2)(u-2)} = \frac{A}{u+2} + \frac{B}{u-2} \implies \frac{2}{(u+2)(u-2)} = \frac{A(u-2)+B(u+2)}{(u+2)(u-2)}$$
$$\implies 2 = A(u-2) + B(u+2)$$

Evaluate the constants:

$$u = 2 \implies 2 = 4B \implies B = \frac{1}{2}$$
$$u = -2 \implies 2 = -4A \implies A = -\frac{1}{2}$$

Therefore

$$\int \frac{1}{x^{\frac{3}{2}} - 4x^{\frac{1}{2}}}\,dx = \int \frac{2}{(u+2)(u-2)}\,du = \int \left(-\frac{1}{2}\cdot\frac{1}{u+2} + \frac{1}{2}\cdot\frac{1}{u-2}\right)du$$
$$= -\frac{1}{2}\ln|u+2| + \frac{1}{2}\ln|u-2| + C = -\frac{1}{2}\ln|\sqrt{x}+2| + \frac{1}{2}\ln|\sqrt{x}-2| + C$$

3. Let $u = e^x$ so $du = e^x\,dx$ and then complete the square:

$$\int \frac{e^x}{\sqrt{e^{2x}+2e^x-5}}\,dx = \int \frac{1}{\sqrt{u^2+2u-5}}\,du = \int \frac{1}{\sqrt{(u+1)^2-1-5}}\,dx = \int \frac{1}{\sqrt{(u+1)^2-6}}\,dx$$

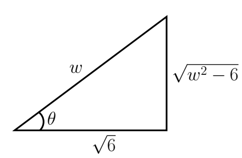
Let $w = u + 1$ so $dw = du$:

$$= \int \frac{1}{\sqrt{w^2-6}}\,dw$$

For $\sqrt{w^2-a^2}$ do the trigonometric substitution $w = \sqrt{6}\sec\theta$ so $dw = \sqrt{6}\sec\theta\tan\theta\,d\theta$:

$$= \int \frac{1}{\sqrt{6\sec^2\theta - 6}} \cdot \sqrt{6}\sec\theta\tan\theta\,d\theta = \int \frac{\sqrt{6}\tan\theta\sec\theta}{\sqrt{6}\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

To return to w note that the substitution implies $\sec\theta = \frac{w}{\sqrt{6}}$. To find $\tan\theta$ a triangle can be drawn, since secant is the reciprocal of cosine, with adjacent side of length $\sqrt{6}$ and hypotenuse of length w. The remaining side is solved for using the Pythagorean Theorem and then the result follows by evaluating the opposite side length over the adjacent side length. Finally we return to u using $w = u + 1$ and to x using $u = e^x$:



$$\int \frac{e^x}{\sqrt{e^{2x}+2e^x-5}}\,dx = \ln\left|\frac{w}{\sqrt{6}} + \frac{\sqrt{w^2-6}}{\sqrt{6}}\right| + C = \ln\left|\frac{u+1}{\sqrt{6}} + \frac{\sqrt{(u+1)^2-6}}{\sqrt{6}}\right| + C$$
$$= \ln\left|\frac{e^x+1}{\sqrt{6}} + \frac{\sqrt{e^{2x}+2e^x-5}}{\sqrt{6}}\right| + C = \ln\left|e^x + 1 + \sqrt{e^{2x}+2e^x-5}\right| + D$$

In the final simplification we used that $\ln|a/\sqrt{6}| = \ln|a| - \ln(\sqrt{6})$ and then combined the latter constant with C to get a new arbitrary constant D.

Further Questions:

Evaluate the following integrals:

1. $\displaystyle\int \frac{e^{3t}}{1+e^{6t}}\,dt$
2. $\displaystyle\int e^{x+e^x}\,dx$
3. $\displaystyle\int \frac{1+e^x}{1-e^x}\,dx$
4. $\displaystyle\int x^2\ln(1+x)\,dx$
5. $\displaystyle\int \tan x\sec^6 x\,dx$
6. $\displaystyle\int \frac{e^{3x}}{1+e^x}\,dx$
7. $\displaystyle\int \frac{\cos^3 x}{\sqrt{1+\sin x}}\,dx$
8. $\displaystyle\int \frac{x}{\csc\left(5x^2\right)}\,dx$
9. $\displaystyle\int \left(2x+2^x+2^\pi\right)\,dx$
10. $\displaystyle\int \frac{7x^2+20x+65}{x^4+4x^3+13x^2}\,dx$

Exercise 3-5

Answers: Page 265

1-10: Evaluate the given integral.

1. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
2. $\displaystyle\int \sin\sqrt{x+5}\,dx$
3. $\displaystyle\int \frac{e^{2x}}{3+e^x}\,dx$
4. $\displaystyle\int \frac{e^{3x}}{5+e^{6x}}\,dx$
5. $\displaystyle\int_1^4 \frac{1}{3+\sqrt{x}}\,dx$
6. $\displaystyle\int \frac{\sin 2x}{\sin^2 x-\sin x-6}\,dx$
7. $\displaystyle\int \frac{x^3+5x}{\left(x^2+1\right)^2}\,dx$
8. $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}+4e^x+6}}\,dx$
9. $\displaystyle\int_0^1 \frac{x+2}{x^2+2x+3}\,dx$
10. $\displaystyle\int \frac{4x}{\left(x^2+2x+9\right)^2}\,dx$

3.6 Improper Integrals

The definite integral due to Riemann which we use involves functions integrated over a closed interval $[a, b]$. Functions which are piecewise continuous where there are only a finite number of jump discontinuities are integrable. We now consider *improper integrals* where these restrictions do not hold. We consider two cases:

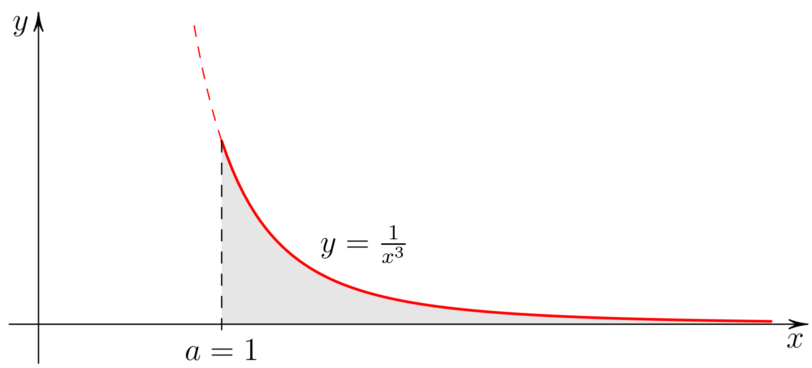
Improper Integrals of the First Kind : The interval of integration is infinite.

Improper Integrals of the Second Kind : The interval of integration contains an infinite discontinuity.

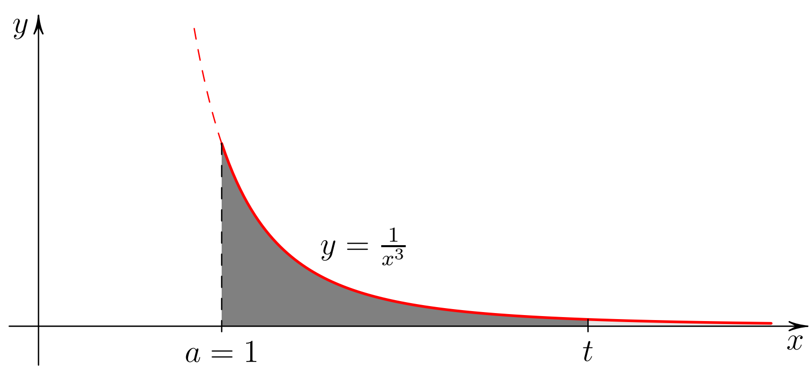
We can define definite integrals under these circumstances by considering suitable limits of integrals over closed intervals.

3.6.1 Improper Integrals of the First Kind

Suppose we wish to find the area under the curve $y = \frac{1}{x^3}$ over the interval $[1, \infty)$ shaded in the following diagram.



Intuitively one would find the area by evaluating the area under the curve (the definite integral) over the closed interval $[1, t]$, and then consider the limit of that as $t \to \infty$:



Should such a (finite) limit exist we would define that to be the area under the curve over the open interval $[1, \infty)$.

The previous discussion prompts the following definition for the improper integral over an infinite interval $[a, \infty)$ and, similarly, over intervals $(-\infty, b]$, and $(-\infty, \infty)$.

Definition: Define the following improper integrals of the first kind :

a) $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$ (Where the latter integrals must exist for every $t \geq a$.)

b) $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$ (Where the latter integrals must exist for every $t \leq b$.)

c) $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$ (Where a is any real number.)

The improper integrals in a) and b) are **convergent** if the limit exists (i.e. is finite) and **divergent** otherwise. For c) the integral is convergent if and only if both integrals on the right side are convergent.

Note that $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ is **not** defined to be $\displaystyle\lim_{t\to\infty}\int_{-t}^t f(x)\,dx$. The integral over $(-\infty,\infty)$ by definition must be broken into two pieces for which independent limits must be taken.

Example 3-19

Determine whether the following integrals are convergent or divergent and evaluate those that are convergent.

1. $\displaystyle\int_0^\infty e^{-4x}\,dx$
2. $\displaystyle\int_{-\infty}^0 \frac{1}{3-2x}\,dx$
3. $\displaystyle\int_{-\infty}^\infty \frac{e^x}{e^{2x}+1}\,dx$

Solution:

These are all improper integrals of the first kind since the x values are approaching ∞, $-\infty$, or both.

1. $\displaystyle\int_0^\infty e^{-4x}\,dx = \lim_{t\to\infty}\int_0^t e^{-4x}\,dx$

   Substitute $u = -4x$ so $du = -4dx \implies -\frac{1}{4}du = dx$

   Limits: $x = 0 \implies u = -4(0) = 0,\; x = t \implies u = -4t$

   The integral becomes:

$$= \lim_{t\to\infty}\int_0^{-4t} e^u\left(-\frac{1}{4}\right)du = -\frac{1}{4}\lim_{t\to\infty}\left[e^u\right|_0^{-4t} = -\frac{1}{4}\lim_{t\to\infty}\left[e^{-4t}-e^0\right]$$
$$= -\frac{1}{4}\lim_{t\to\infty}\left[\frac{1}{e^{4t}}-1\right] = -\frac{1}{4}[0-1] = \frac{1}{4}$$

   Here we used the known properties of the exponential function, that $\displaystyle\lim_{x\to\infty} e^x = \infty$, to evaluate the limit. Therefore the improper integral is convergent with value $\frac{1}{4}$.

2. $\displaystyle\int_{-\infty}^0 \frac{1}{3-2x}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{3-2x}\,dx$

   Substitute $u = 3-2x$ so $du = -2dx \implies -\frac{1}{2}du = dx$

   Limits: $x = t \implies u = 3-2t,\; x = 0 \implies u = 3-2(0) = 3$

The integral becomes:

$$= \lim_{t\to-\infty} \int_{3-2t}^{3} \frac{1}{u} \cdot \left(-\frac{1}{2}\right) du = -\frac{1}{2} \lim_{t\to-\infty} \ln|u| \Big|_{3-2t}^{3}$$
$$= -\frac{1}{2} \lim_{t\to-\infty} [\ln|3| - \ln|3-2t|]$$
$$= -\frac{1}{2} [\ln 3 - \infty] = \infty$$

Here we used the property of the logarithm, that $\lim_{x\to\infty} \ln x = +\infty$, to evaluate the limit. Therefore the integral is divergent.

3. For this integral we must first break the infinite domain into two separate integrals:

$$\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1}\, dx = \int_{-\infty}^{0} \frac{e^x}{e^{2x}+1}\, dx + \int_{0}^{\infty} \frac{e^x}{e^{2x}+1}\, dx$$

Since the integrand is the same for both integrals, let us first evaluate the following indefinite integral with the substitution $u = e^x$ so $du = e^x dx$:

$$\int \frac{e^x}{e^{2x}+1}\, dx = \int \frac{1}{u^2+1}\, du = \tan^{-1} u + C = \tan^{-1}(e^x) + C$$

The integrals become:

$$\int_{-\infty}^{0} \frac{e^x}{e^{2x}+1}\, dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{e^x}{e^{2x}+1}\, dx = \lim_{t\to-\infty} \left[\tan^{-1}(e^x) \right|_{t}^{0}$$
$$= \lim_{t\to-\infty} \left[\tan^{-1}(e^0) - \tan^{-1}(e^t)\right] = \lim_{t\to-\infty} \left[\tan^{-1}(1) - \tan^{-1}(e^t)\right]$$
$$= \frac{\pi}{4} - \tan^{-1}(0) = \frac{\pi}{4}$$

$$\int_{0}^{\infty} \frac{e^x}{e^{2x}+1}\, dx = \lim_{t\to\infty} \int_{0}^{t} \frac{e^x}{e^{2x}+1}\, dx = \lim_{t\to\infty} \left[\tan^{-1}(e^x) \right|_{0}^{t}$$
$$= \lim_{t\to\infty} \left[\tan^{-1}(e^t) - \tan^{-1}(e^0)\right] = \lim_{t\to\infty} \tan^{-1}(e^t) - \tan^{-1}(1)$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},$$

where we used that $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$. The original integral therefore is

$$\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1}\, dx = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

and hence convergent.

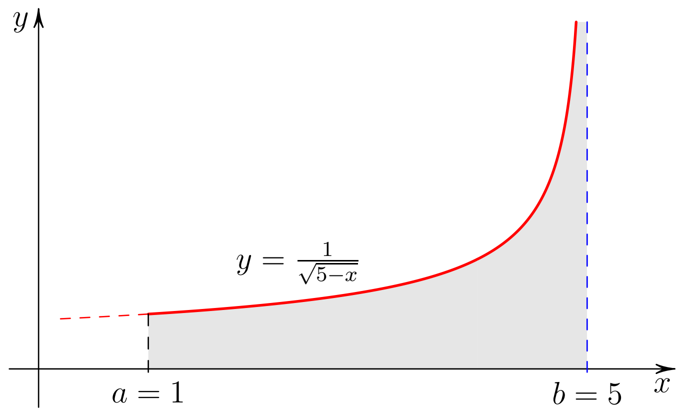
Further Questions:

Determine whether the following integrals converge or diverge. Find the value of any convergent integral.

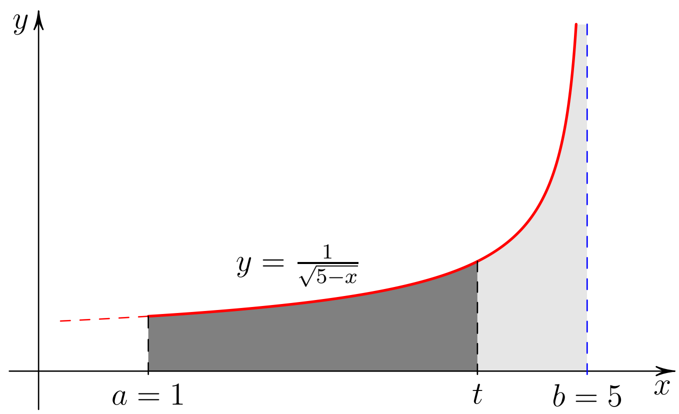
1. $\displaystyle\int_{1}^{\infty} \frac{1}{x^3}\, dx$
2. $\displaystyle\int_{2}^{\infty} \frac{1}{x-1}\, dx$
3. $\displaystyle\int_{-\infty}^{0} xe^{x}\, dx$
4. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$
5. $\displaystyle\int_{-\infty}^{0} xe^{-x^2}\, dx$
6. $\displaystyle\int_{1}^{\infty} \frac{\ln x}{x}\, dx$

3.6.2 Improper Integrals of the Second Kind

In the second case we consider those situations where the function being integrated has an infinite discontinuity at some point over which we want to integrate. Consider the area under the curve $y = \dfrac{1}{\sqrt{5-x}}$ between $x = 1$ and $x = 5$. The situation is shown in the following diagram.



The function has an infinite discontinuity at the right endpoint ($b = 5$). Intuitively we can imagine finding the area under the curve over the closed interval $[1, t]$ with $t < b$ and then consider the limit as $t \to b$:



This discussion suggests the following definition for improper integrals involving infinite integrands. Our example illustrated an integral where the right endpoint had the discontinuity. Similarly integrals with a discontinuity at the left endpoint or within the interval are defined.

Definition: Define the following improper integrals of the second kind :

a) Suppose $f(x)$ is continuous on $[a, b)$ but discontinuous at $x = b$ then:
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

b) Suppose $f(x)$ is continuous on $(a, b]$ but discontinuous at $x = a$ then:
$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

c) Suppose $f(x)$ is continuous on $[a,b]$ except at a value c in (a,b) then:
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The improper integrals in a) and b) are **convergent** if the limit exists (i.e. is finite) and **divergent** otherwise. For c) the integral is convergent if and only if both integrals on the right side are convergent.

Example 3-20

Determine whether the following integrals are convergent or divergent. Find the value of any convergent integral.

1. $\displaystyle\int_0^1 \frac{\ln x}{x^2}\,dx$ 2. $\displaystyle\int_{-4}^0 \frac{x}{\sqrt{x+4}}\,dx$ 3. $\displaystyle\int_0^2 \frac{x}{x-1}\,dx$

Solution: These are improper integrals of the second kind and we need first to identify where the integrand is discontinuous.

1. Here the integrand is undefined at 0 both because $\ln x$ is undefined there and also because we cannot divide by zero. The improper integral is therefore defined by
$$\int_0^1 \frac{\ln x}{x^2}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{x^2}\,dx$$
Use Integration by Parts to evaluate the integral:
$$u = \ln x \quad dv = x^{-2}\,dx$$
$$\implies du = \frac{1}{x}\,dx \quad v = -\frac{1}{x}$$
The integral then equals
$$\begin{aligned}\int_0^1 \frac{\ln x}{x^2}\,dx &= \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{x^2}\,dx = \lim_{t\to 0^+}\left[-\frac{\ln x}{x}\bigg|_t^1 + \int_t^1 \frac{1}{x^2}\,dx\right]\\ &= \lim_{t\to 0^+}\left[-\frac{\ln x}{x} - \frac{1}{x}\bigg|_t^1\right] = \lim_{t\to 0^+}\left[\frac{-\ln 1}{1} - \frac{1}{1} + \frac{\ln t}{t} + \frac{1}{t}\right]\\ &= -0 - 1 + \lim_{t\to 0^+}\left(\frac{\ln t}{t} + \frac{1}{t}\right) \quad (-\infty + \infty \text{ form})\\ &= -1 + \lim_{t\to 0^+}\frac{\ln(t)+1}{t} \quad \left(\frac{\infty}{\infty}\text{ form}\right)\\ &\underset{\text{LH}}{=} -1 + \lim_{t\to 0^+}\frac{1/t}{1} = \infty\end{aligned}$$
Therefore the integral is divergent.

2. The integrand is discontinuous at $x = -4$ so the improper integral is defined by
$$\int_{-4}^0 \frac{x}{\sqrt{x+4}}\,dx = \lim_{t\to -4^+}\int_t^0 \frac{x}{\sqrt{x+4}}\,dx$$
Substitute $u = x+4$ so $du = dx,\ x = u-4$

Limits : $x = 0 \implies u = 0+4 = 4,\ x = t \implies u = t+4$

The integral becomes:

$$\int_{-4}^{0} \frac{x}{\sqrt{x+4}}\,dx = \lim_{t\to -4^+}\int_{t}^{0} \frac{x}{\sqrt{x+4}}\,dx = \lim_{t\to -4^+}\int_{t+4}^{4} \frac{u-4}{\sqrt{u}}\,du = \lim_{t\to -4^+}\int_{t+4}^{4} \left(\sqrt{u}-4u^{-\frac{1}{2}}\right)du$$

$$= \lim_{t\to -4^+}\left[\frac{2}{3}u^{\frac{3}{2}}-8u^{\frac{1}{2}}\right|_{t+4}^{4} = \lim_{t\to -4^+}\left[\frac{2}{3}(4)^{\frac{3}{2}}-8(4)^{\frac{1}{2}}-\frac{2}{3}(t+4)^{\frac{3}{2}}+8(t+4)^{\frac{1}{2}}\right]$$

$$= \frac{2}{3}(8)-8(2)-\frac{2}{3}(-4+4)^{\frac{3}{2}}+8(-4+4)^{\frac{1}{2}} = \frac{16}{3}-16-0+0 = \frac{16-48}{3}$$

$$= -\frac{32}{3}$$

Therefore it is convergent.

3. The integrand is undefined at $x=1$ so we must break the improper integral into two integrals:

$$\int_{0}^{2} \frac{x}{x-1}\,dx = \int_{0}^{1} \frac{x}{x-1}\,dx + \int_{1}^{2} \frac{x}{x-1}\,dx$$

Evaluate the first integral:

$$\int_{0}^{1} \frac{x}{x-1}\,dx = \lim_{t\to 1^-}\int_{0}^{t} \frac{x}{x-1}\,dx \quad (\leftarrow \text{ an improper rational function})$$

$$= \lim_{t\to 1^-}\int_{0}^{t} \frac{x-1+1}{x-1}\,dx = \lim_{t\to 1^-}\int_{0}^{t}\left(\frac{x-1}{x-1}+\frac{1}{x-1}\right)dx$$

$$= \lim_{t\to 1^-}\int_{0}^{t}\left(1+\frac{1}{x-1}\right)dx = \lim_{t\to 1^-}\left[x+\ln|x-1|\right|_{0}^{t}$$

$$= \lim_{t\to 1^-}\Big[t+\ln|t-1|-0-\ln|-1|\Big]$$

$$= 1-\infty-0-0 = -\infty\,,$$

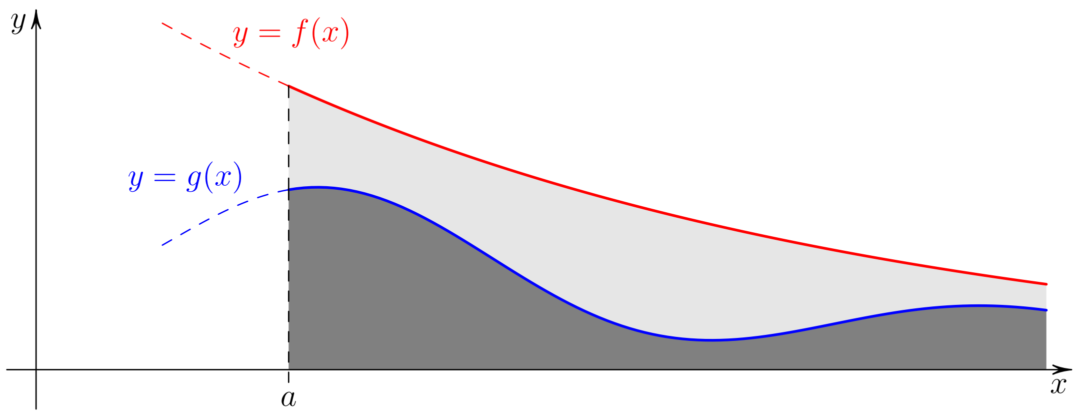
where here we used that $\lim_{x\to 0^+}\ln x = -\infty$. Since the fist integral is divergent the given integral $\int_{0}^{2}\frac{x}{x-1}\,dx$ is also divergent. (There is no need to evaluate the second composite integral.)

Further Questions:

Determine whether the following integrals converge or diverge. Find the value of any convergent integral.

1. $\int_{1}^{5} \frac{1}{\sqrt{5-x}}\,dx$
2. $\int_{0}^{1} x\ln x\,dx$
3. $\int_{-2}^{7} \frac{1}{(x+1)^{\frac{2}{3}}}\,dx$
4. $\int_{0}^{2} \frac{1}{x^2-4x+3}\,dx$

A consideration of the areas represented by improper integrals in the following diagram makes the following theorem plausible:



Theorem 3-2: Let f and g be continuous functions satisfying $f(x) \geq g(x) \geq 0$ for all $x \geq a$. If $\int_a^\infty f(x)\,dx$ is convergent then $\int_a^\infty g(x)\,dx$ is convergent. If $\int_a^\infty g(x)\,dx$ is divergent then $\int_a^\infty f(x)\,dx$ is divergent.

Analagous theorems for the infinite intervals $(-\infty, b]$ and $(-\infty, \infty)$ as well as for improper integrals of the second kind may also be written. The theorems are useful for determining convergence or divergence of functions that are difficult to integrate.

Example 3-21

Determine whether the following integrals are convergent or divergent.

1. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^4+5}}$
2. $\displaystyle\int_0^1 \frac{e^x}{x^2}\,dx$

Solution:

1. The integrand in the improper integral $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^4+5}}$ has no obvious antiderivative. However, for $x \geq 1$ we have

$$\sqrt{x^4+5} \geq \sqrt{x^4} = x^2 > 0 \implies 0 < \frac{1}{\sqrt{x^4+5}} \leq \frac{1}{x^2}\,.$$

The following integral

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right|_1^t = \lim_{t\to\infty}\left[-\frac{1}{t}+\frac{1}{1}\right] = 0+1 = 1$$

is convergent. Thus, since our desired positive integrand lies below a function whose integral converges we have, by the Comparison Test for Integrals, that the given integral is convergent.

2. The integral $\displaystyle\int_0^1 \frac{e^x}{x^2}\,dx$ is improper since the integrand is undefined at zero. Since the integrand has no obvious antiderivative we try a comparison with an integrable function. For $0 \leq x \leq 1$ we have

$$e^x \geq 1 > 0 \implies \frac{e^x}{x^2} \geq \frac{1}{x^2} > 0\,.$$

The following integral

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\left[-\frac{1}{x}\right|_t^1 = \lim_{t\to 0^+}\left[-1+\frac{1}{t}\right] = -1+\infty=\infty$$

is divergent. Since our desired integrand lies above a positive function whose integral diverges, we have, by the Comparison Test for Integrals, that the given integral is also divergent.

Further Questions:

Determine whether the following integrals are convergent or divergent.

1. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$
2. $\displaystyle\int_0^1 \frac{e^{-x}}{x^{\frac{2}{3}}}\,dx$

Exercise 3-6

Answers: Page 266

1-10: Determine whether the given integral is convergent or divergent.

1. $\displaystyle\int_3^\infty \frac{1}{(x-2)^2}\,dx$
2. $\displaystyle\int_5^\infty \frac{1}{x-4}\,dx$
3. $\displaystyle\int_{-\infty}^0 x^3\,e^{-x^4}\,dx$
4. $\displaystyle\int_{-\infty}^\infty \frac{x}{x^4+16}\,dx$
5. $\displaystyle\int_{-\infty}^0 \frac{1}{x^2-4x+3}\,dx$
6. $\displaystyle\int_0^5 \frac{1}{\sqrt{5-x}}\,dx$
7. $\displaystyle\int_1^2 \frac{1}{(x-1)^{2/3}}\,dx$
8. $\displaystyle\int_0^2 \frac{1}{x^2-4x+3}\,dx$
9. $\displaystyle\int_1^e \frac{1}{x\,(\ln x)^2}\,dx$
10. $\displaystyle\int_0^{\pi/2} \tan^2 x\,dx$

Answers: Page 266

Chapter 3 Review Exercises

1-12: Evaluate the given integral.

1. $\int x \tan^{-1} x \, dx$
2. $\int \tan x \sec^3 x \, dx$
3. $\int_0^2 \ln(2+x) \, dx$
4. $\int \frac{1}{(x^2+36)^{5/2}} \, dx$
5. $\int \frac{x-2}{(x+1)^5} \, dx$
6. $\int \frac{4}{x^3+2x} \, dx$
7. $\int \frac{1}{\sqrt{x^2+6x+12}} \, dx$
8. $\int_0^{\ln 2} \frac{e^{3x}}{1+e^x} \, dx$
9. $\int \frac{4x^2-8x-6}{(x-1)(x-2)(x-3)} \, dx$
10. $\int \frac{6x^3-4x^2+5}{x^4+5x^2+4} \, dx$
11. $\int x^{5/2} \ln x \, dx$
12. $\int \frac{\cos x}{\sqrt{1+\sin^2 x}} \, dx$

13-15: Determine whether the given integral is convergent or divergent.

13. $\int_0^{\infty} x \, e^{-3x} \, dx$
14. $\int_0^{\pi} \frac{\sin x}{\sqrt{1+\cos x}} \, dx$
15. $\int_{-\infty}^{\infty} \frac{1}{5+x^2} \, dx$

Chapter 4: Sequences and Series

4.1 Sequences

Definition: An ordered list of numbers:

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$$

is called a **sequence**. The numbers are called **terms** with a_1 here being the *first term*, and, more generally, a_n being the n^{th} term in the sequence.

The above sequence may represented by the compact notation $\boldsymbol{\{a_n\}}$ or sometimes with the index limits made explicit as $\boldsymbol{\{a_n\}_{n=1}^{\infty}}$. An explicit index is useful if we start enumerating the sequence from a value other than 1.

Some texts will distinguish **finite** and **infinite** sequences depending on whether the sequence terminates or not. For our purposes we will be assuming infinite sequences unless otherwise noted.

An equivalent way of thinking of a sequence is as a function f whose domain is the positive integers. In this case $a_n = f(n)$. Writing a_n as just such a function of the index is a convenient way of representing a sequence.

Example 4-1

The following are several ways to represent the same sequences.

1. $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\} = \left\{\frac{1}{n}\right\} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$, $a_n = \frac{1}{n}$
   Note this sequence could also have been represented by $\left\{\frac{1}{n+1}\right\}_{n=0}^{\infty}$
2. $\left\{\frac{1}{2}, \frac{-4}{5}, \frac{9}{8}, \ldots, \frac{(-1)^{n+1}n^2}{3n-1}, \ldots\right\} = \left\{\frac{(-1)^{n+1}n^2}{3n-1}\right\}_{n=1}^{\infty}$, $a_n = \frac{(-1)^{n+1}n^2}{3n-1}$
3. $\{4, 4, 4, \ldots, 4, \ldots\} = \{4\}_{n=1}^{\infty}$, $a_n = 4$

A sequence may not have a simple defining function in terms of the index n.

Example 4-2

The sequence generated by the digits of $\pi = 3.14159\ldots$

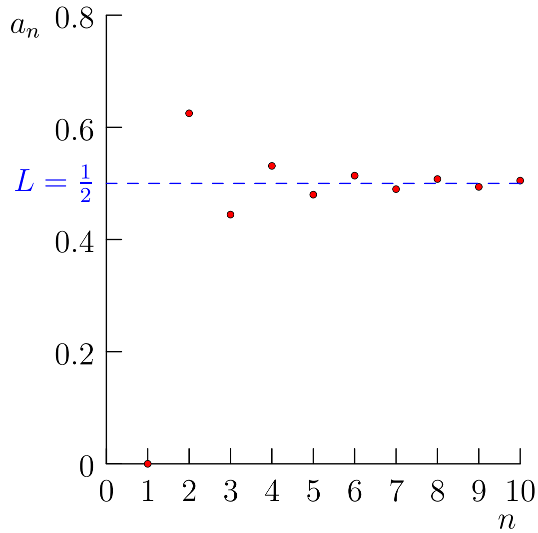
$$\{3, 1, 4, 1, 5, 9, \ldots\}$$

is not representable by a simple function $f(n)$.

Since theoretically any sequence is a function $a_n = f(n)$ on the set of positive integers we can graphically represent it by plotting the coordinate points (n, a_n).

Example 4-3

A graph of the sequence $\left\{\dfrac{\cos(n\pi) + n^2}{2n^2}\right\}$ is as follows:



The above graph clearly approaches the value 1/2 as n gets large. In symbols we would write

$$\lim_{n\to\infty} \frac{\cos(n\pi) + n^2}{2n^2} = \frac{1}{2}$$

This limit is analagous to the limit of a function $f(x)$ as $x \to \infty$ with the only difference being that n is restricted to positive integers. This discussion motivates the following definition for the limit of a sequence.[1]

Definition: If the terms a_n of sequence $\{a_n\}$ get arbitrarily close to the value L for sufficiently large n then we say the sequence **converges** to **limit** $\boldsymbol{L}$ or **is convergent** with **limit** $\boldsymbol{L}$. Symbolically $a_n \to L$ as $n \to \infty$ or

$$\lim_{n\to\infty} a_n = L \ .$$

If a sequence is not convergent (i.e. it has no limit) then the sequence **diverges** or **is divergent**.

A divergent sequence may have a trend to infinity.[2]

Definition: If the terms a_n of sequence $\{a_n\}$ get arbitrarily large (positively) for sufficiently large n we say that the sequence $\{a_n\}$ **diverges to infinity** and we write

$$\lim_{n\to\infty} a_n = \infty$$

An analagous definition holds for a sequence to diverge to $-\infty$.

Example 4-4

The *Fibonacci Sequence* satisfies $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-2} + a_{n-1}$ for $n > 2$, i.e.

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

The sequence diverges to ∞ $\left(\lim_{n\to\infty} a_n = \infty\right)$.

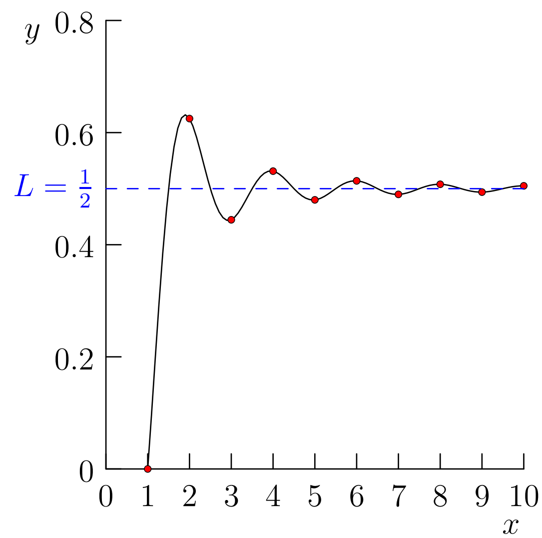
[1] A more rigorous definition of the limit of a sequence is that $a_n \to L$ as $n \to \infty$ if and only if for any $\epsilon > 0$ there exists $m > 0$ such that $n > m$ implies $|a_n - L| < \epsilon$.

[2] A more rigorous definition for $\lim_{n\to\infty} a_n = \infty$ is that for any $M > 0$ there exists an index $m > 0$ such that $n > m$ implies $a_n > M$.

The limit of a sequence with $a_n = f(n)$ is essentially the limit of $f(x)$ as $x \to \infty$ with x restricted to the positive integers (instead of the continuous real axis).

Example 4-5

If we plot $y = f(x) = \dfrac{\cos(x\pi) + x^2}{2x^2}$ over our earlier sequence we have:



The limit of $f(n)$, with n an integer, clearly cannot differ from that of $f(x)$ if the latter exists, thereby leading to the following theorem.

Theorem 4-1: If $\lim\limits_{x\to\infty} f(x) = L$ then the limit of sequence $\{a_n\}$ with $a_n = f(n)$ is also L,

$$\lim_{n\to\infty} a_n = L \ .$$

(Note the converse of this theorem is not true, $\lim\limits_{n\to\infty} a_n = L \not\Rightarrow \lim\limits_{x\to\infty} f(x) = L$.)

For a sequence which converges to $L = 0$ we have the following result:

Theorem 4-2: $\lim\limits_{n\to\infty} a_n = 0$ if and only if $\lim\limits_{n\to\infty} |a_n| = 0$.

These theorems are convenient for evaluating the limits of certain sequences.

Example 4-6

Find the limit of the following sequences:

1. $\left\{\dfrac{n^2+4}{3n^2+5n+1}\right\}$
2. $\left\{\dfrac{n^2+\ln n}{4n^2}\right\}$
3. $\left\{(-1)^n \cdot \dfrac{\ln n}{n^2}\right\}$

Solution:

1. To evaluate the limit pull out the highest power of n (here n^2) from each of the numerator and denominator.

$$\lim_{n\to\infty} \frac{n^2+4}{3n^2+5n+1} = \lim_{n\to\infty} \frac{n^2}{n^2} \cdot \frac{\frac{n^2}{n^2} + \frac{4}{n^2}}{\frac{3n^2}{n^2} + \frac{5n}{n^2} + \frac{1}{n^2}} = \lim_{n\to\infty} 1 \cdot \frac{1 + \frac{4}{n^2}}{3 + \frac{5}{n} + \frac{1}{n^2}} = \frac{1+0}{3+0+0} = \frac{1}{3}$$

2. Here the n^{th} term $a_n = f(n)$ where $f(n) = \dfrac{n^2 + \ln n}{4n^2}$. Considering the function as a function of a continuous variable x, so $f(x) = \dfrac{x^2 + \ln x}{4x^2}$, we have:

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x^2 + \ln x}{4x^2} \quad \left(\frac{\infty}{\infty} \text{ form}\right) \\
&\underset{\text{LH}}{=} \lim_{x\to\infty} \frac{2x + \frac{1}{x}}{8x} \quad \left(\frac{\infty}{\infty} \text{ form}\right) \\
&\underset{\text{LH}}{=} \lim_{x\to\infty} \frac{2 - \frac{1}{x^2}}{8} = \frac{2-0}{8} = \frac{1}{4}
\end{aligned}$$

Therefore $\displaystyle\lim_{n\to\infty} \frac{n^2 + \ln n}{4n^2} = \frac{1}{4}$ by Theorem 4-1.

3. Consider the limit of the absolute value of $a_n = \dfrac{(-1)^n \ln n}{n^2}$:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|(-1)^n \cdot \frac{\ln n}{n^2}\right| = \lim_{n\to\infty} \frac{\ln n}{n^2} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

Letting $f(x) = \dfrac{\ln x}{x^2}$, a continuous function of x, we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x^2} \quad \left(\frac{\infty}{\infty}\text{form}\right) \underset{\text{LH}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$$

$$\implies \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{\ln n}{n^2} = 0 \text{ (by Theorem 4-1)}$$

$$\implies \lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \frac{\ln n}{n^2} = 0 \text{ (by Theorem 4-2)}$$

Further Questions:

Find the limits of the following sequences:

1. $\left\{\dfrac{2n}{5n-3}\right\}$
2. $\left\{\dfrac{5n}{e^{2n}}\right\}$
3. $\left\{\dfrac{(-1)^n(3n+1)}{n^2+5}\right\}$

Theorem 4-3: Given sequences $\{a_n\}$ and $\{b_n\}$ are convergent, c is a constant, and f a function defined at a_n and continuous at $L = \lim_{n\to\infty} a_n$, then the following hold:

1. $\displaystyle\lim_{n\to\infty} c = c$
2. $\displaystyle\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
3. $\displaystyle\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$
4. $\displaystyle\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right)$
5. $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

   (Here we require $\lim_{n\to\infty} b_n \neq 0$.)
6. $\displaystyle\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$

By the last item of the previous theorem applied to $f(x) = x^k$ one has the corollary

Theorem 4-4: Given non-negative sequence $\{a_n\}$ (i.e. $a_n \geq 0$) and power $k > 0$ one has

$$\lim_{n\to\infty} (a_n)^k = \left(\lim_{n\to\infty} a_n\right)^k \ .$$

Here the sequence must have $a_n \geq 0$ for $(a_n)^k$ to be defined and the power k cannot be negative to accommodate sequences for which $\lim_{n\to\infty} a_n = 0$.

Definition: A sequence is **geometric** if it has the form $\{ar^n\}_{n=0}^{\infty} = \left\{a, ar, ar^2, ar^3, \ldots, ar^n, \ldots\right\}$ $(a \neq 0)$. Here a is the first term in the sequence and r is called the common ratio since $\frac{a_{n+1}}{a_n} = r$ (constant) for a geometric sequence.

One notes that a geometric sequence is of the form $\left\{ar^{n-1}\right\}_{n=1}^{\infty}$ if we start with index $n = 1$.

The following theorem results from consideration of the behaviour of the limit of the exponential function $\lim_{x\to\infty} r^x$ when $r \geq 0$ in Theorem 4-1 and use of Theorem 4-2 noting that $\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n$ when $-1 < r < 0$.

Theorem 4-5: The geometric sequence $\{ar^n\}_{n=0}^{\infty} = \left\{a, ar, ar^2, ar^3, \ldots, ar^n, \ldots\right\}$ $(a \neq 0)$ is convergent when $-1 < r \leq 1$ with

$$\lim_{n\to\infty} ar^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ a & \text{if } r = 1 \end{cases} \ .$$

It is divergent for all other values of r, diverging to infinity (∞) for $r > 1$.

Example 4-7

Determine whether the following sequences are divergent or convergent. For convergent sequences determine the limit.

1. $\left\{5\left(\frac{1}{10}\right)^n\right\}_{n=0}^{\infty}$
2. $\{3(-5)^n\}$
3. $\left\{\frac{2n + \ln n}{5n + 3\ln n}\right\}$

3 **Solution:**

1. This is a geometric sequence with $a = 5$ and $r = \frac{1}{10}$. Since $|r| = \left|\frac{1}{10}\right| < 1$ it is convergent with

$$\lim_{n\to\infty} 5\left(\frac{1}{10}\right)^n = 0 \,.$$

2. Without explicit index labels we infer $\{3(-5)^n\} = \{3(-5)^n\}_{n=1}^{\infty}$. We can put it in standard form for a geometric sequence starting at $n = 1$ by rewriting it:

$$\{3(-5)^n\}_{n=1}^{\infty} = \left\{3(-5)(-5)^{n-1}\right\}_{n=1}^{\infty} = \left\{-15(-5)^{n-1}\right\}_{n=1}^{\infty} \ .$$

This is a geometric sequence with $a = -15$ and $r = -5$. Since $r = -5 < -1$ it is divergent.

Alternatively we can show the sequence is geometric by evaluating

$$\frac{a_{n+1}}{a_n} = \frac{3(-5)^{n+1}}{3(-5)^n} = -5$$

proving it has a constant (common) ratio of $r = -5$. The first term is $a = 3(-5)^1 = -15\,$.

3. Let $f(x) = \dfrac{2x + \ln x}{5x + 3\ln x}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x + \ln x}{5x + 3\ln x} \underset{\text{LH}}{=} \lim_{x\to\infty} \frac{2 + \frac{1}{x}}{5 + \frac{3}{\ln x}} = \frac{2+0}{5+0} = \frac{2}{5}$$

$$\implies \lim_{n\to\infty} \frac{2n + \ln n}{5n + 3\ln n} = \frac{2}{5}$$

Therefore the sequence is convergent.

Further Questions:

Determine whether the following sequences are divergent or convergent. For convergent sequences determine the limit.

1. $\left\{\left(\dfrac{n+1}{8n}\right)^{\frac{1}{3}}\right\}$
2. $\left\{5\left(\dfrac{1}{2}\right)^{n}\right\}$
3. $\{2^n\}$
4. $\left\{\left(-\dfrac{1}{3}\right)^{n}\right\}$
5. $\{(-3)^n\}$

A special class of sequences are those that are **monotonic**.

Definition: A sequence is **monotonic** if it is either

increasing : $a_n < a_{n+1}$ for all n, or
decreasing : $a_n > a_{n+1}$ for all n, or
nondecreasing : $a_n \leq a_{n+1}$ for all n, or
nonincreasing : $a_n \geq a_{n+1}$ for all n.

Note that increasing sequences are, by definition, nondecreasing as are decreasing sequences nonincreasing. An example of a nondecreasing sequence that is not an increasing sequence is

$$\{1, 1, 2, 2, 3, 3, 4, 4, \ldots\}$$

Example 4-8

Classify the monotonicity of the following sequences.

1. $\left\{\dfrac{n^2+3}{n+5}\right\}$
2. $\left\{\sqrt{n+2}-\sqrt{n}\right\}$

Solution:

1. We illustrate two approaches:
   - Method 1: Using the Definition

$$
\begin{aligned}
a_n &< a_{n+1} \quad (n \geq 1) \\
\iff \frac{n^2+3}{n+5} &< \frac{(n+1)^2+3}{(n+1)+5} \quad (n \geq 1) \\
\iff \frac{n^2+3}{n+5} &< \frac{n^2+2n+4}{n+6} \quad (n \geq 1) \\
\iff (n^2+3)(n+6) &< (n^2+2n+4)(n+5) \quad (n \geq 1) \\
\iff n^3+6n^2+3n+18 &< n^3+5n^2+2n^2+10n+4n+20 \quad (n \geq 1) \\
\iff 0 &< n^2+11n+2 \quad (n \geq 1)
\end{aligned}
$$

   Since the final statement is true we have proven that $a_n < a_{n+1}$ and thus $\left\{\dfrac{n^2+3}{n+5}\right\}$ is an increasing sequence.

   - Method 2: Using the Derivative

   Consider $f(x) = \dfrac{x^2+3}{x+5}$. Then

$$
f'(x) = \frac{2x(x+5)-(x^2+3)(1)}{(x+5)^2} = \frac{2x^2+10x-x^2-3}{(x+5)^2} = \frac{x^2+10x-3}{(x+5)^2} > 0 \text{ for } x \geq 1\,.
$$

   Therefore, $f(x)$ is an increasing function for $x \geq 1$. Thus $\left\{\dfrac{n^2+3}{n+5}\right\}$ is an increasing sequence.

   Since the sequence is increasing it is also nondecreasing.

2. Considering $f(x) = \sqrt{x+2}-\sqrt{x}$ we have

$$
\begin{aligned}
f'(x) &= \frac{1}{2}(x+2)^{-\frac{1}{2}}(1) - \frac{1}{2}x^{-\frac{1}{2}} \\
&= \frac{1}{2\sqrt{x+2}} - \frac{1}{2\sqrt{x}} < 0 \text{ for } x \geq 1\,.
\end{aligned}
$$

   Therefore $f(x)$ is a decreasing function for $x \geq 1$. Thus $\left\{\sqrt{n+2}-\sqrt{n}\right\}$ is a decreasing sequence, and also, therefore, a nonincreasing sequence.

Further Questions:

Classify the monotonicity of the following sequences.

1. $\left\{\dfrac{2}{n+3}\right\}$
2. $\left\{\dfrac{2n+3}{3n+5}\right\}$
3. $\left\{\dfrac{3n+2}{2n+1}\right\}$

Definition: Sequence $\{a_n\}$ is **bounded below** if there exists some N such that $N \leq a_n$ for all n. The sequence $\{a_n\}$ is **bounded above** if there exists some M such that $a_n \leq M$ for all n. The sequence $\{a_n\}$ is **bounded** if it is bounded both below and above.

Example 4-9

Determine whether the following sequences are bounded above, bounded below, or bounded.

1. $\{e^n\}$
2. $\{-n^4\}$
3. $\left\{\dfrac{2n^2+1}{n^2+3}\right\}$

Solution:

1. Since $e^n > 0$ for all $n \geq 1$ the sequence is bounded below. $\lim\limits_{n\to\infty} e^n = \infty$ shows that it is not bounded above.
2. Since $-n^4 < 0$ for all $n \geq 1$ the sequence is bounded above. $\lim\limits_{n\to\infty} -n^4 = -\infty$ shows that it is not bounded below.
3. Clearly $\dfrac{2n^2+1}{n^2+3} > 0$ for all n ≥ 1 so the sequence is bounded below. Considering the limit of the sequence

$$\lim_{n\to\infty} \frac{2n^2+1}{n^2+3} = \lim_{n\to\infty} \frac{n^2}{n^2} \cdot \frac{\frac{2n^2}{n^2} + \frac{1}{n^2}}{\frac{n^2}{n^2} + \frac{3}{n^2}} = \lim_{n\to\infty} 1 \cdot \frac{2 + \frac{1}{n^2}}{1 + \frac{1}{n^2}} = \frac{2+0}{1+0} = 2$$

suggests it may have an upper bound around 2. The inequality

$$\frac{2n^2+1}{n^2+3} < 2 \iff 2n^2+1 < 2(n^2+3) \iff 0 < 5$$

confirms 2 is an upper bound for the sequence. Thus $0 < \dfrac{2n^2+1}{n^2+3} < 2$ and the sequence is bounded.

Further Questions:

Determine whether the following sequences are bounded above, bounded below, or bounded.

1. $\{n^2\}$
2. $\left\{\dfrac{n}{n+1}\right\}$

The following theorem can be used to determine whether a monotonic sequence converges or not.

Theorem 4-6: A monotonic sequence converges if and only if it is bounded.

Answers: Page 267

Exercise 4-1

1-3: Find the first four terms of the infinite sequence and evaluate $\lim_{n\to\infty} a_n$ if it exists.

1. $a_n = \dfrac{n^3 + 3n^2}{2n^3 + 5}$
2. $a_n = \dfrac{2e^n}{3e^n + 1}$
3. $a_n = (-1)^n \dfrac{n^2 + 3}{n^3 + 2n + 4}$

4-7: Evaluate $\lim_{n\to\infty} a_n$ if it exists and determine whether the infinite sequence is convergent or divergent.

4. $a_n = \dfrac{e^n}{2e^n + n}$
5. $a_n = \dfrac{\tan^{-1} n}{n}$
6. $a_n = \dfrac{\ln n}{3n}$
7. $a_n = (-1)^n \dfrac{n+1}{n^2 + 5}$

8-10: Determine whether the infinite sequence is increasing, decreasing or not monotonic.

8. $a_n = ne^{-n^2}$
9. $a_n = \dfrac{n^2 + 3}{n^2 + 5}$
10. $a_n = \dfrac{e^n}{e^n + 2}$

4.2 Series

Definition: Given a sequence $\{a_k\}$, the summation of its terms,

$$a_1 + a_2 + a_3 + \cdots + a_k + \cdots$$

is called an **(infinite) series**. A series is abbreviated in **sigma notation** as $\sum_{k=1}^{\infty} a_k$ or sometimes without explicit index limits as $\sum a_k$.

Due to the sum being over an infinite number of terms it need not exist.

Example 4-10

The series

$$\sum_{k=1}^{\infty} 1 = 1 + 1 + \cdots + 1 + \cdots$$

clearly cannot approach a number when added.

To rigorously define what we mean by the value of the sum of a series we introduce the following.

Definition: Given the series $\sum_{k=1}^{\infty} a_k$ define the sum of the first n terms of the series to be the **n^{th} partial sum S_n**:

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

Example 4-11

For the series $1 + 1 + 1 + \cdots + 1 + \cdots$ the n^{th} partial sums are

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + 1 = 2 \\ S_3 &= 1 + 1 + 1 = 3 \\ &\vdots \\ S_n &= \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ times}} = n \end{aligned}$$

The partial sums S_n for a series $\sum a_k$ themselves form a sequence $\{S_n\}$ the limit of which we will consider the sum of the series.

Definition: Let series $\sum a_k$ have n^{th} partial sums S_n. If the sequence $\{S_n\}$ is convergent, so

$$\lim_{n\to\infty} S_n = S \ ,$$

then we say that the series $\sum a_k$ is **convergent** and call S the **sum** of the series,

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots + a_k + \cdots = S \ .$$

If a series is not convergent then it is **divergent** .

Example 4-12

The series $\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$ can be written using partial fraction decomposition as $\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)$. The n^{th} partial sum is therefore

$$S_n = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{1}{2}-\frac{1}{n+2}$$

Since $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right)=\frac{1}{2}$ the series is convergent with sum 1/2, i.e.

$$\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+2)}=\frac{1}{2}\,.$$

Because of the cancellation arising in the partial sum the series is called a **telescoping series**. The series collapses like a spyglass in which the tubes disappear into each other to shorten the telescope.

Example 4-13

We saw the series $\sum_{k=1}^{\infty} 1$ has partial sum $S_n = n$. Therefore $\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$ and so the sequence $\{S_n\}$ and hence the series $\sum_{k=1}^{\infty} 1$ are divergent (as expected).

Theorem 4-7: The **geometric series**

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots + ar^k + \cdots = a\left(1+r+r^2+\cdots+r^k+\cdots\right)$$

is convergent if $-1 < r < 1$ with sum $\frac{a}{1-r}$ and is otherwise divergent.[3]

One notes that the geometric series also has the form $\sum_{k=1}^{\infty} ar^{k-1}$ if the lower limit is taken to be $k=1$ rather than $k=0$.

Example 4-14

Determine whether the following series are convergent and, if so, find the sum.

1. $2+6+18+54+\cdots$
2. $\sum_{k=0}^{\infty}\frac{(-1)^k 6^{k+1}}{5^k}$
3. $\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$

[3] Proof follows by noting that if $S_n = 1 + ar + \cdots + ar^{n-1}$ then

$$(1-r)S_n = a + ar + ar^2 + \cdots + ar^{n-1} - \left(ar + ar^2 + ar^3 + \cdots + ar^n\right) = a - ar^n,$$

and so $S_n = \frac{a(1-r^n)}{1-r}$. Then $S = \lim_{n\to\infty} S_n = \frac{a}{1-r}$ since $\lim_{n\to\infty} r^n = 0$ for $-1 < r < 1$.

Solution:

1. The series has first term $a = 2$ and the ratios of subsequent terms

$$\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = \cdots$$

all equal to 3 shows the series is geometric with common ratio $r = 3$. Since $|r| = 3 > 1$ the geometric series is divergent. Alternatively we note that we can write the series in sigma notation as $\sum_{k=0}^{\infty} 2(3)^k$ or $\sum_{k=1}^{\infty} 2(3)^{k-1}$ to prove it is geometric and to identify $a = 2$ and $r = 3$.

2. The series starts at $k = 0$ and so the standard form for that geometric series is $\sum_{k=0}^{\infty} ar^k$. Rewriting the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k 6^{k+1}}{5^k} = \sum_{k=0}^{\infty} 6 \frac{(-1)^k 6^k}{5^k} = \sum_{k=0}^{\infty} 6 \left(-\frac{6}{5}\right)^k$$

proves the series is geometric with first term $a = 6$ and common ratio $r = -\frac{6}{5}$. Since $|r| = \frac{6}{5} > 1$ the geometric series is divergent.

3. The series starts at $n = 1$ so put it in the standard form $\sum_{n=1}^{\infty} ar^{n-1}$:

$$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^{n-1} \implies a = \frac{1}{4},\ r = \frac{1}{4}$$

Then $|r| < 1$ implies the series converges to sum

$$S = \frac{a}{1-r} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4} \cdot \frac{4}{3} = \frac{1}{3}$$

Further Questions:

Determine whether the following series are convergent and, if so, find the sum.

1. $2 + \frac{2}{2} + \frac{2}{4} + \cdots + 2\left(\frac{1}{2}\right)^{n-1} + \cdots$

2. $\sum_{n=1}^{\infty} (3)^{n-1}$

3. $1 + x + x^2 + x^3 + \cdots$ (for $|x| < 1$)

A necessary (but not sufficient) requirement for a series to converge is that its terms must approach zero as $n \to \infty$.

Theorem 4-8: If series $\displaystyle\sum_{k=1}^{\infty} a_k$ is convergent then $\displaystyle\lim_{k\to\infty} a_k = 0$.

Note the converse of the last theorem, namely that if $\displaystyle\lim_{k\to\infty} a_k = 0$ then $\displaystyle\sum_{k=1}^{\infty} a_k$ is convergent, is **not** true, as demonstrated in the following example.

Example 4-15

The **harmonic series**,

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots$$

is divergent. To see this note that the harmonic series

$$1 \;+\; \frac{1}{2} \;+\; \frac{1}{3} + \frac{1}{4} \;+\; \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \;+\; \frac{1}{9} + \cdots$$

is strictly greater than

$$\underbrace{\frac{1}{1}}_{>\frac{1}{2}} + \underbrace{\frac{1}{2}}_{=\frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{=\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{=\frac{1}{2}} + \frac{1}{16} + \cdots,$$

which diverges to infinity as one can exceed any multiple of 1/2 one wants by taking enough terms. Specifically, the partial sums of the harmonic series form an increasing sequence in which $S_{2^n} > \frac{1}{2}(n+1)$ and therefore the sequence $\{S_n\}$ is unbounded (and hence divergent).

The contrapositive of Theorem 4-8 (which logically must be true) provides a useful method to test if some series are divergent:

Theorem 4-9: The Term Test for Divergence:

If the terms a_k of series $\sum a_k$ approach a non-zero limit $\left(\displaystyle\lim_{k\to\infty} a_k = L \neq 0\right)$ or $\displaystyle\lim_{k\to\infty} a_k$ does not exist, then $\sum a_k$ is divergent.

We note that by Theorem 4-2 it follows that if $\displaystyle\lim_{k\to\infty} |a_k| \neq 0$ or does not exist then $\displaystyle\lim_{k\to\infty} a_k$ must either be non-zero or not exist and vice versa. As such, when applying the Term Test for Divergence, it is sufficient to test $\displaystyle\lim_{k\to\infty} |a_k|$ which may be easier to evaluate.

Example 4-16

Determine whether the series

$$\sum_{k=1}^{\infty} \cos\left(\frac{1}{k}\right)$$

converges or diverges.

Solution:

Let $a_k = \cos\left(\frac{1}{k}\right)$ be the k^{th} term in the series. Consider $\cos\left(\frac{1}{x}\right)$, a continuous function of x. Then

$$\lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \left(\lim_{x\to\infty} \frac{1}{x}\right) = \cos(0) = 1$$

implies, by Theorem 4-1, $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \cos\left(\frac{1}{k}\right) = 1 \neq 0$. By the Term Test for Divergence the series $\sum a_k$ is divergent.

Further Question:

Determine whether the series

$$\frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \cdots + \frac{k}{2k+1} + \cdots$$

converges or diverges.

Note that if we find $\lim_{n\to\infty} a_k = 0$ we know nothing about the convergence or divergence of series $\sum a_k$.

Theorem 4-10: If c is any constant and $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ are convergent series then the following series are convergent with the given results:

1. $\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k$
2. $\sum_{k=1}^{\infty}(a_k \pm b_k) = \sum_{k=1}^{\infty} a_k \pm \sum_{k=1}^{\infty} b_k$

If $\sum_{k=1}^{\infty} a_k$ is divergent and $c \neq 0$ then $\sum_{k=1}^{\infty} ca_k$ is divergent. If one of $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ is convergent and one is divergent then $\sum_{k=1}^{\infty}(a_k \pm b_k)$ is divergent.

Example 4-17

Prove that the following series converges and find its sum.

$$\sum_{i=1}^{\infty}\left(\frac{1}{5^i} + \frac{3}{10^i}\right)$$

Solution:

$$\begin{aligned}
\sum_{i=1}^{\infty}\left(\frac{1}{5^i} + \frac{3}{10^i}\right) &= \sum_{i=1}^{\infty}\frac{1}{5^i} + \sum_{i=1}^{\infty}\frac{3}{10^i} \\
&= \underbrace{\sum_{i=1}^{\infty}\frac{1}{5}\cdot\left(\frac{1}{5}\right)^{i-1}}_{\substack{\text{geometric series} \\ a=\frac{1}{5},\, r=\frac{1}{5} \\ |r|<1 \text{ (convergent)}}} + \underbrace{\sum_{i=1}^{\infty}\frac{3}{10}\cdot\left(\frac{1}{10}\right)^{i-1}}_{\substack{\text{geometric series} \\ a=\frac{3}{10},\, r=\frac{1}{10} \\ |r|<1 \text{ (convergent)}}} \\
&= \frac{\frac{1}{5}}{1-\frac{1}{5}} + \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{\frac{1}{5}}{\frac{4}{5}} + \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{1}{5}\cdot\frac{5}{4} + \frac{3}{10}\cdot\frac{10}{9} = \frac{1}{4} + \frac{1}{3} = \frac{3+4}{12} = \frac{7}{12}
\end{aligned}$$

Thus the given series is convergent with sum $\frac{7}{12}$.

Further Question:

Prove that the following series converges and find its sum.

$$\sum_{k=1}^{\infty}\left[\frac{2}{3^{k-1}}+\frac{7}{k(k+1)}\right]$$

Example 4-18

Determine whether each series is convergent or divergent. For convergent series, find its sum.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2+5}{6n^2+3n+1}$
2. $\displaystyle\sum_{n=1}^{\infty}\frac{4}{(n+1)(n+3)}$
3. $\displaystyle\sum_{n=2}^{\infty}\left[\left(\frac{3}{4}\right)^n+\frac{n}{\ln n}\right]$

Solution:

1. Let a_n be the n^{th} term in the series and consider $\lim\limits_{n\to\infty} a_n$:

$$\lim_{n\to\infty}\frac{2n^2+5}{6n^2+3n+1}=\frac{n^2}{n^2}\cdot\lim_{n\to\infty}\frac{\frac{2n^2}{n^2}+\frac{5}{n^2}}{\frac{6n^2}{n^2}+\frac{3n}{n^2}+\frac{1}{n^2}}=\lim_{n\to\infty}1\cdot\frac{2+\frac{5}{n^2}}{6+\frac{3}{n}+\frac{1}{n^2}}$$
$$=\frac{2+0}{6+0+0}=\frac{2}{6}=\frac{1}{3}\neq 0$$

Therefore the given series $\sum a_n$ is divergent by the Term Test for Divergence.

2. Using partial fraction decomposition, the series can be written as

$$\sum_{n=1}^{\infty}\frac{4}{(n+1)(n+3)}=\sum_{n=1}^{\infty}\left(\frac{2}{n+1}-\frac{2}{n+3}\right).$$

The n^{th} partial sum may be evaluated due to the telescoping terms:

$$S_n=\left(\frac{2}{2}-\frac{2}{4}\right)+\left(\frac{2}{3}-\frac{2}{5}\right)+\left(\frac{2}{4}-\frac{2}{6}\right)+\left(\frac{2}{5}-\frac{2}{7}\right)+\ldots+\left(\frac{2}{n+1}-\frac{2}{n+3}\right)$$
$$=\frac{2}{2}+\frac{2}{3}-\frac{2}{n+3}=1+\frac{2}{3}-\frac{2}{n+3}=\frac{5}{3}-\frac{2}{n+3}$$

Taking the limit gives

$$S=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(\frac{5}{3}-\frac{2}{n+3}\right)=\frac{5}{3}-0=\frac{5}{3},$$

so the given series is convergent with sum $\frac{5}{3}$.

Note that the series $\sum\frac{2}{n+1}$ and $\sum\frac{2}{n+3}$ are both separately divergent since they are, up to a scalar multiplier and a few missing initial terms, both harmonic series. So separating a series into two divergent components does not imply the original series is divergent.

3. Consider the series as a sum of two series:

$$\sum_{n=2}^{\infty}\left[\left(\frac{3}{4}\right)^n+\frac{n}{\ln n}\right]=\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n+\sum_{n=2}^{\infty}\frac{n}{\ln n}.$$

The first series is geometric with $r = \frac{3}{4}$. Since $|r| < 1$ that series converges. However the second series diverges since

$$\lim_{x\to\infty} \frac{x}{\ln x} \underset{\text{LH}}{=} \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty\,,$$

which implies $\lim_{n\to\infty} \frac{n}{\ln n} = \infty$ and the second series diverges by the Term Test for Divergence. Since the original series is the sum of a convergent and divergent series, it is divergent.

Further Questions:

Determine whether each series is convergent or divergent. For convergent series, find the sum.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{3k}{5k-1}$
2. $\displaystyle\sum_{k=1}^{\infty} k!$
3. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{3^k} - \frac{1}{4^k}\right)$
4. $\displaystyle\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} + \cdots$
5. $\displaystyle\sum_{k=1}^{\infty} \left[\left(\frac{3}{2}\right)^k + \left(\frac{2}{3}\right)^k\right]$
6. $\displaystyle\sum_{k=0}^{\infty} \frac{6^k}{7^{k+1}}$
7. $\displaystyle\sum_{n=3}^{\infty} \ln\left(\frac{2n}{3n-7}\right)$

Here the factorial $k! = k \cdot (k-1) \cdots (1)$ for $k \geq 1$ (and $0!$ is defined to be 1).

Exercise 4-2

Answers: Page 267

1-11: Determine whether the infinite series converges or diverges. If it converges, find its sum.

1. $\displaystyle 3 + \frac{3}{2} + \frac{3}{2^2} + \cdots + \frac{3}{2^{n-1}} + \cdots$
2. $\displaystyle 1 + \frac{3}{e} + \left(\frac{3}{e}\right)^2 + \cdots + \left(\frac{3}{e}\right)^{n-1} + \cdots$
3. $\displaystyle\sum_{n=1}^{\infty} 2^{-n}3^{n-1}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{5^{n-1}}$
5. $\displaystyle\sum_{n=1}^{\infty} (-3)^{1-n}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n}$
7. $\displaystyle\sum_{n=1}^{\infty} \left[\frac{1}{2^n} + \frac{5}{n(n+1)}\right]$
8. $\displaystyle\sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{2}{n(n+1)}\right]$
9. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{3n^2+4}{2n^2+1}\right)$
10. $\displaystyle\sum_{n=1}^{\infty} \tan^{-1} n$
11. $\displaystyle\sum_{k=1}^{\infty} \left[\sin\left(\frac{1}{k}\right) - \sin\left(\frac{1}{k+1}\right)\right]$

4.3 Testing Series with Positive Terms

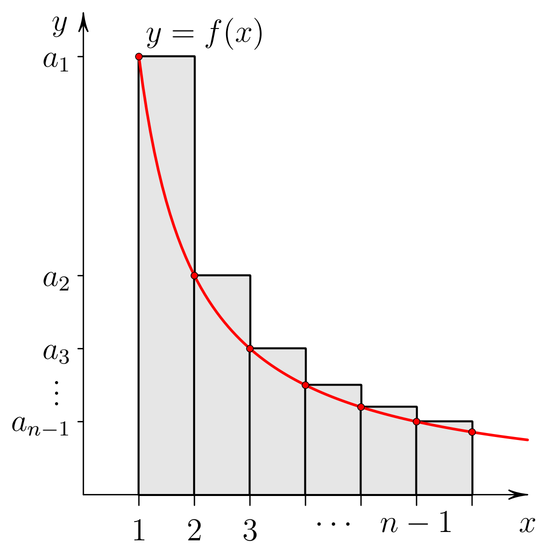
It is often difficult to find an exact sum of a series. In most cases a simple formula for the partial sum S_n cannot be found. It is therefore of interest to develop techniques to test whether a series is convergent or divergent. We start by considering series $\sum a_k$ with positive ($a_k > 0$) terms. Because the terms are positive the sequence of partial sums, $\{S_n\}$, is increasing.

4.3.1 The Integral Test

Suppose a series $\sum_{k=1}^{\infty} a_k$ has terms $a_k = f(k)$ written in terms of a function $f(x)$ that is continuous, positive, and decreasing for $x \geq 1$. The integral $\int_1^n f(x)\,dx$ will be smaller than the partial sum S_{n-1},

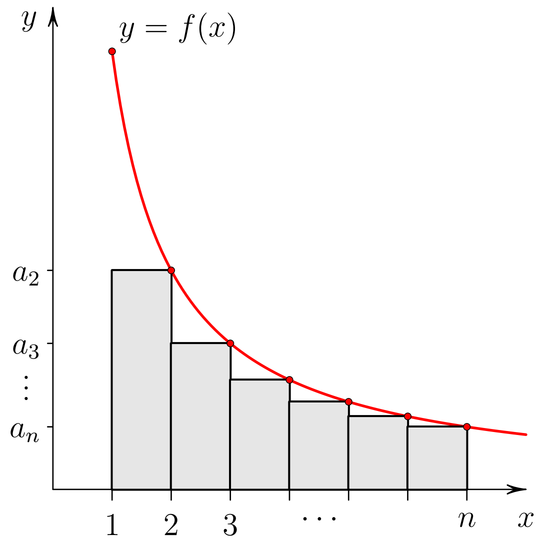
$$S_{n-1} = \sum_{k=1}^{n-1} a_k = a_1 + a_2 + \cdots + a_{n-1} = (a_1)(1) + (a_2)(1) + \cdots + (a_{n-1})(1)\,,$$

since the latter can be considered the total area of rectangles of height a_k and width 1 for $k = 1$ to $k = n-1$ as shown in the following diagram:



Here we are considering the a_k to be the height on the left side of the rectangles. Consider the case that $\int_1^{\infty} f(x)\,dx$ is divergent. Since $f(x)$ is positive, $\int_1^t f(x)\,dx$ is an increasing function of t and $\int_1^{\infty} f(x)\,dx = +\infty$. Suppose that increasing sequence $\{S_n\}$ were bounded with upper bound M. Then the relationship $\int_1^n f(x)\,dx < S_{n-1}$ implies that $\int_1^n f(x)\,dx < M$ for any integer n and therefore $\int_1^t f(x)\,dx < M$ for any real $t \geq 1$, a contradiction to the divergence of $\int_1^{\infty} f(x)\,dx$. Hence $\{S_n\}$ must be an unbounded monotonic sequence and therefore is divergent. Thus if $\int_1^{\infty} f(x)\,dx$ is divergent then $\sum a_k$ is divergent.

Alternatively if we consider rectangles with height being a_k on the right (so $k = 2$ to $k = n$) we have the following diagram:



In this case it follows that the integral $\int_1^n f(x)\,dx$ must be greater than:

$$(a_2)(1) + (a_3)(1) + \cdots + (a_n)(1) = a_2 + a_3 + \cdots + a_n = S_n - a_1$$

Consider the case where $\int_1^\infty f(x)\,dx$ is convergent. Let M be the value of the integral. Since $f(x)$ is positive, $\int_1^n f(x)\,dx < M$. From $S_n - a_1 < \int_1^n f(x)\,dx$ it follows that for any n, $S_n < M + a_1$ and thus monotonic sequence $\{S_n\}$ is bounded and therefore convergent. Thus if $\int_1^\infty f(x)$ is convergent, then $\sum a_k$ is convergent.

We summarize our result in the following theorem.

Theorem 4-11: The Integral Test:

Let $f(x)$ be a continuous positive decreasing function for $x \geq 1$ and $\sum_{k=1}^{\infty} a_k$ be a series with $a_k = f(k)$.

1. If $\int_1^\infty f(x)\,dx$ is convergent then $\sum_{k=1}^{\infty} a_k$ is also convergent.

2. If $\int_1^\infty f(x)\,dx$ is divergent then $\sum_{k=1}^{\infty} a_k$ is also divergent.

Example 4-19

Determine whether the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+16}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$

Solution:

1. Setting $f(x) = \dfrac{1}{x^2+16}$ we have that $f(x)$ is continuous and positive for $x \geq 1$. Also

$$f'(x) = \frac{-2x}{(x^2+16)^2} < 0 \text{ for } x > 0$$

shows $f(x)$ is decreasing for $x \geq 1$.

$$\int_1^\infty \frac{1}{x^2+16}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+16}\,dx = \lim_{t\to\infty}\left[\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)\right|_1^t$$
$$= \frac{1}{4}\lim_{t\to\infty}\left[\tan^{-1}\left(\frac{t}{4}\right) - \tan^{-1}\left(\frac{1}{4}\right)\right] = \frac{1}{4}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{4}\right)\right]$$

Integral $\int_1^\infty \frac{1}{x^2+16}\,dx$ is convergent and therefore the series $\sum_{n=1}^{\infty}\frac{1}{n^2+16}$ is convergent by the Integral Test.

2. Let $f(x) = \frac{(\ln x)^2}{x}$. Then $f(x)$ is continuous and positive for $x \geq 1$. Consider the derivative:

$$f'(x) = \frac{2(\ln x)\frac{1}{x}\cdot x - (\ln x)^2(1)}{x^2} = \frac{2(\ln x) - (\ln x)^2}{x^2} = \frac{(\ln x)(2-\ln x)}{x^2}$$

For $x \geq 2$ both $\ln x$ and x^2 are positive. Solve the inequality for the remaining factor to see where it decreases:

$$2 - \ln x < 0 \implies 2 < \ln x \implies e^2 < e^{\ln x} \implies e^2 < x \implies x > e^2 \approx 7.3891$$

(Note here we used that e^x is an increasing function so that the inequality was preserved under exponentiation.) Thus $f(x)$ is decreasing for $x > e^2$. Evaluating the improper integral from $x = 8$ onward gives:

$$\int_8^\infty \frac{(\ln x)^2}{x}\,dx = \lim_{t\to\infty}\int_8^t \frac{(\ln x)^2}{x}\,dx \quad (\leftarrow u = \ln x \text{ so } du = \frac{1}{x}dx)$$
$$= \lim_{t\to\infty}\int_{\ln 8}^{\ln t} u^2\,du = \lim_{t\to\infty}\left[\frac{1}{3}u^3\right|_{\ln 8}^{\ln t} = \lim_{t\to\infty}\left[\frac{1}{3}(\ln t)^3 - \frac{1}{3}(\ln 8)^3\right] = \infty$$

Therefore the improper integral is divergent and the series $\sum_{n=8}^{\infty}\frac{(\ln n)^2}{n}$ must also diverge by the Integral Test. The original series $\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n}$ differs from the latter by a finite sum of seven terms and so it also diverges.

In practice, when applying the Integral Test and, indeed, other series convergence tests in this chapter, we need only apply the test to the infinite tail of the sequence starting at any finite index (like 8 here). The requirements of the theorem have to only be met from that index forward. The convergence (or not) of a series can only depend on the tail of the series as the initial sum of a finite number of terms is necessarily finite and cannot, therefore, impact the convergence properties of the series as a whole.

Further Questions:

Determine whether each of the following series is convergent or divergent.

1. $\sum_{n=1}^{\infty} ne^{-n^2}$
2. $\sum_{k=1}^{\infty}\frac{1}{k}$ (The harmonic series)
3. $\sum_{k=1}^{\infty}\frac{1}{k^p}$ (The hyperharmonic or p-series)
4. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$
5. $\sum_{n=1}^{\infty}\frac{5}{\sqrt{n}}$
6. $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$

We summarize our results for the p-series from the Further Questions of the last example in the following theorem.

Theorem 4-12: The ***p*-series** $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^p}$ is convergent for $p>1$ and divergent otherwise.

Notes on the Integral Test:

1. The Integral Test can be relaxed to consider a function $f(x)$ that is continuous positive and decreasing only on $x \geq n$ with the corresponding integral $\int_n^\infty f(x)\,dx$. This determines convergence or not of the series $\sum_{k=n}^{\infty} a_k$ but this in turn determines convergence of the entire series $\sum_{k=1}^{\infty} a_k$ since these two series differ only by a finite number of terms having a finite sum.

2. It follows from the Integral Test that the improper integral and the series either are both convergent or both divergent. This means that determining the convergence or not of a series can be used to determine the convergence properties of the improper integral of a continuous positive decreasing function $f(x)$ if that were desired.

Estimating the Series Sum

Even if a series is convergent it may be impossible to sum due to the impossibility of finding a closed form for the partial sum S_n for which we can take the limit. In that case one may resort to numerically calculating the partial sum S_n itself, for "large" n as an approximation for the sum of the series, $S \approx S_n$. The error in the approximation is the **remainder** $R_n = S - S_n$ which is the sum of the terms that were not included:

$$S=\sum_{k=1}^{\infty}a_k=\underbrace{a_1+a_2+\cdots+a_n}_{S_n=\sum_{k=1}^{n}a_k}+\underbrace{a_{n+1}+a_{n+2}+\cdots}_{R_n=\sum_{k=n+1}^{\infty}a_k}=S_n+R_n$$

For a convergent series $\sum a_k$ with $a_k = f(k)$ where $f(x)$ is a continuous positive decreasing function one can place bounds on the size of the remainder, thereby estimating the error in the numerical approximation. In our previous discussion we found that the n^{th} partial sum satisfied

$$S_n - a_1 < \int_1^n f(x)\,dx < S_{n-1}$$

In the case of convergence these inequalities imply

$$S - a_1 \leq \int_1^\infty f(x)\,dx \leq S\ .$$

If we start summing at the n^{th} term instead of the first this generalizes to

$$(a_n + a_{n+1} + a_{n+2} + \cdots) - a_n \leq \int_n^\infty f(x)\,dx \leq a_n + a_{n+1} + a_{n+2} + \cdots\ ,$$

from which it follows that

$$R_n \leq \int_n^\infty f(x)\,dx \leq R_{n-1}\ .$$

Thus $\int_n^\infty f(x)\,dx$ is an upper bound for the error R_n. The substitution $n-1 \to n$ for the inequality on the right implies $\int_{n+1}^\infty f(x)\,dx \leq R_n$, thereby providing a lower bound on the remainder (error) as well. We summarize the result in the following theorem:

Theorem 4-13: For convergent series $\sum_{k=1}^{\infty} a_k$ with $a_k = f(x)$ where $f(x)$ is a continuous positive decreasing function, the remainder $R_n = \sum_{k=n+1}^{\infty} a_k = S - S_n$ satisfies:

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx \ ,$$

and therefore

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \leq S \leq S_n + \int_{n}^{\infty} f(x)\,dx \ .$$

Example 4-20

1. Estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+16}$ using the third partial sum S_3.
2. Find bounds on the error (the remainder R_3) that you are making by using that approximation.

Solution:

1. From Example 4-19 Problem 1 we know that the series is convergent. The sum of the series S is approximately:

$$S_3 = \sum_{n=1}^{3} \frac{1}{n^2+16} = \frac{1}{(1)^2+16} + \frac{1}{(2)^2+16} + \frac{1}{(3)^2+16} = \frac{1}{17} + \frac{1}{20} + \frac{1}{25} = \frac{253}{1700} \approx 0.1488$$

2. $S = S_3 + R_3$ where R_3 satisfies

$$\int_{3+1}^{\infty} \frac{1}{x^2+16} \leq R_3 \leq \int_{3}^{\infty} \frac{1}{x^2+16} \,.$$

Evaluate the integrals as we did in Example 4-19 .

$$\int_{4}^{\infty} \frac{1}{x^2+16}\,dx = \ldots = \frac{1}{4}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{4}{4}\right)\right] = \frac{1}{4}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{\pi}{16} \approx 0.1963$$
$$\int_{3}^{\infty} \frac{1}{x^2+16}\,dx = \ldots = \frac{1}{4}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{3}{4}\right)\right] \approx 0.2318$$

Thus $0.1963 < R_3 < 0.2318$. Our estimate of 0.1488 is therefore too low by somewhere between 0.1963 and 0.2318. In other words, the actual sum S of the series must lie between $0.1488 + 0.1963 = 0.3451$ and $0.1488 + 0.2318 = 0.3806$.

Further Questions:

Leonhard Euler was able to show the sum of the p-series with $p = 2$ is

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \cdots = \frac{\pi^2}{6} = 1.644934\ldots$$

1. Find S_4 .
2. Find the remainder R_4 .
3. Show R_4 falls within the bounds of the last theorem.
4. What partial sum S_n is required to be in error less than 0.01?

4.3.2 The Basic Comparison Test

We have seen several examples of series with their associated convergence properties:

geometric: $\displaystyle\sum_{k=1}^{\infty} ar^{k-1}$ is convergent for $|r| < 1$, divergent for $|r| \geq 1$

telescoping: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ (for example) is convergent

harmonic: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent

***p*-series:** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent for $p > 1$, divergent for $p \leq 1$

We now develop some series convergence tests that use the known convergence properties of one series to determine that of another. The first test is a discrete analogue to our improper integral test found in Theorem 3-2.

Theorem 4-14: The Basic Comparison Test:
Let $\sum a_k$ and $\sum b_k$ be series with positive terms satisfying $a_k \leq b_k$ for all k.

1. If $\sum b_k$ is convergent then $\sum a_k$ is convergent.
2. If $\sum a_k$ is divergent then $\sum b_k$ is divergent.

Proof: Let $S_n = \sum_{k=1}^{n} a_k$ and $T_n = \sum_{k=1}^{n} b_k$ denote the partial sums of the series $\sum a_k$ and $\sum b_k$ respectively. Since $a_k \leq b_k$ it follows that $S_n \leq T_n$ for any n. Furthermore sequences $\{S_n\}$ and $\{T_n\}$ are both increasing (and hence monotonic) since terms a_k and b_k are positive.

Part 1 of the theorem follows from noting that monotonic sequence $\{T_n\}$ has upper bound $T = \sum_{k=1}^{\infty} b_k$ since the series $\sum b_k$, and hence the sequence $\{T_k\}$ converges. The sequence $\{S_n\}$ must also then have this upper bound since $S_n \leq T_n \leq T$. Thus $\{S_n\}$ is a monotonic bounded sequence and hence converges to S. Therefore $\sum a_k$ is convergent.

Part 2 follows from noting that if $\sum a_k$ is divergent then monotonic sequence $\{S_n\}$ is unbounded, which implies it has no upper bound as it is bounded below by 0. Now $S_n \leq T_n$ implies that monotonic sequence $\{T_n\}$ has no upper bound and hence does not converge, thereby proving $\sum b_k$ is divergent.

Note that the **Basic Comparison Test** is also known as the **Direct Comparison Test**.

Example 4-21

Determine whether the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{5^n + n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n (n+2)^2}{n^2 - 3}$

Solution:

1. The given series is $\displaystyle\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{4^n}{5^n + n}$ and is therefore a positive series.

Consider $\sum_{n=1}^{\infty} b_n$ where $b_n = \dfrac{4^n}{5^n} = \left(\dfrac{4}{5}\right)^n$. Then

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n = \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)\left(\frac{4}{5}\right)^{n-1}$$

shows $\sum b_n$ is a geometric series with $r = \dfrac{4}{5}$. Since $|r| < 1$ it is a convergent series. We now show that $a_n \leq b_n$ for $n \geq 1$:

$$\begin{aligned} a_n \leq b_n &\iff \frac{4^n}{5^n + n} \leq \frac{4^n}{5^n} \\ &\iff (4^n)(5^n) \leq (4^n)(5^n + n) \\ &\iff 0 \leq n(4^n) \quad \text{(which is true for } n \geq 1) \end{aligned}$$

Therefore, by the Basic Comparison Test, the given positive series $\sum a_n$ is convergent because it lies below a convergent series.

2. The given series is $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{4^n(n+2)^2}{n^2 - 3}$ and is therefore a positive series for $n \geq 2$.

   Noting that for large n both the polynomials in the numerator and denominator will be dominated by their quadratic (n^2) term, consider $b_n = \dfrac{4^n \cdot n^2}{n^2} = 4^n$. Then

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 4^n = \sum_{n=1}^{\infty} 4(4^{n-1})$$

   shows $\sum b_n$ is a geometric series with $a = 4$ and $r = 4 > 1$ and therefore is divergent. Now we must show that $a_n \geq b_n$ for $n \geq 2$:

$$\begin{aligned} a_n \geq b_n &\iff \frac{4^n(n+2)^2}{n^2-3} \geq 4^n \iff \frac{(n+2)^2}{n^2-3} \geq 1 \iff (n+2)^2 \geq n^2 - 3 \\ &\iff n^2 + 4n + 4 \geq n^2 - 3 \iff 4n + 7 \geq 0 \\ &\iff n \geq -\frac{7}{4} \text{ (which is true for } n \geq 2) \end{aligned}$$

   Therefore, by the Basic Comparison Test, the given series $\sum a_n$ is divergent because it lies above a divergent series that is ultimately positive (for $n \geq 2$). Once again one notes that the conditions of the test need only apply for the infinite tail of the given series and one can ignore a finite number of initial terms (here the first term $a_1 = -18$) as these do not affect the convergence of the series.

Further Questions:

Determine whether the series converges or diverges.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2 + 5^k}$
2. $\displaystyle\sum_{n=2}^{\infty} \frac{3}{\sqrt{n} - 1}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n3^n}$

Remainder Estimate

Note that if one uses convergent series $\sum b_k$ to show $\sum a_k$ is convergent by the Basic Comparison Test, it follows, since $a_k \le b_k$, that the remainder $R_n = \sum_{k=n+1}^{\infty} a_k$ is less than or equal to the remainder $\tilde{R}_n = \sum_{k=n+1}^{\infty} b_k$. Hence if we have an estimate for the size of the error $\tilde{R}_n \le \epsilon$ for series $\sum b_k$, this implies $R_n \le \epsilon$ for series $\sum a_k$.

4.3.3 The Limit Comparison Test

Our next test involves taking the limit of the ratio of terms of two series, one of whose convergence properties are presumed known.

Theorem 4-15: The Limit Comparison Test:

Let $\sum a_k$ and $\sum b_k$ be series with positive terms.

1. If $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = L > 0$ then both series are convergent or both divergent.
2. If $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = 0$ and $\sum b_k$ is convergent then $\sum a_k$ is convergent.
3. If $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = \infty$ and $\sum b_k$ is divergent then $\sum a_k$ is divergent.

The convergence conclusions of the Limit Comparison Test can be remembered by noting that the limit condition effectively suggests that the tail of the series satisfies $a_k = Lb_k$, in other words the series tail is effectively a multiple of that of the other series by a constant $c = L$. For $c = L \neq 0$ we saw in Theorem 4-10 that the new series has the same convergence properties as the original.

Proof: Consider the case where $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = L > 0$. Then there exists $\tilde{M} > 0$ and $\tilde{N} > 0$ such that

$$\tilde{M} < \frac{a_k}{b_k} < \tilde{N} \quad \text{for } k > n$$

Let M be the minimum of the finite set of numbers $\left\{ \frac{a_k}{b_k} | k \le n \right\}$ and the number $\tilde{M}$. Similarly let N be the maximum of the finite set of numbers $\left\{ \frac{a_k}{b_k} | k \le n \right\}$ and the number $\tilde{N}$. It follows that for all k:

$$M < \frac{a_k}{b_k} < N$$

Since $b_k > 0$ we have, for all k,

$$Mb_k < a_k < Nb_k \ .$$

If $\sum b_k$ converges then $\sum Nb_k$ converges and $a_k < Nb_k$ implies $\sum a_k$ converges by the Basic Comparison Test. Similarly if $\sum b_k$ diverges then $\sum Mb_k$ diverges and $Mb_k < a_k$ implies $\sum a_k$ diverges by the Basic Comparison Test.

In the case $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = 0$ we can only argue that $a_k/b_k < N$ and so only $\sum b_k$ convergent implies $\sum a_k$ convergent. In the case $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = \infty$ we can only argue that $M < a_k/b_k$ and so only $\sum b_k$ divergent implies $\sum a_k$ divergent.

Example 4-22

Determine whether the following series are convergent or divergent.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{5^n - 6n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 2\ln n}{n^4 + 3n^2 + 5}$

Solution:

1. The given series is $\displaystyle\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{4^n}{5^n - 6n}$. This is a positive series for $n \geq 2$ since the denominator becomes positive at $n = 2$ (equal to 13) and only increases (so remains positive) after that since $\frac{d}{dx}(5^x - 6x) = 5^x \ln 5 - 6 > 0$ when $x \geq 2$. Since convergence of a series depends only on its infinite tail we can identify a suitable comparison series by looking at the large n properties of a_n. We note that the denominator is dominated by the exponential 5^n which suggests considering $\sum_{n=1}^{\infty} b_n$ where $b_n = \frac{4^n}{5^n} = \left(\frac{4}{5}\right)^n$. Then

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n = \sum_{n=1}^{\infty} \frac{4}{5}\left(\frac{4}{5}\right)^{n-1}$$

is a geometric series with $a = \frac{4}{5}$ and $|r| = \left|\frac{4}{5}\right| = \frac{4}{5} < 1$ and therefore convergent.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{4^n}{5^n - 6n}}{\frac{4^n}{5^n}} = \lim_{n\to\infty} \frac{4^n}{5^n - 6n} \cdot \frac{5^n}{4^n} = \lim_{n\to\infty} \frac{5^n}{5^n - 6n}$$
$$= \lim_{n\to\infty} \frac{5^n}{5^n} \cdot \frac{\frac{5^n}{5^n}}{\frac{5^n}{5^n} - \frac{6n}{5^n}} = \lim_{n\to\infty} \frac{1}{1 - \frac{6n}{5^n}} = \lim_{n\to\infty} \frac{1}{1 - 0} = 1$$

Here we used L'Hôpital's Rule in evaluating the last step of the limit:

$$\lim_{x\to\infty} \frac{6x}{5^x} \underset{\text{LH}}{=} \lim_{x\to\infty} \frac{6}{5^x \ln 5} = \frac{6}{\infty} = 0$$

The latter limit shows that 5^n dominates $6n$ as claimed above and indicates why we factored out that term (and not $6n$) when evaluating our original limit. Therefore, since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$, by the Limit Comparison Test, the given series $\sum a_n$ is also convergent.

As an aside, note that this series is very similar to Problem 1 in Example 4-21. However attempting to use the Basic Comparison Test in the current example using the same comparison series $\sum b_n$ will fail to be conclusive since here, as the reader may confirm, $a_n > b_n$ for $n \geq 2$. In general the Limit Comparison Test will often work to establish convergence of a series where the Basic Comparison Test will not.

2. The given series is $\displaystyle\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{n^2 + 2\ln n}{n^4 + 3n^2 + 5}$ and therefore positive. Consideration of the limiting behaviour of the terms at large n suggests a comparison with the series $\sum b_n$ where $b_n = \frac{n^2}{n^4}$. Then

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p-series with $p = 2 > 1$ and therefore convergent. Taking the limit of the ratio of terms gives:

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{\frac{n^2+2\ln n}{n^4+3n^2+5}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{(n^2+2\ln n)n^2}{n^4+3n^2+5} = \lim_{n\to\infty}\frac{n^4+2n^2\ln n}{n^4+3n^2+5}\\ &= \lim_{n\to\infty}\frac{n^4}{n^4}\cdot\frac{\frac{n^4}{n^4}+\frac{2n^2\ln n}{n^4}}{\frac{n^4}{n^4}+\frac{3n^2}{n^4}+\frac{5}{n^4}} = \lim_{n\to\infty}\frac{1+\frac{2\ln n}{n^2}}{1+\frac{3}{n^2}+\frac{5}{n^4}} = \frac{1+\lim_{n\to\infty}\frac{2\ln n}{n^2}}{1+0+0}\\ &= 1+\lim_{n\to\infty}\frac{2\ln n}{n^2} = 1\end{aligned}$$

Here we evaluated the limit at the last step using L'Hôpital's Rule on the corresponding continuous function $f(x) = \dfrac{2\ln x}{x^2}$:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{2\ln x}{x^2} = \lim_{x\to\infty}\frac{2\cdot\frac{1}{x}}{2x} = \lim_{x\to\infty}\frac{1}{x^2} = 0$$

This limit confirms that the power function dominates the logarithm at large n as claimed above and why it is the power, and not the logarithm, which we factor out of the above limit.

Therefore $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = 1 > 0$ and the given series $\sum a_n$ is also convergent by the Limit Comparison Test.

As a final note, the Limit Comparison Test is seen to be superior to the Basic Comparison Test here again as the presence of the logarithm and fourth order polynomial in a_n would make evaluation of the required inequality in the latter test difficult.

Further Questions:

Determine whether the following series are convergent or divergent.

1. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt[3]{k^2+1}}$
2. $\displaystyle\sum_{n=1}^{\infty}\frac{3n^2+5n}{2^n\,(n^2+1)}$
3. $\displaystyle\sum_{k=1}^{\infty}\frac{1+2^k}{1+3^k}$
4. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2\ln n}$

Note that since convergence is entirely determined by the infinite tail of a series, we can relax the Basic and Limit Comparison Tests to require that they only have positive terms for $k > n$ for some fixed n and, in the case of the Basic Comparison Test, that additionally $a_k \leq b_k$ for $k > n$.

Answers:
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Exercise 4-3

1-10: Determine whether the infinite series is convergent or divergent.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2+3n^2}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[5]{\ln n}}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+5n}{5n^3+3}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{9n^4+5n}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^6}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{2+4^n}{3+5^n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{2n^3+5}}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{5+\cos 2n}{n^3}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+5)}}$

4.4 The Alternating Series Test

We now consider series where all the terms are not positive. An alternating series is a special case of such a series.

Definition: An **alternating series** is a series of either of the forms

$$a_1 - a_2 + a_3 - \cdots + (-1)^{k-1}a_k + \cdots = \sum_{k=1}^{\infty}(-1)^{k-1}a_k \ ,$$

$$-a_1 + a_2 - a_3 + \cdots + (-1)^k a_k + \cdots = \sum_{k=1}^{\infty}(-1)^k a_k \ ,$$

where a_k is positive for all k.

Theorem 4-16: The Alternating Series Test:
If an alternating series of the form $\sum_{k=1}^{\infty}(-1)^{k-1}a_k$ or $\sum_{k=1}^{\infty}(-1)^k a_k$ with $a_k > 0$ satisfies

1. $a_{k+1} \le a_k$ for all k ,
2. $\lim_{k\to\infty} a_k = 0$,

then the alternating series is convergent.

Note that $a_{k+1} \le a_k$ is equivalent to $a_{k+1} - a_k \le 0$ and $\dfrac{a_{k+1}}{a_k} \le 1$.

Proof: Suppose we have an alternating series of the form $\sum_{k=1}^{\infty}(-1)^{k-1}a_k$. Consider the even partial sums S_2, S_4,..., where, in general the $(2n)^{\text{th}}$ partial sum is S_{2n} for n a positive integer given by:

$$S_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \cdots + a_{2n-1} - a_{2n} \ .$$

Then grouping the terms in pairs one has

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) \cdots + (a_{2n-1} - a_{2n}) \ ,$$

where each term in parentheses is nonnegative since $a_k \ge a_{k+1}$. This implies $\{S_{2n}\}$ is a nondecreasing sequence ($S_2 \le S_4 \le S_6 \le \ldots \le S_{2n} \le \ldots$). The terms of the even partial sum S_{2n} may be regrouped as

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \ ,$$

where, once again, the terms in parentheses are positive. This shows $S_{2n} < a_1$ for all n. The monotonic sequence of even partial sums $\{S_n\}$ is bounded and hence has limit S.

The odd partial sums are S_{2n+1} for n a positive integer and may be written

$$S_{2n+1} = S_{2n} + a_{2n+1} \ .$$

Taking the limit of the odd partial sum sequence $\{S_{2n+1}\}$ gives

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty}(S_{2n} + a_{2n+1}) = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = S + 0 = S$$

Since both even and odd partial sum sequences approach the same limit S, the sequence $\{S_n\}$ approaches S as well and alternating sequence $\sum_{k=1}^{\infty}(-1)^{k-1}a_k$ is convergent. Since alternating sequence $\sum_{k=1}^{\infty}(-1)^k a_k = (-1)\sum_{k=1}^{\infty}(-1)^{k-1}a_k$ this completes the proof for the other possible case.

Example 4-23

The alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$$

is convergent since $a_k = \frac{1}{k}$ satisfies $a_{k+1} = \frac{1}{k+1} \le \frac{1}{k} = a_k$ and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0$.

Example 4-24

Determine whether the following alternating series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2+2}{n^4+1}$
2. $\displaystyle\sum_{n=2}^{\infty} (-1)^{n} \frac{n}{\ln n}$

Solution:

1. The given series is $\displaystyle\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ where $c_n = (-1)^{n-1} a_n$ and $a_n = \dfrac{n^2+2}{n^4+1} > 0$. The series is therefore an alternating series. Consider $f(x) = \dfrac{x^2+2}{x^4+1}$. Then

$$f'(x) = \frac{2x(x^4+1) - (x^2+2)(4x^3)}{(x^4+1)^2} = \frac{2x^5 + 2x - 4x^5 - 8x^3}{(x^4+1)^2} = \frac{-2x^5 - 8x^3 + 2x}{(x^4+1)^2}$$

implies $f'(x) < 0$ for $x \ge 1$ and hence $f(x)$ decreases there. This implies $f(n+1) \le f(n)$ for $n \ge 1$ and hence $a_{n+1} \le a_n$. Next consider the limit

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2+2}{x^4+1} \underset{\text{LH}}{=} \lim_{x\to\infty} \frac{2x}{4x^3} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$$

where we used L'Hôpital's Rule on the $\frac{\infty}{\infty}$ indeterminate form. Thus $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = 0$. Since both conditions are met, the series $\sum c_n$ is convergent by the Alternating Series Test.

2. The given series is $\displaystyle\sum_{n=2}^{\infty} c_n = \sum_{n=2}^{\infty} (-1)^{n} a_n$ where $c_n = (-1)^n a_n$ and $a_n = \dfrac{n}{\ln n} > 0$. The series is therefore an alternating series. Consider $f(x) = \dfrac{x}{\ln x}$. Then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\ln x} \underset{\text{LH}}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty$$

where we used L'Hôpital's Rule on the $\frac{\infty}{\infty}$ indeterminate form. Thus $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = \infty$. Since $\displaystyle\lim_{n\to\infty} a_n = \infty \ne 0$ we cannot draw any conclusions about the convergence or divergence of the given series using the Alternating Series Test. However since $a_n = |c_n|$ we can conclude by Theorem 4-2 that $\displaystyle\lim_{n\to\infty} c_n \ne 0$ or does not exist. Therefore, the given series $\sum c_n$ is divergent by the Term Test for Divergence.

As this example illustrates, in practice it makes sense, when applying the Alternating Series Test to series $\sum c_n$, to first determine whether the requirement that the limit of $a_n = |c_n|$

goes to zero holds. If it does not, then, as shown in this example, the series must diverge by the Term Test for Divergence. If the limit does go to zero, proceed to determine whether the other condition, that a_n decreases, is met or not. If it is met, then the series converges by the Alternating Series Test. If it is not met, then that test is inconclusive.

Further Questions:

Determine whether the following alternating series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n^2-3}$
2. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k}\frac{2k}{4k-3}$
3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{n}$
4. $\displaystyle\sum_{k=1}^{\infty}\cos(k\pi)\frac{3k^2+2}{2k^2+1}$

Remainder Estimate

An estimate of the error made when approximating the sum S of an alternating series with the n^{th} partial sum S_n is given by the following theorem.

Theorem 4-17: For an alternating series with terms of absolute value $a_k > 0$ the remainder $R_n = S - S_n$ satisfies $|R_n| < a_{n+1}$.

Exercise 4-4

Answers: Page 267

1-10: Determine whether the infinite series is convergent or divergent.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n+1}{n^2+5}$
2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\ln\left(1+\frac{1}{n}\right)$
3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\ln\left(\frac{5n^2+2}{4n^2+3}\right)$
4. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{e^{2n}}{n^2}$
5. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+5}{4^n}$
6. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3n+2}{\sqrt{n}+1}$
7. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{3n+10}}{5n+7}$
8. $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^n}{n^3}$
9. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{(\ln n)^2}$
10. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

4.5 Tests of Absolute Convergence

For series $\sum a_k$ whose terms have mixed sign, one can consider the convergence properties of the series $\sum |a_k|$ with nonnegative terms generated by taking the absolute value of the terms of the original series.

4.5.1 Absolute Convergence

Definition: A series $\sum a_k$ is **absolutely convergent** if the series $\sum |a_k|$ is convergent.

A series may be convergent that is not absolutely convergent, prompting the following definition.

Definition: A series $\sum a_k$ that is convergent but not absolutely convergent is called **conditionally convergent** .

Example 4-25

The alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent because it converges but the harmonic series, which is the series of the absolute values of its terms, does not.

The following theorem shows a convergent series can only be absolutely or conditionally convergent.

Theorem 4-18: If a series $\sum a_k$ is absolutely convergent then it is convergent.

The theorem also shows that convergence of some series $\sum a_k$ may be determined by considering convergence of $\sum |a_k|$.

We remind the reader of some properties of absolute value that will be frequently encountered:

$$|xy| = |x||y| \qquad \left|\frac{x}{y}\right| = \frac{|x|}{|y|} \qquad |x+y| \le |x| + |y|$$

and for $a > 0$:

$$\begin{aligned} |x| < a &\iff -a < x < a \\ |x| = a &\iff x = \pm a \\ |x| > a &\iff x < -a \text{ or } x > a \\ |x| = 0 &\iff x = 0 \end{aligned}$$

Example 4-26

Determine whether the following series are absolutely convergent or conditionally convergent.

1. $\displaystyle\sum_{n=1}^{\infty} (-1)^n e^{-n}$

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$

Solution:

1. Consider the series of absolute values of the given series $\displaystyle\sum_{n=1}^{\infty} (-1)^n e^{-n}$, that is

$$\sum_{n=1}^{\infty} \left|(-1)^n e^{-n}\right| = \sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{1}{e} \left(\frac{1}{e}\right)^{n-1} .$$

This is a geometric series with $a = \dfrac{1}{e}$ and $|r| = \left|\dfrac{1}{e}\right| = \dfrac{1}{e} < 1$ and therefore convergent. Thus the given series is absolutely convergent and therefore also convergent.

2. The given series is $\displaystyle\sum_{n=1}^{\infty} c_n$ where $c_n = (-1)^{n-1}\dfrac{\sqrt{n}}{n+1}$. Consider the series of absolute values $\sum a_n$ where $a_n = |c_n| = \dfrac{\sqrt{n}}{n+1}$. Consideration of the large-n behaviour of a_n suggests a comparison with series $\sum b_n$ where

$$b_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}\,.$$

Then $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} < 1$ and therefore divergent. Taking the limit

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{n}{n}\cdot\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1 > 0$$

shows the series of absolute values, $\displaystyle\sum_{n=1}^{\infty} a_n$, is also divergent by the Limit Comparison Test.

Returning to the original series we see that it is $\displaystyle\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$ with $a_n = \dfrac{\sqrt{n}}{n+1}$ and so is an alternating series. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\sqrt{n}}{n+1} = \lim_{n\to\infty}\frac{\sqrt{n}}{n}\cdot\frac{1}{1+\frac{1}{n}} = \frac{1}{\sqrt{n}}\cdot\frac{1}{1+\frac{1}{n}} = 0\cdot\frac{1}{1+0} = 0\,.$$

Also for $n \geq 1$ one has

$$\begin{aligned}
a_{n+1} \leq a_n \iff & \frac{\sqrt{n+1}}{n+2} \leq \frac{\sqrt{n}}{n+1} \\
\iff & (n+1)^{\frac{3}{2}} \leq \sqrt{n}(n+2) \quad (\Leftarrow \text{Cross-multiplying by positive terms}) \\
\iff & (n+1)^3 \leq n(n+2)^2 \quad (\Leftarrow \text{Squaring of positive sides preserves inequality}) \\
\iff & n^3 + 3n^2 + 3n + 1 \leq n^3 + 4n^2 + 4n \\
\iff & -n^2 - n + 1 \leq 0 \text{ (which is true for all } n \geq 1)
\end{aligned}$$

Therefore, the given series $\sum c_n$ is convergent by the Alternating Series Test. Since the series of its absolute values $\sum |c_n| = \sum a_n$ is divergent, the convergence of $\sum c_n$ is conditional.

Further Questions:

Determine whether the following series are absolutely convergent or conditionally convergent.

1. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^2}$
2. $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$
3. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt{k}}$

4.5.2 The Ratio Test

The following convergence test considers the limit of the ratio of terms within a series.

Theorem 4-19: The Ratio Test:

Suppose the ratio of consecutive terms of series $\sum_{k=1}^{\infty} a_k$ has limit

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L \ ,$$

then

1. If $L < 1$ the series $\sum a_k$ is absolutely convergent (and hence convergent).
2. If $L = 1$ the test is inconclusive.
3. If $L > 1$ the series $\sum a_k$ is divergent.

If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \infty$ then $\sum a_k$ is also divergent.

Note that if the Ratio Test is inconclusive ($L = 1$) this means that $\sum a_k$ is potentially absolutely convergent, conditionally convergent, or divergent.

The convergence conclusions of the Ratio Test can be remembered by noting that the limit condition effectively suggests that the tail of the series satisfies $a_{k+1} = La_k$, in other words it behaves like a geometric series with $r = L$. From this it follows $r = L < 1$ should converge and $r = L > 1$ should diverge.

Example 4-27

Determine whether the following series are absolutely convergent, conditionally convergent or divergent.

1. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{5^n}{n!(n+4)}$
2. $\sum_{n=1}^{\infty}(-1)^{n}\frac{2^n}{n^4}$
3. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$

Solution:

1. The series is $\sum_{n=1}^{\infty} a_n$ where $a_n = (-1)^{n-1}\frac{5^n}{n!(n+4)}$. The presence of the factorial suggests trying the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^n\cdot 5^{n+1}}{(n+1)!(n+5)}}{\frac{(-1)^{n-1}5^n}{n!(n+4)}}\right| = \lim_{n\to\infty}\frac{5^{n+1}}{(n+1)!(n+5)}\cdot\frac{n!(n+4)}{5^n}$$

$$= \lim_{n\to\infty}\frac{5^n\cdot 5}{(n+1)n!(n+5)}\cdot\frac{n!(n+4)}{5^n} \quad (\Leftarrow (n+1)! = (n+1)(n)\cdots(1) = (n+1)n!)$$

$$= \lim_{n\to\infty}\frac{5(n+4)}{(n+1)(n+5)} = \lim_{n\to\infty}\frac{5n+20}{n^2+6n+5} = \lim_{n\to\infty}\frac{n}{n^2}\cdot\frac{\frac{5n}{n}+\frac{20}{n}}{\frac{n^2}{n^2}+\frac{6n}{n^2}+\frac{5}{n^2}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\cdot\frac{5+\frac{20}{n}}{1+\frac{6}{n}+\frac{5}{n^2}} = 0\cdot\frac{5+0}{1+0+0} = 0 < 1$$

Therefore the given series is absolutely convergent by the Ratio Test.

2. The series is $\sum_{n=1}^{\infty} a_n$ where $a_n = (-1)^n \frac{2^n}{n^4}$. Applying the Ratio Test gives:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{2^{n+1}}{(n+1)^4}}{(-1)^n\frac{2^n}{n^4}}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)^4}\cdot\frac{n^4}{2^n} = \lim_{n\to\infty}\frac{2n^4}{(n+1)^4}\\ &= 2\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^4 = 2\left(\lim_{n\to\infty}\frac{n}{n+1}\right)^4 = 2\left(\lim_{n\to\infty}\frac{n}{n}\cdot\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right)^4\\ &= 2\left(\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}\right)^4 = 2\left(\frac{1}{1+0}\right)^4 = 2 > 1\end{aligned}$$

Therefore the given series is divergent by the Ratio Test.

3. The series is $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{\sqrt{n+1}}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{1}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{1} = \lim_{n\to\infty}\sqrt{\frac{n+1}{n+2}} = \sqrt{\lim_{n\to\infty}\frac{n+1}{n+2}}\\ &= \sqrt{\lim_{n\to\infty}\frac{n}{n}\cdot\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = \sqrt{\frac{1+0}{1+0}} = 1\end{aligned}$$

Therefore the Ratio Test is inconclusive and a different test is required. Since $a_n = \frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{1+\frac{1}{n}}}$ shows that $a_n \approx b_n = \frac{1}{\sqrt{n}}$ for large n, using the Limit Comparison Test with the divergent ($p = 1/2$) p-series $\sum b_n$ will show $\sum a_n$ is also divergent. Quite generally the Ratio Test (and also the Root Test introduced below) will always fail to be conclusive for any series that behaves like a p-series at large n.

Further Questions:

Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{k=1}^{\infty}(-1)^k\frac{3^k}{k!}$

2. $\sum_{n=1}^{\infty}\frac{4^n}{n^2}$

3. $\sum_{k=1}^{\infty}e^{-k}k!$

4. $\sum_{n=1}^{\infty}\frac{2^n}{2n^2+1}$

4.5.3 The Root Test

The next convergence test considers the limit of the k^{th} root of $|a_k|$.

Theorem 4-20: The Root Test:
Suppose the terms of series $\sum_{k=1}^{\infty} a_k$ satisfy

$$\lim_{k\to\infty}\sqrt[k]{|a_k|} = L\ ,$$

then

1. If $L < 1$ the series $\sum a_k$ is absolutely convergent (and hence convergent).
2. If $L = 1$ the test is inconclusive.
3. If $L > 1$ the series $\sum a_k$ is divergent.

If $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \infty$ then $\sum a_k$ is also divergent.

The convergence conclusions of The Root Test can be remembered by noting that the limit condition effectively suggests that the tail of the series behaves like $a_k = L^k$, in other words like a geometric series with $r = L$. From this it follows that $r = L < 1$ should converge and $r = L > 1$ should diverge.

Example 4-28

Determine the convergence or divergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+3}\right)^n$
2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n+1)^n}{e^{2n}}$

Solution:

1. The given series is $\displaystyle\sum_{n=1}^{\infty} a_n$ with $a_n = \left(\frac{n+1}{2n+3}\right)^n$. The presence of the power n in the term suggests trying the Root Test:

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n+1}{2n+3}\right)^n\right|} = \lim_{n\to\infty} \frac{n+1}{2n+3} \\ &= \lim_{n\to\infty} \frac{n}{n} \cdot \frac{\frac{n}{n} + \frac{1}{n}}{\frac{2n}{n} + \frac{3}{n}} = \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{2 + \frac{3}{n}} = \frac{1+0}{2+0} = \frac{1}{2} < 1\end{aligned}$$

Therefore the given series is absolutely convergent by the Root Test.

2. The given series is $\displaystyle\sum_{n=1}^{\infty} a_n$ with $a_n = (-1)^{n+1} \frac{(n+1)^n}{e^{2n}}$. Applying the Root Test gives:

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|(-1)^{n+1} \frac{(n+1)^n}{e^{2n}}\right|} &= \lim_{n\to\infty} \sqrt[n]{\frac{(n+1)^n}{e^{2n}}} = \lim_{n\to\infty} \left(\frac{(n+1)^n}{e^{2n}}\right)^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{(n+1)^{\frac{n}{n}}}{e^{\frac{2n}{n}}} = \lim_{n\to\infty} \frac{n+1}{e^2} = \infty\end{aligned}$$

Therefore the given series is divergent by the Root Test.

Further Questions:

Determine the convergence or divergence of the following series.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{2^{3k+1}}{k^k}$
2. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

4.5.4 Rearrangement of Series

For a finite summation of terms the order in which we add the numbers does not matter, i.e. $1+4+2 = 4+1+2$, or, in symbols $a_1 + a_2 + a_3 = a_2 + a_1 + a_3$. The second summation is a **rearrangement** of the first. To make the idea precise we note that the indices on the second summation, $(2, 1, 3)$ are a **permutation** of those on the first $(1, 2, 3)$. A permutation on the infinite set of positive indices $(1, 2, 3, \ldots)$ of a series can similarly be defined thereby making the intuitive definition of a **rearrangement of a series** precise.

If a series is absolutely convergent then we get the same sum regardless of the order in which the terms are added (as expected), as summarized in the following theorem.

Theorem 4-21: Any rearrangment of absolutely convergent series $\sum a_k$ has the same sum as the original series.

However for a series that is only conditionally convergent the order in which we add the terms does matter. Indeed we get the following remarkable result:

Theorem 4-22: Riemann Rearrangement Theorem: Let $\sum a_k$ be a conditionally convergent series with sum S. Then for any real number R there exists a rearrangement of series $\sum a_k$ having sum R. Additionally there exist rearrangements of series $\sum a_k$ which diverge to $+\infty$, $-\infty$, and which fail to approach any limit, finite or infinite.

Example 4-29

By the latter theorem it follows that the (conditionally convergent) alternating harmonic series can be rearranged to sum to any number or to diverge.

Exercise 4-5

Answers: Page 268

1-6: Determine whether the infinite series is convergent or divergent.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)!}{e^{2n}}$

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2n+25}{3^n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{5^{n+1}}{n^n}$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n 2^{3n}}{7^{n-1}}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)^n}{\left(4n+\frac{5}{n}\right)^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$

7-12: Determine whether the infinite series is absolutely convergent, conditionally convergent, or divergent.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{20-n}{n!}$

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{5^n}{n 10^n}$

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{3n^2+2}{2n^4+5}\right)^n$

10. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\left(n^2+3\right)^n e^n}{(n+2)^n}$

11. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{(3n-4)!}$

12. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{n!}$

4.6 Procedure for Testing Series

We have seen several methods for testing for the convergence and divergence of series. The form of the series should suggest the type of test to be used. The following steps will be helpful for determining convergence and divergence of series.

1. Recognize known series with associated convergence and divergence properties:

   telesoping series: Series for which n^{th} partial sum $S_n = \displaystyle\sum_{k=1}^{n} a_k$ can be directly evaluated and $\displaystyle\lim_{n\to\infty} S_n$ taken to determine convergence.

   geometric series: $\displaystyle\sum_{k=1}^{\infty} ar^{k-1} = \sum_{k=0}^{\infty} ar^k$ is convergent for $|r| < 1$ and otherwise divergent.

   ***p*-series:** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent for $p > 1$ and otherwise divergent.
   (Note that $p = 1$ is $\sum \frac{1}{k}$ the (divergent) **harmonic** series.)

2. If $\displaystyle\lim_{k\to\infty} a_k \neq 0$ or that limit does not exist then the series is divergent by the **Term Test for Divergence**.

3. If $\displaystyle\lim_{k\to\infty} a_k = 0$ then proceed as follows:

   (a) If the terms of the series are **positive**, use one of the following tests.

   Basic Comparison Test: Useful when a_k is a rational or algebraic function of k (i.e. involving roots of polynomials). Consider a suitable geometric or p-series for comparison. Remember any comparison series must be positive.

   Limit Comparison Test: Same criteria as the Basic Comparison Test. Choose this one if evaluating the limit of the ratio of comparing terms is easier than proving an inequality between them as required in the Basic Comparison Test.

   Ratio Test: Useful for series involving factorials or other products (including a constant raised to power k). Do not use this test for rational or algebraic functions of k as these result in inconclusive ($L = 1$) results.

   Root Test: Useful if a_k may be written $a_k = (b_k)^k$.

   Integral Test: Useful if $a_k = f(k)$ for positive, continuous, decreasing $f(x)$ and $\int_1^\infty f(x)\,dx$ is easily evaluated.

   (b) If the series is **alternating** (either $\sum(-1)^k a_k$ or $\sum(-1)^{k-1} a_k$ for $a_k > 0$) either:

   i. Use the **Alternating Series Test** .

   ii. Apply a positive series test from 3(a) above to the absolute value of the alternating series (either $\sum \left|(-1)^k a_k\right| = \sum a_k$ or $\sum \left|(-1)^{k-1} a_k\right| = \sum a_k$) since the convergence of the latter implies the convergence of the alternating series.

   (c) If the terms of the series $\sum a_k$ are neither positive nor alternating, apply a test from 3(a) above to the absolute value of the series, $\sum |a_k|$. If $\sum |a_k|$ is convergent then $\sum a_k$ is also convergent.

   (d) If the series only satisfies theorem criteria (positivity, decreasing, alternating, etc.) after a certain point (i.e. for $k \geq n$ for some n) apply the above steps to the tail series $\sum_{k=n}^{\infty} a_k$. The convergence or divergence of the entire series will be the same as that of the tail series since they only differ by a finite number of terms of finite sum.

Example 4-30

Determine the convergence (absolute or conditional) or divergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n+1)^5}$

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{e^n}$

3. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{10^n} + \frac{5}{n^3} \right)$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)^3}{(n^3+5)^2}$

5. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+1}{2n+5} \right)^n$

Solution:

1. The given series is $\displaystyle\sum_{n=1}^{\infty} a_n$ with $a_n = \dfrac{(-1)^n n!}{(2n+1)^5}$. The presence of the factorial suggests trying the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+1)!}{(2n+3)^5}}{\frac{(-1)^n n!}{(2n+1)^5}} \right| = \lim_{n\to\infty} \frac{(n+1)!(2n+1)^5}{n!(2n+3)^5} \\
&= \lim_{n\to\infty} \frac{(n+1)n!(2n+1)^5}{n!(2n+3)^5} = \lim_{n\to\infty} \frac{(n+1)(2n+1)^5}{(2n+3)^5} \\
&= \lim_{n\to\infty} (n+1) \lim_{n\to\infty} \left(\frac{2n+1}{2n+3} \right)^5 = \lim_{n\to\infty} (n+1) \left(\lim_{n\to\infty} \frac{2n+1}{2n+3} \right)^5 \\
&= \lim_{n\to\infty} (n+1) \left(\lim_{n\to\infty} \frac{n}{n} \cdot \frac{\frac{2n}{n} + \frac{1}{n}}{\frac{2n}{n} + \frac{3}{n}} \right)^5 = \lim_{n\to\infty} (n+1) \lim_{n\to\infty} \left(\frac{2 + \frac{1}{n}}{2 + \frac{3}{n}} \right)^5 \\
&= \infty \cdot 1 = \infty
\end{aligned}$$

Therefore the given series is divergent by the Ratio Test.

2. The given series is $\displaystyle\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} (-1)^n a_n$ where $c_n = (-1)^n a_n$ and $a_n = \dfrac{n}{e^n} > 0$. So the series is an alternating series. Let $f(x) = \dfrac{x}{e^x}$, then

$$\begin{aligned}
\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^x} &\underset{\text{LH}}{=} \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0 \\
&\implies \lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = 0 \\
f'(x) = \frac{(1)e^x - xe^x}{e^{2x}} &= \frac{e^x(1-x)}{e^{2x}} = \frac{1-x}{e^x} < 0 \quad \text{for} \quad x > 1 \\
&\implies a_{n+1} = f(n+1) \le f(n) = a_n \quad \text{for} \quad n \ge 1
\end{aligned}$$

Therefore the given series $\sum c_n$ is convergent by the Alternating Series Test.

Now, consider the series of absolute values: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{e^n}$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)e^n}{ne^{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)e^n}{ne^ne} = \lim_{n\to\infty}\frac{n+1}{ne}\\ &= \frac{1}{e}\lim_{n\to\infty}\frac{n}{n}\cdot\frac{\frac{n}{n}+\frac{1}{n}}{\frac{n}{n}} = \frac{1}{e}\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1} = \frac{1}{e}\cdot\frac{1+0}{1} = \frac{1}{e} < 1\end{aligned}$$

Therefore the series $\sum a_n = \sum |c_n|$ is convergent by the Ratio Test. Thus, the series $\sum c_n = \sum_{n=1}^{\infty}(-1)^n\frac{n}{e^n}$ is absolutely convergent.

The astute reader will notice that because the series turned out to be absolutely convergent it was unnecessary to apply the Alternating Series Test. Had we just applied the Ratio Test to $\sum c_n$ directly we could have concluded that series absolutely converged (and hence converged). As a general rule, if you must determine not only the convergence of a series but the type of convergence (conditional or absolute) always consider the convergence of the absolute series first. If that converges the original series is absolutely convergent and no further consideration of the series is required.

3. The given series is $\sum_{n=1}^{\infty} a_n$ with The given series can be written as:

$$\begin{aligned}\sum_{i=1}^{\infty}\left(\frac{1}{10^n}+\frac{5}{n^3}\right) &= \sum_{n=1}^{\infty}\frac{1}{10^n} + \sum_{n=1}^{\infty}\frac{5}{n^3}\\ &= \underbrace{\sum_{n=1}^{\infty}\frac{1}{10}\cdot\left(\frac{1}{10}\right)^{n-1}}_{\substack{\text{geometric series}\\ a=\frac{1}{10},\, r=\frac{1}{10}\\ |r|<1 \text{ (convergent)}}} + \underbrace{5\sum_{n=1}^{\infty}\frac{1}{n^3}}_{\substack{p\text{-series}\\ p=3\\ p>1 \text{ (convergent)}}}\end{aligned}$$

Thus the given series is convergent as it is the sum of two convergent series. The convergence is absolute as the series is already positive.

4. The given series is $\sum_{n=1}^{\infty} a_n$ with $a_n = \frac{(n+2)^3}{(n^3+5)^2}$. Consideration of the order of the polynomials suggests a comparison with $\sum b_n$ where $b_n = \frac{n^3}{(n^3)^2} = \frac{n^3}{n^6} = \frac{1}{n^3}$. Then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent p-series since $p = 3 > 1$. Also,

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{\frac{(n+2)^3}{(n^3+5)^2}}{\frac{1}{n^3}} = \lim_{n\to\infty}\frac{n^3(n+2)^3}{(n^3+5)^2} = \lim_{n\to\infty}\frac{n^3(n^3+6n^2+12n+8)}{n^6+10n^3+25}\\ &= \lim_{n\to\infty}\frac{n^6+6n^5+12n^4+8}{n^6+10n^3+25} = \lim_{n\to\infty}\frac{n^6}{n^6}\cdot\frac{\frac{n^6}{n^6}+\frac{6n^5}{n^6}+\frac{12n^4}{n^6}+\frac{8}{n^6}}{\frac{n^6}{n^6}+\frac{10n^3}{n^6}+\frac{25}{n^6}}\\ &= \lim_{n\to\infty}\frac{1+\frac{6}{n}+\frac{12}{n^2}+\frac{8}{n^6}}{1+\frac{10}{n^3}+\frac{25}{n^6}} = \frac{1+0+0+0}{1+0+0} = 1 > 0\end{aligned}$$

Therefore the given series is also convergent by the Limit Comparison Test. Since the series is positive it is absolutely convergent.

5. The given series is $\sum_{n=1}^{\infty} a_n$ with $a_n = \left(\frac{2n+1}{2n+5}\right)^n$. The power in the term suggests trying the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n+1}{2n+5}\right)^n\right|} = \lim_{n\to\infty} \frac{2n+1}{2n+5} = \lim_{n\to\infty} \frac{n}{n} \cdot \frac{\frac{2n}{n}+\frac{1}{n}}{\frac{2n}{n}+\frac{5}{n}}$$
$$= \lim_{n\to\infty} \frac{2+\frac{1}{n}}{2+\frac{5}{n}} = \frac{2+0}{2+0} = 1$$

Therefore the Root Test is inconclusive. Next consider trying the Term Test for Divergence. To evaluate the limit let $f(x) = \left(\frac{2x+1}{2x+5}\right)^x$. Then $\lim_{x\to\infty} f(x)$ is the indeterminate form 1^∞. Taking the logarithm gives $\ln f(x) = x \ln\left(\frac{2x+1}{2x+5}\right)$ and so

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} x \ln\left(\frac{2x+1}{2x+5}\right) \quad (\infty \cdot 0 \text{ form})$$
$$= \lim_{x\to\infty} \frac{\ln(\frac{2x+1}{2x+5})}{x^{-1}} \quad \left(\frac{0}{0} \text{ form}\right)$$
$$\underset{\text{LH}}{=} \lim_{x\to\infty} \frac{\frac{1}{\frac{2x+1}{2x+5}} \cdot \frac{2(2x+5)-2(2x+1)}{(2x+5)^2}}{-x^{-2}} = \lim_{x\to\infty} \frac{\frac{4x+10-4x-2}{(2x+5)(2x+1)}}{-x^{-2}}$$
$$= -\lim_{x\to\infty} \frac{8x^2}{(2x+5)(2x+1)} = -\lim_{x\to\infty} \frac{8x^2}{4x^2+12x+5}$$
$$\underset{\text{LH}}{=} -\lim_{x\to\infty} \frac{16x}{8x+12} \underset{\text{LH}}{=} -\lim_{x\to\infty} \frac{16}{8} = -2\,.$$

Thus $\lim_{x\to\infty} f(x) = e^{-2}$ and therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = e^{-2} \neq 0\,.$

Thus the given series is divergent by the Term Test for Divergence.

Further Questions:

Determine the convergence (absolute or conditional) or divergence of the following series.

1. $\sum_{k=1}^{\infty} \frac{2k^2}{k^2+1}$
2. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$
3. $\sum_{n=1}^{\infty} \frac{1000-n}{n!}$
4. $\sum_{k=1}^{\infty} e^{-2k}$
5. $\sum_{n=1}^{\infty} \frac{5^n}{n^6\,3^{n+1}}$
6. $\sum_{k=1}^{\infty} \sin\left(\frac{\pi}{2}+\frac{1}{k}\right)$
7. $\sum_{k=1}^{\infty} \frac{k^k}{10^k}$
8. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n^2+1}$
9. $\sum_{n=1}^{\infty} \left(\frac{1}{3^n}+\frac{5}{\sqrt{n}}\right)$

Answers: Page 268

Exercise 4-6

1-14: Determine whether the infinite series is absolutely convergent, conditionally convergent, or divergent.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{5n^3+4n+2}{7n^3+2n^2+6}$

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{5n^3+6}$

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^n}{(n!)^n}$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^n 2^n}{n3^n}$

5. $\displaystyle\sum_{n=1}^{\infty} \tan^{-1} n$

6. $\displaystyle\sum_{n=1}^{\infty} \tan\left(\frac{n\pi+5}{4n+7}\right)$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^3+4}{3n^5+6}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{50}{\sqrt{n}+10}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos\left(\frac{1}{n}\right)}{n^2}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$

11. $\displaystyle\sum_{n=1}^{\infty} [\ln(2n+3) - \ln(n+2)]$

12. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4}{3^n} + \frac{3}{4^n}\right)$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$

14. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{5n+7}{2n^3+25}$

4.7 Power Series

We have seen that in many cases the terms of the series $\sum a_k$ may be written as a function of the summation index, namely $a_k = f(k)$, for some function f. However if the series $\sum a_k$ is convergent its value is a number, namely its sum. The final sum does not depend on the index k in the same way that a definite integral $\int_a^b f(t)\,dt$ results in a number independent of the dummy variable t.

Consider, however, the situation where the terms a_k depend additionally on an actual variable, say x, different from the summation index, present in the series (i.e. $a_k = a_k(x)$). In this case the sum of the series (and indeed its convergence) depends on the value of x.

Example 4-31

The geometric series with $a = 1$ and $r = x$ is given by

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots + x^k + \cdots$$

Here $a_k(x) = x^k$. The sum is now a function of x, namely $\dfrac{1}{1-x}$, and is valid for $|x| < 1$ for which the series is convergent.

We could introduce the variable x into the terms of a series in many ways, for instance

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k!}\,.$$

If we choose to introduce it as in our geometric series above, namely so that the terms of the series look like terms in a polynomial, $c_k x^k$, we have a **power series**.

Definition: Let x be a variable. A series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + \cdots\,,$$

where c_k are real constants (for $k = 0, 1, 2\ldots$) is a **power series in x** . The constants c_k are called the **coefficients** of the series.

Example 4-32

The geometric series with $r = x$ above, $1 + x + x^2 + \cdots + x^k + \cdots$, is a power series in x with coefficients $c_k = 1$ for all k.

If we choose to make the k^{th} term of the series have the more general form $c_k(x-a)^k$ we get the following.

Definition: Given real constant coefficients c_k and real constant a the **power series in $(x-a)$** is

$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_k(x-a)^k + \cdots\,.$$

The series is also known as the **power series about a** or **centred on a** .

A power series in x is just a special case of this last definition with $a = 0$.

Since the convergence of a power series (and indeed its sum should it converge) will, in general, depend on the value of the variable x, an obvious question is to find the values of x for which the power series is convergent.

Example 4-33

Find the values of x for which the following power series are convergent.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} x^n$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{3^n} x^n$
3. $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$
4. $\displaystyle\sum_{n=1}^{\infty} n^n (x+3)^n$

Solution:

1. The power series is $\displaystyle\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{(-1)^n}{(n+1)!} x^n$. Consider the following Ratio Test limit:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1}}{(n+2)!} x^{n+1}}{\frac{(-1)^n}{(n+1)!} x^n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+2)!} \frac{(n+1)!}{x^n}\right|$$
$$= \lim_{n\to\infty} \frac{(n+1)!}{(n+2)(n+1)!} |x| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

Note that here, since the limit is in n, the value x is treated as a constant for the purpose of evaluating the limit. Since the limit of 0 is less than 1 we have, by the Ratio Test, that the power series converges for all real values of x.

2. The power series is $\displaystyle\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{2n+1}{3^n} x^n$. The Ratio Test limit is:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{2n+3}{3^{n+1}} x^{n+1}}{\frac{2n+1}{3^n} x^n}\right| = \lim_{n\to\infty} \left|\frac{(2n+3)x^{n+1}}{3^{n+1}} \frac{3^n}{(2n+1)x^n}\right| = \lim_{n\to\infty} \frac{2n+3}{3(2n+1)} |x|$$
$$= \lim_{n\to\infty} \frac{n}{n} \cdot \frac{\frac{2n}{n} + \frac{3}{n}}{3\left(\frac{2n}{n} + \frac{1}{n}\right)} |x| = \lim_{n\to\infty} \frac{2 + \frac{3}{n}}{3 + \left(2 + \frac{1}{n}\right)} |x| = \frac{2+0}{3(2+0)} |x| = \frac{1}{3} |x|$$

Thus, by the Ratio Test, the series is convergent if the limit satisfies

$$\frac{1}{3}|x| < 1 \implies |x| < 3 \implies -3 < x < 3\,.$$

The Ratio Test is inconclusive if $\frac{1}{3}|x| = 1 \implies |x| = 3$ (so $x = 3$ or $x = -3$). We must study these two values separately.

- If $x = 3$ then the series becomes:

$$\sum_{n=1}^{\infty} \frac{2n+1}{3^n} 3^n = \sum_{n=1}^{\infty} (2n+1)\,.$$

Then $\displaystyle\lim_{n\to\infty} (2n+1) = \infty$ shows that when $x = 3$ the series diverges by the Term Test for Divergence.

- If $x = -3$ then the series becomes:

$$\sum_{n=1}^{\infty} \frac{2n+1}{3^n}(-3)^n = \sum_{n=1}^{\infty}(-1)^n(2n+1)\,.$$

Then the fact that $\lim_{n\to\infty}(-1)^n(2n+1)$ does not exist shows that when $x = -3$ the series is divergent by the Term Test for Divergence.

Therefore the given power series is convergent if $-3 < x < 3$ and divergent otherwise.

3. The power series is $\sum_{n=2}^{\infty} a_n$ where $a_n = \dfrac{x^n}{\ln n}$. The Ratio Test limit is:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{\ln(n+1)}}{\frac{x^n}{\ln n}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\ln(n+1)}\cdot\frac{\ln n}{x^n}\right| = \lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}|x|$$

Again $|x|$ here is just a constant with respect to the limit and we need to evaluate $\lim_{n\to\infty}\dfrac{\ln n}{\ln(n+1)}$.

If $f(y) = \dfrac{\ln y}{\ln(y+1)}$, then

$$\lim_{y\to\infty} f(y) = \lim_{y\to\infty}\frac{\ln y}{\ln(y+1)} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$\underset{\text{LH}}{=} \lim_{y\to\infty}\frac{\frac{1}{y}}{\frac{1}{y+1}} = \lim_{y\to\infty}\frac{y+1}{y} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right) = 1+0 = 1\,.$$

Thus,

$$\lim_{x\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}|x| = (1)\ |x| = |x|$$

Therefore, the series is convergent if $|x| < 1 \implies -1 < x < 1$ by the Ratio Test. The Ratio Test is inconclusive when $|x| = 1$ (so $x = 1$ and $x = -1$) and we must study these two values separately.

- If $x = 1$ then the series becomes:

$$\sum_{n=2}^{\infty}\frac{1^n}{\ln n} = \sum_{n=1}^{\infty}\frac{1}{\ln n}\,.$$

Let $a_n = \dfrac{1}{\ln n}$, $b_n = \dfrac{1}{n}$. Then $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty}\dfrac{1}{n}$ is a divergent harmonic series. Also for $n \geq 2$ we have

$$0 < \ln n \leq n \implies \frac{1}{\ln n} \geq \frac{1}{n} \text{ for } n \geq 2$$

Thus when $x = 1$ the series is divergent by the Basic Comparison Test.

- If $x = -1$ then the series becomes an alternating series:

$$\sum_{n=2}^{\infty}\frac{1}{\ln n}(-1)^n = \sum_{n=2}^{\infty}(-1)^n a_n \text{ where } a_n = \frac{1}{\ln n} > 0$$

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln n} = \frac{1}{\infty} = 0$$

If $f(y) = \dfrac{1}{\ln y}$, then $f'(y) = -\dfrac{1}{y(\ln y)^2} < 0$ for $y \geq 2$ so $a_n = f(n)$ is a decreasing sequence. Thus when $x = -1$ the series is convergent by the Alternating Series Test.

Therefore the given power series is convergent if $-1 \leq x < 1$.

4. The power series is $\displaystyle\sum_{n=1}^{\infty} a_n$ where $a_n = n^n(x+3)^n$. Since $(x+3) = (x-(-3))$ this is a power series centred at $a = -3$. Consider the following Root Test limit:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|n^n(x+3)^n|} = \lim_{n\to\infty} \sqrt[n]{n^n}\,\sqrt[n]{|x+3|^n} = \lim_{n\to\infty} n|x+3|$$
$$= \begin{cases} 0 & \text{if } x = -3 \\ \infty & \text{if } x \neq -3 \end{cases}$$

Notice that while x and hence $x+3$ are effectively constants when evaluating the limit, the case where $|x+3| = 0$ (so $x = -3$) does result in a unique limit of zero since $a_n = 0$ when $x = -3$. By the Root Test only in that case is the limit less than 1. Hence the power series converges only for $x = -3$.

As an aside, while the Root Test can be used to analyze power series convergence, the Ratio Test is almost always preferred. The Root Test usually results in limits with indeterminate forms requiring evaluation. This example was an exception.

Further Questions:

Find the values of x for which the following power series are convergent.

1. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$
2. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n} x^n$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{e^n}(x-e)^n$

As suggested by the previous example a power series about a will converge on an interval centred on a as detailed in the following theorem.

Theorem 4-23: The power series $\displaystyle\sum_{k=0}^{\infty} c_k(x-a)^k$ will either:

1. Converge only at $x = a$.
2. Converge for $|x-a| < R$ and diverge for $|x-a| > R$ for some positive real number R.
3. Converge for all x.

Definition: The **radius of convergence** $\boldsymbol{R}$ for a power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ is the value R if Part 2 of Theorem 4-23 applies. For Part 1 the radius of convergence is defined to be $R = 0$ and for Part 3 the radius of convergence is defined to be $R = \infty$.

Definition: The **interval of convergence** $\boldsymbol{I}$ of a power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ is the set of values of x for which the series converges. For the three possibilities of convergence one has:

1. The set containing the single value $x = a$ (i.e. $\{a\}$) for $R = 0$.
2. One of $[a-R, a+R]$, $[a-R, a+R)$, $(a-R, a+R]$, or $(a-R, a+R)$ for $R > 0$ finite.
3. $(-\infty, \infty)$ for $R = \infty$.

The choice of interval in the second case depends upon the convergence or not of the series at the interval endpoint values $x = a \pm R$.

Example 4-34

For Example 4-33 find the radii and intervals of convergence of each series.

Solution:

Consideration of the solutions of Example 4-33 shows the radii and intervals of convergence are:

1. $R = \infty$, $I = (-\infty, \infty)$
2. $R = 3$, $I = (-3, 3)$
3. $R = 1$, $I = [-1, 1)$
4. $R = 0$, $I = \{-3\}$

Further Questions:

For Example 4-33 find the radii and intervals of convergence of each series in the Further Questions.

Example 4-35

Find the radius and interval of convergence of each of the following series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n 3^n}{(n!)^2}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x-2)^n}{5^n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n(3x-2)^n}{n^2}$
4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-4)^n}{\sqrt{n+1}}$

Solution:

1. The power series is $\displaystyle\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{x^n 3^n}{(n!)^2}$. The Ratio Test limit is

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}3^{n+1}}{[(n+1)!]^2}}{\frac{x^n 3^n}{(n!)^2}}\right| = \lim_{n\to\infty}\frac{(n!)^2(3)}{[(n+1)!]^2}|x| = \lim_{n\to\infty}\frac{3n!n!}{[(n+1)n!]^2}|x|$$
$$= \lim_{n\to\infty}\frac{3}{(n+1)^2}|x| = 0\cdot|x| = 0 < 1 \text{ for all } x\,.$$

Therefore, the power series converges for all x. Thus the radius of convergence is $R = \infty$ and the interval of convergence is $I = (-\infty, \infty)$.

2. The power series is $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{n!(x-2)^n}{5^n}$. The Ratio Test limit is

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!(x-2)^{n+1}}{5^{n+1}}}{\frac{n!(x-2)^n}{5^n}}\right| = \lim_{n\to\infty}\frac{5^n(n+1)n!}{5^{n+1}n!}|x-2| = \lim_{n\to\infty}\frac{n+1}{5}|x-2|$$
$$= \begin{cases} 0 & \text{if } x=2 \\ \infty & \text{if } x\neq 2 \end{cases}$$

This power series is convergent only if $x = 2$. Therefore, the radius of convergence is $R = 0$ and the interval of convergence is $I = \{2\}$. Note that the *interval* is centred at 2 as expected for a power series about $a = 2$.

3. The power series is $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{(-5)^n(3x-2)^n}{n^2}$. The Ratio Test limit is

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(5)^{n+1}(3x-2)^{n+1}}{(n+1)^2}}{\frac{(-5)^n(3x-2)^n}{n^2}}\right| = \lim_{n\to\infty}\frac{5n^2}{(n+1)^2}|3x-2| = 5\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2|3x-2|$$
$$= 5\lim_{n\to\infty}\left(\frac{n}{n}\cdot\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right)^2|3x-2| = 5\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^2|3x-2|$$
$$= 5\left(\frac{1}{1+0}\right)^2|3x-2| = 5|3x-2|$$

The series is convergent by the Ratio Test if:

$$5|3x-2|<1 \implies 15\left|x-\frac{2}{3}\right|<1 \implies \left|x-\frac{2}{3}\right|<\frac{1}{15}$$
$$\implies -\frac{1}{15}<x-\frac{2}{3}<\frac{1}{15} \implies -\frac{1}{15}+\frac{2}{3}<x<\frac{1}{15}+\frac{2}{3}$$
$$\implies \frac{3}{5}<x<\frac{11}{15}$$

The Ratio Test is inconclusive (limit equals 1) at the endpoints $x = \frac{3}{5}$ and $x = \frac{11}{15}$. We must test these two values separately.

- If $x = \frac{3}{5}$ then the series becomes:

$$\sum_{n=1}^{\infty}\frac{(-5)^n\left(\frac{9}{5}-2\right)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-5)^n(-\frac{1}{5})^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

  This is a convergent p-series since $p = 2 > 1$.

- If $x = \frac{11}{15}$ then the series becomes:

$$\sum_{n=1}^{\infty}\frac{(-5)^n\left(\frac{33}{15}-2\right)^n}{n^2}\sum_{n=1}^{\infty}\frac{(-5)^n(\frac{1}{5})^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

  Consider $a_n = \frac{1}{n^2}$ and define $f(y) = \frac{1}{y^2}$ then:

$$\lim_{y\to\infty} f(y) = \lim_{y\to\infty}\frac{1}{y^2} = 0 \implies \lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = 0\,.$$
$$f'(y) = -\frac{2}{y^3} < 0 \text{ for } y>0 \implies a_n = f(n) \text{ decreases}\,.$$

Thus the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is convergent by the Alternating Series Test.

Therefore the radius of convergence of the power series is $R = \frac{1}{15}$ and the interval of convergence is $I = \left[\frac{3}{5}, \frac{11}{15}\right]$. Notice that the interval is centred at $a = \frac{2}{3}$. That the original power series is centred at this value can be seen by writing $(3x-2)^n = 3^n(x-2/3)^n$ in the original series.

4. The power series is $\sum_{n=1}^{\infty} a_n$ where $a_n = (-1)^{n+1}\frac{(x-4)^n}{\sqrt{n+1}}$. The Ratio Test limit is

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(x-4)^{n+1}}{\sqrt{n+2}}}{\frac{(-1)^{n+1}(x-4)^n}{\sqrt{n+1}}}\right| = \lim_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{n+2}}|x-4| = \lim_{n\to\infty}\sqrt{\frac{n+1}{n+2}}|x-4|$$
$$= \lim_{n\to\infty}\sqrt{\frac{n}{n}\cdot\frac{\frac{n}{n}+\frac{1}{n}}{\frac{n}{n}+\frac{2}{n}}}|x-4| = \lim_{n\to\infty}\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}}|x-4| = |x-4|$$

The series is convergent by the Ratio Test if $|x-4| < 1 = R$. Then

$$|x-4| < 1 \implies -1 < x-4 < 1 \implies 3 < x < 5$$

which is as expected since the series is centred at $a = 4$ with radius 1. The Ratio Test is inconclusive if $x = 3$ or $x = 5$ and we test these values separately.

- If $x = 3$ then the series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{\sqrt{n+1}} = -\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$$

  Let $a_n = \frac{1}{\sqrt{n+1}}$ and $b_n = \frac{1}{\sqrt{n}}$. Then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} \leq 1$ and hence divergent. Taking the limit

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = \sqrt{\lim_{n\to\infty}\frac{n}{n+1}}$$
$$= \sqrt{\lim_{n\to\infty}\frac{n}{n}\cdot\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}} = \sqrt{\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}} = \sqrt{\frac{1}{1+0}} = 1 > 0$$

  shows the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$ is divergent by the Limit Comparison Test.

- If $x = 5$ then the series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1)^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n+1}}$$

  Then $\lim_{n\to\infty}\frac{1}{\sqrt{n+1}} = \frac{1}{\infty} = 0$. Let $f(y) = \frac{1}{\sqrt{y+1}}$ then:

$$f'(y) = -\frac{1}{2(y+1)^{\frac{3}{2}}} < 0 \text{ for } x \geq 1$$

  shows the magnitude of the terms, $f(n)$, is decreasing. Therefore the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n+1}}$ converges by the Alternating Series Test.

Therefore the radius of convergence of the power series is $R = 1$ and the interval of convergence is $I = (3, 5]$.

Further Questions:

Find the radius and interval of convergence of each of the following series.

1. $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{1}{k+1}(x-3)^k$
2. $\displaystyle\sum_{n=0}^{\infty} n^3(x-5)^n$
3. $\displaystyle\sum_{k=0}^{\infty} k!(2x-1)^k$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-3)^n}{n\,3^n}$

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Exercise 4-7

1-5: Find the interval and the radius of convergence of the power series.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+4}{2n^3+5} x^n$
2. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}(x-1)^n$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{(3n)!} x^{3n}$
4. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n3^n}(4x-1)^n$
5. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{10^n}(x-1)^n$

4.8 Representing Functions with Power Series

The sum of a power series for a given x from its interval of convergence I results in a number. As such it is natural to consider the power series as defining a function $f(x)$ of x on I with the value of the function being the sum. We write

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \quad (x \in I)$$

If the sum of the power series can be written in a closed form, then the power series can be considered a representation of that function valid on I.

Example 4-36

The power series $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$ converges for $|x| < 1$, and therefore is a function of x on $I = (-1, 1)$. On that interval the sum for given x is $\frac{1}{1-x}$. The power series thus is a representation of the function $\frac{1}{1-x}$ on this restricted domain:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k \quad \text{for } |x| < 1$$

One can find power series representations of other functions using known power series.

Example 4-37

Find representations of the following functions with power series. Indicate the values of x for which the representatives are valid.

1. $\dfrac{1}{2-5x}$
2. $\dfrac{x}{1+4x^2}$
3. $\dfrac{x^2}{4+3x^2}$

Solution:

1. The given functions can be written as:

$$\frac{1}{2-5x} = \frac{1}{2(1-\frac{5}{2}x)}\,.$$

The power series representation of $\dfrac{1}{1-x}$ is:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \; |x| < 1\,.$$

Thus

$$\frac{1}{2-5x} = \frac{1}{2(1-\frac{5}{2}x)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{5}{2}x\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{5}{2}\right)^n x^n = \sum_{n=0}^{\infty}\frac{5^n}{2^{n+1}}x^n\,,$$

which converges when

$$\left|\frac{5}{2}x\right| < 1 \implies |x| < \frac{2}{5} \implies -\frac{2}{5} < x < \frac{2}{5}\,.$$

2. This function can be written as

$$\frac{x}{1+4x^2} = x\cdot\frac{1}{1-(-4x^2)} = x\sum_{n=0}^{\infty}\left(-4x^2\right)^n,\ |-4x^2|<1$$
$$= \sum_{n=0}^{\infty}(-1)^n 4^n x^{2n+1}.$$

Note here that the variable x is effectively a constant with respect to the series in n and can therefore be brought in and out of the summation without affecting its convergence properties.

The series converges if

$$|-4x^2|<1 \implies |4x^2|<1 \implies 4x^2<1 \implies x^2<\frac{1}{4} \implies |x|<\frac{1}{2}.$$

3. This function can be written as

$$\frac{x^2}{4+3x^2} = \frac{x^2}{4(1+\frac{3}{4}x^2)} = \frac{x^2}{4}\cdot\frac{1}{1-\left(-\frac{3}{4}x^2\right)}$$
$$= \frac{x^2}{4}\sum_{n=0}^{\infty}\left(-\frac{3}{4}x^2\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{3^n}{4^{n+1}}x^{2n+2}.$$

The power series converges if

$$\left|-\frac{3}{4}x^2\right|<1 \implies \frac{3}{4}x^2<1 \implies x^2<\frac{4}{3} \implies |x|<\frac{2}{\sqrt{3}}.$$

Further Questions:

Find representations of the following functions with power series. Indicate the values of x for which the representations are valid.

1. $\frac{1}{1+x}$
2. $\frac{1}{1-x^3}$
3. $\frac{x}{2-3x}$

As functions power series behave like polynomials. They are continuous and can be termwise differentiated and integrated to produce new functions of x as detailed in the following theorem.

Theorem 4-24: Suppose $f(x)$ is defined by the power series

$$f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_k(x-a)^k + \cdots$$

with radius of convergence $R>0$ (finite or infinite). Then on the interval $(a-R, a+R)$:

1. $f(x)$ is continuous.
2. $f(x)$ is differentiable with derivative

$$f'(x) = \sum_{k=0}^{\infty} kc_k(x-a)^{k-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + kc_k(x-a)^{k-1} + \cdots$$

3. $f(x)$ is integrable with integral

$$\int f(x)\,dx = C+\sum_{k=0}^{\infty}\frac{c_k}{k+1}(x-a)^{k+1} = C+c_0(x-a)+\frac{c_1}{2}(x-a)^2+\frac{c_2}{3}(x-a)^3+\cdots+\frac{c_k}{k+1}(x-a)^{k+1}+\cdots$$

The series resulting from differentiation and integration both have radius of convergence R.

Note that the above theorem shows that for power series the calculus operations of differentiation and integration can be exchanged with summation, just as occurs with finite sums.

$$\frac{d}{dx}\sum c_k(x-a)^k = \sum \frac{d}{dx}\left[c_k(x-a)^k\right]$$
$$\int\left[\sum c_k(x-a)^k\right]\,dx = \sum\left[\int c_k(x-a)^k\,dx\right]$$

If the power series is a representation of a function with a closed form, differentiation and integration can be used to find power series representations of other functions.

Example 4-38

Find a power series representation for each of the following functions. Indicate the interval for which the representation is valid.

1. $\dfrac{1}{(4-x)^2}$
2. $\ln(1-x)$

Solution:

1. To find the power series representation of $\dfrac{1}{(4-x)^2}$ we note that it is just the derivative of the function $\dfrac{1}{4-x}$ which has the representation

$$\frac{1}{4-x} = \frac{1}{4(1-\frac{x}{4})} = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{4^{n+1}}$$
$$\implies \frac{1}{4-x} = \sum_{n=0}^{\infty}\frac{x^n}{4^{n+1}},\ \left|\frac{x}{4}\right| < 1\,.$$

Differentiating both sides with respect to x we have

$$\frac{d}{dx}\left(\frac{1}{4-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{4^{n+1}} = \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{x^n}{4^{n+1}}\right)$$
$$\frac{1}{(4-x)^2} = \sum_{n=0}^{\infty}\frac{1}{4^{n+1}}nx^{n-1} = \sum_{n=1}^{\infty}\frac{n}{4^{n+1}}x^{n-1}\,,$$

where we changed the starting index to $n=1$ since the $n=0$ term equals zero. The power series converges if

$$\left|\frac{x}{4}\right| < 1 \implies |x| < 4\,.$$

2. For the geometric series we have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n,\ |x| < 1\,.$$

We desire a power series representation of $\ln|1-x|$ which is, up to a factor of -1, just the integral of this function. Integrating both sides with respect to x we obtain

$$\begin{aligned}\underbrace{\int \frac{1}{1-x}\,dx}_{u=1-x,du=-dx} &= \int \left(\sum_{n=0}^{\infty} x^n\right) dx = \sum_{n=0}^{\infty} \int x^n\,dx \\ \implies -\ln|1-x| &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C \\ \implies \ln|1-x| &= -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} - C\end{aligned}$$

For fixed x the function on the left is a number and so the series on the right cannot contain an arbitrary constant. To determine what it equals evaluate both sides at $x=0$ to get

$$\ln|1| = -0 - C \implies C = 0\,.$$

Thus

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\,,$$

which converges for $|x|<1$ as follows by Theorem 4-24. Since $1-x>0$ in that interval the absolute value bars have been dropped.

The series representation can be simplified by changing to a new index. Similar to introducing a new variable in an integral, here let $m=n+1$. The new limits of the series then become

$$\begin{aligned} n=\infty &\implies m=\infty+1=\infty \\ n=0 &\implies m=0+1=1\end{aligned}$$

and our series representation simplifies to

$$\ln(1-x) = -\sum_{m=1}^{\infty} \frac{x^m}{m},\ |x|<1\,.$$

Just like an integration variable we can change the index name back to n now if we so desire. Reindexing in this manner has several uses. In addition to simplifying a series, one can also compare two series whose lower limits are not the same by reindexing one to have the same limits as the other. So for example one can show the following two forms of the geometric series are equivalent,

$$\sum_{n=0}^{\infty} ar^n = \sum_{m=1}^{\infty} ar^{m-1}\,,$$

by letting $m=n+1$ in the original sequence, changing the limits as above, and noting that $m=n+1 \implies n=m-1$ when rewriting the general term of the first series in terms of the new index m. Reindexing also allows one to align the limits of two series to allow their addition. Alternatively one may reindex power series so that they all have the same power of x^n so that they may be added and simplified.

Our new version of the representation $\ln(1-x)$ shows that at $x=1$ the series representation diverges as it reduces to the harmonic series. This is unsurprising as $\ln(1-1)=\ln(0)$ is undefined as well. At $x=-1$ the series becomes the alternating harmonic series and therefore converges conditionally at that value. The representation is, in fact, valid at $x=-1$ as its sum may be shown to be $\ln(1-(-1))=\ln 2$. As such the series representation of $\ln(1-x)$ is valid on $[-1,1)$.

Further Questions:

Find a power series representation for each of the following functions. Indicate the interval for which the representation is valid.

1. $\dfrac{1}{(1+x)^2}$
2. $\ln(1+x)$
3. $\tan^{-1}x$

Because power series representations are easy to integrate, they may be used as a means to integrate difficult functions. The integral will only be valid for x lying within $(a-R, a+R)$.

Example 4-39

Integrate each of the following using a power series.

1. $\displaystyle\int \frac{1}{1+x^6}\,dx$
2. $\displaystyle\int \frac{\ln(1-x)}{x}\,dx$

Solution:

1. Starting with the geometric series $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,\ |x|<1$ first get a representation for the integrand:

$$\frac{1}{1+x^6} = \frac{1}{1-(-x^6)} = \sum_{n=0}^{\infty}(-x^6)^n = \sum_{n=0}^{\infty}(-1)^n x^{6n}\,,$$

which converges for $|-x^6|<1 \implies |x|<1$. Thus

$$\int \frac{1}{1+x^6}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{6n}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+1}}{6n+1} + C = \sum_{n=0}^{\infty}\frac{(-1)^n}{6n+1}x^{6n+1} + C\,,$$

which is valid for $|x|<1$.

2. In Example 4-38 Problem 2 we found $\displaystyle\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n},\ |x|<1$. Thus

$$\begin{aligned}\int \frac{\ln(1-x)}{x}\,dx &= -\int \frac{1}{x}\sum_{n=1}^{\infty}\frac{x^n}{n}\,dx = -\sum_{n=1}^{\infty}\frac{1}{n}\int x^{n-1}\,dx = -\sum_{n=1}^{\infty}\frac{1}{n}\frac{x^{(n-1)+1}}{(n-1)+1} + C\\ &= -\sum_{n=1}^{\infty}\frac{1}{n^2}x^n + C\,,\end{aligned}$$

which is valid for $|x|<1$.

These indefinite integrals could be used to approximate a definite integral by suitably truncating the series at large n and subtracting its evaluation at the limits of the integral as we would for any antiderivative. The integration limits would need to be within the interval $(-1,1)$ due to the representation of these antiderivatives only being valid there.

Further Questions:

Integrate each of the following using a power series.

1. $\displaystyle\int \frac{1}{1+x^3}\,dx$
2. $\displaystyle\int \frac{x}{1-x^4}\,dx$

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Exercise 4-8

1-5: Find a power series representation for $f(x)$ and specify the interval of convergence.

1. $f(x) = \dfrac{1}{1+x^2}$
2. $f(x) = \dfrac{x}{5-4x}$
3. $f(x) = \dfrac{x}{2+x^2}$
4. $f(x) = \dfrac{1}{(2+3x)^2}$
5. $f(x) = \ln(3+2x)$

4.9 Maclaurin Series

We expect some functions $f(x)$ can be represented by an infinite series $\sum_{k=0}^{\infty} c_k x^k$ for a certain, potentially restricted, domain. While we verified that $\frac{1}{1-x}$ could be represented by the geometric series, we would like a general mechanism for determining the coefficients of the power series that correspond to an arbitrary $f(x)$. Suppose $f(x)$ has a power series expansion:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \ .$$

One observes that if we set $x = 0$ that $f(0) = c_0$. In other words, we can determine c_0 by evaluating f at $x = 0$. We can differentiate the power series above term by term to get:

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

If we evaluate this at $x = 0$ we get $c_1 = f'(0)$. The next derivative is:

$$f''(x) = 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots$$

and so $c_2 = \frac{f''(0)}{2}$. Repeated differentiation and evaluation relates the k^{th} coefficient c_k to the derivative $f^{(k)}$ evaluated at $x = 0$ as follows:

$$c_k = \frac{f^{(k)}(0)}{k!}$$

where recall the **factorial** is defined by $\boldsymbol{k! = k \cdot (k-1) \cdots 2 \cdot 1}$. The formula is true for $k = 0$ as well with the convention $f^{(0)}(x) = f(x)$ and noting that $\mathbf{0! = 1}$ by definition. Plugging our c_k into the original power series we get the following definition.

Definition: Given function $f(x)$ differentiable to all orders at $x = 0$ the power series in x given by

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \frac{f^{(k)}(0)}{k!}x^k + \cdots$$

is the **Maclaurin series** for $f(x)$.

As a power series, the Maclaurin series must converge for $|x| < R$ for some radius of convergence R dependent upon the function.[4]

Example 4-40

Find the Maclaurin series and its integral of convergence for each of the following functions.

1. $f(x) = e^{2x}$
2. $f(x) = \cos x$

Solution:

1. Since the Maclaurin series is $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ we compute the following derivatives and

[4] We will discuss shortly whether the Maclaurin series of $f(x)$ actually is a valid representation of the function on this interval.

evaluate them at zero:

$$\begin{aligned} f(x) &= e^{2x}\ , & f^{(0)}(0) &= f(0) = e^0 = 1 \\ f'(x) &= 2e^{2x}\ , & f'(0) &= 2e^0 = 2 \\ f''(x) &= 4e^{2x}\ , & f''(0) &= 4e^0 = 4 \\ f'''(x) &= 8e^{2x}\ , & f'''(0) &= 8e^0 = 8 \\ f^{(4)}(x) &= 16e^{2x}\ , & f^{(4)}(0) &= 16e^0 = 16 \\ f^{(5)}(x) &= 32e^{2x}\ , & f^{(5)}(0) &= 32e^0 = 32 \end{aligned}$$

Since $0! = 1$ and $1! = 1$ the Maclaurin series is given by:

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \frac{1}{4!}f^{(4)}(0)x^4 + \frac{1}{5!}f^{(5)}(0)x^5 + \dots \\ \implies e^{2x} &= 1 + 2x + \frac{1}{2!}(4)x^2 + \frac{1}{3!}(8)x^3 + \frac{1}{4!}(16)x^4 + \frac{1}{5!}(32)x^5 + \dots \\ \implies e^{2x} &= 1 + 2x + \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 + \frac{2^4}{4!}x^4 + \frac{2^5}{5!}x^5 + \dots \\ \implies e^{2x} &= \sum_{n=0}^{\infty} \frac{2^n}{n!}x^n \end{aligned}$$

Alternatively if one only wants the series in sigma notation one can observe that $f^{(n)}(0) = 2^n$ and substitute that immediately into our Maclaurin formula.

To find the interval of convergence we will use the Ratio Test.

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}x^{n+1}}{(n+1)!}}{\frac{2^n x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}}{(n+1)n!}\frac{n!}{2^n x^n}\right| = 2\lim_{n\to\infty}\frac{1}{n+1}|x| \\ &= 2(0)|x| = 0 < 1 \quad \text{for all } x \end{aligned}$$

Thus the integral of convergence of the Maclaurin series is $I = (-\infty, \infty)$.

2. We compute the following derivatives and evaluate them:

$$\begin{aligned} f(x) &= \cos x\ , & f(0) &= \cos 0 = 1 \\ f'(x) &= -\sin x\ , & f'(0) &= -\sin 0 = 0 \\ f''(x) &= -\cos x\ , & f''(0) &= -\cos 0 = -1 \\ f'''(x) &= \sin x\ , & f'''(0) &= \sin 0 = 0 \\ f^{(4)}(x) &= \cos x\ , & f^{(4)}(0) &= \cos 0 = 1 \\ &\vdots & &\vdots \quad \text{(pattern repeats)} \end{aligned}$$

Thus the Macluarin series is

$$\begin{aligned} \cos x &= 1 + \frac{1}{2!}(-1)x^2 + \frac{1}{4!}(1)x^4 + \frac{1}{6!}(-1)x^6 + \dots \\ \implies \cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots \\ \implies \cos x &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\ . \end{aligned}$$

To find the interval of convergence, we will use the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!}}{\frac{(-1)^n x^{2n}}{(2n)!}}\right| = \lim_{n\to\infty}\frac{(2n)!}{(2n+2)!}x^2 = \lim_{n\to\infty}\frac{(2n)!}{(2n+2)(2n+1)(2n)!}x^2$$
$$= \lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)}x^2 = 0\cdot x^2 = 0 < 1 \quad \text{for all } x$$

Therefore the interval of convergence is $I = (-\infty, \infty)$.

One notes that the fact that cosine is an even function is reflected in the disappearance of odd powers of x in its Maclaurin series which make it look like an even polynomial with an infinite number of terms.

Also note that the n index in our found series is not the same as the n that appears in the Maclaurin formula. The series found skips the terms with zero coefficient. This is not merely for convenience. It was required so that the Ratio Test could be applied to the series to find convergence.

Further Questions:

Find the Maclaurin series and its interval of convergence for each of the following functions.

1. $f(x) = \dfrac{1}{1-x}$
2. $f(x) = e^x$
3. $f(x) = \sin x$
4. $f(x) = \ln(1+x)$

Some important Maclaurin series and their domains of validity are:

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \quad ;\ (-1,1)\\
e^x &= \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad ;\ (-\infty,\infty)\\
\sin x &= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad ;\ (-\infty,\infty)\\
\cos x &= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad ;\ (-\infty,\infty)\\
\tan^{-1} x &= \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad ;\ [-1,1]\\
\ln(1+x) &= \sum_{k=0}^{\infty}(-1)^k\frac{x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad ;\ (-1,1]
\end{aligned}$$

With these series representations we now have a means of numerically calculating these functions for any value of x in the interval of convergence to arbitrary precision!

As we saw with the more general power series, we can derive series of new functions from known Maclaurin series.

Example 4-41

For the following functions we have the power series:

1. $e^{x^3} = \sum_{k=0}^{\infty} \frac{\left(x^3\right)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{3k}}{k!}$
   Series is valid for x^3 in $(-\infty, \infty)$ which implies for x in $(-\infty, \infty)$.

2. $$\begin{aligned}\sin^2 x &= \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \frac{(2x)^{2k}}{(2k)!} = \frac{1}{2} + \sum_{k=0}^{\infty}(-1)^{k+1}\frac{2^{2k-1}x^{2k}}{(2k)!} \\ &= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^{2k-1}x^{2k}}{(2k)!}\end{aligned}$$

   (Here we simplified our expression in the last line by noting the $k = 0$ term in the series equals $-1/2$ and so cancels the $1/2$.) Series is valid for $2x$ in $(-\infty, \infty)$ and so for x in $(-\infty, \infty)$.

These series can be confirmed to be the Maclaurin series of the given functions by direct computation.

Answers: Page 269

Exercise 4-9

1-5: Find the Maclaurin series for $f(x)$ and state the radius of convergence.

1. $f(x) = e^{-2x}$
2. $f(x) = x^2 e^{2x}$
3. $f(x) = \cos^2 x$
4. $f(x) = x^2 \sin 4x$
5. $f(x) = \cos\left(x^3\right)$

4.10 Taylor Series

Maclaurin series can be generalized by expanding in powers of $(x-a)^k$ rather than x^k for some constant a. A similar argument to that before gives the following definition.

Definition: Given function $f(x)$ differentiable to all orders at $x=a$ the power series in $(x-a)$ given by

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots\frac{f^{(k)}(a)}{k!}(x-a)^k+\cdots$$

is the **Taylor Series** for the function $f(x)$ **at** $\boldsymbol{a}$ (or about a or centred on a).

The Taylor series expansion at a will converge within some radius R about a, i.e. for $|x-a|<R$.

We note that Maclaurin series is just a special case of the Taylor series when $a=0$.

Example 4-42

Find the Taylor series of the given function at the specified value and determine the interval of convergence.

1. $f(x)=\cos x$ at $a=\dfrac{\pi}{2}$
2. $f(x)=e^{-2x}$ at $a=-1$
3. $f(x)=\dfrac{1}{x}$ at $a=1$

Solution:

1. Since the Taylor series about $x=a$ is $f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ we compute the following derivatives and evaluate them at $x=a=\dfrac{\pi}{2}$:

$$\begin{aligned}
f(x)&=\cos x\ , & f\left(\frac{\pi}{2}\right)&=\cos\frac{\pi}{2}=0\\
f'(x)&=-\sin x\ , & f'\left(\frac{\pi}{2}\right)&=-\sin\frac{\pi}{2}=-1\\
f''(x)&=-\cos x\ , & f''\left(\frac{\pi}{2}\right)&=-\cos\frac{\pi}{2}=0\\
f'''(x)&=\sin x\ , & f'''\left(\frac{\pi}{2}\right)&=\sin\frac{\pi}{2}=1\\
f^{(4)}(x)&=\cos x\ , & f^{(4)}\left(\frac{\pi}{2}\right)&=\cos\frac{\pi}{2}=0\\
&\vdots & &\vdots\ \text{(pattern repeats)}
\end{aligned}$$

Thus the Taylor series is given by:

$$\begin{aligned}
f(x)&=f\left(\frac{\pi}{2}\right)+f'\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right)+\frac{1}{2!}f''\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right)^2+\frac{1}{3!}f'''\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right)^3\\
&\quad+\frac{1}{4!}f^{(4)}\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right)^4+\dots\\
\implies \cos x&=(-1)\left(x-\frac{\pi}{2}\right)+\frac{1}{3!}(1)\left(x-\frac{\pi}{2}\right)^3+\frac{1}{5!}(-1)\left(x-\frac{\pi}{2}\right)^5+\dots\\
\implies \cos x&=-\left(x-\frac{\pi}{2}\right)+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3-\frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5+\dots\\
\implies \cos x&=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}\left(x-\frac{\pi}{2}\right)^{2n+1}
\end{aligned}$$

To find the interval of convergence we will use the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}\left(x-\frac{\pi}{2}\right)^{2n+3}}{(2n+3)!}}{\frac{(-1)^{n+1}\left(x-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}}\right| = \lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)}\left(x-\frac{\pi}{2}\right)^2$$
$$= 0\cdot\left(x-\frac{\pi}{2}\right)^2 = 0 < 1 \text{ for all } x$$

Thus the interval of convergence is $I = (-\infty, \infty)$.

2. We compute the following derivatives and evaluate them at $x = a = -1$:

$$\begin{aligned} f(x) &= e^{-2x}\ , & f(-1) &= e^2 \\ f'(x) &= -2e^{-2x}\ , & f'(-1) &= -2e^2 \\ f''(x) &= 4e^{-2x}\ , & f''(-1) &= 4e^2 \\ f'''(x) &= -8e^{-2x}\ , & f'''(-1) &= -8e^2 \end{aligned}$$

Therefore the Taylor series is

$$f(x) = f(-1) + f'(-1)(x-(-1)) + \frac{1}{2!}f''(-1)(x-(-1))^2 + \frac{1}{3!}f'''(-1)(x-(-1))^3 + \ldots$$
$$\implies e^{-2x} = e^2 + \left(-2e^2\right)(x+1) + \frac{1}{2!}\left(4e^2\right)(x+1)^2 + \frac{1}{3!}\left(-8e^2\right)(x+1)^3 + \ldots$$
$$\implies e^{-2x} = e^2\left[1 - 2(x+1) + \frac{2^2}{2!}(x+1)^2 - \frac{2^3}{3!}(x+1)^3 + \ldots\right]$$
$$\implies e^{-2x} = e^2\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{n!}(x+1)^n\,.$$

Alternatively, if only the sigma form of the series is desired, one can substitute the result $f^{(n)}(-1) = (-1)^n(2)^ne^2$ obtained from analyzing the pattern of the derivative evaluation directly into the Taylor series formula.

To find the interval of convergence, we will use the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}2^{n+1}(x+1)^{n+1}}{(n+1)!}}{\frac{(-1)^n2^n(x+1)^n}{n!}}\right| = \lim_{n\to\infty}\frac{2}{n+1}|x+1|$$
$$= 0\cdot|x+1| = 0 < 1 \text{ for all } x$$

Therefore the interval of convergence is $I = (-\infty, \infty)$.

3. We compute the following derivatives and evaluate them at $x = a = 1$:

$$\begin{aligned} f(x) &= \frac{1}{x}\ , & f(1) &= \frac{1}{1} = 1 \\ f'(x) &= -\frac{1}{x^2}\ , & f'(1) &= -1 \\ f''(x) &= \frac{2}{x^3}\ , & f''(1) &= 2 \\ f'''(x) &= -\frac{6}{x^4}\ , & f'''(1) &= -6 \\ f^{(4)}(x) &= \frac{24}{x^5}\ , & f^{(4)}(1) &= 24 \end{aligned}$$

Thus the Taylor series is

$$f(x) = f(1) + f'(1)(x-1) + \frac{1}{2!}f''(1)(x-1)^2 + \frac{1}{3!}f'''(1)(x-1)^3 + \frac{1}{4!}f^{(4)}(1)(x-1)^4 + \dots$$

$$\implies \frac{1}{x} = 1 + (-1)(x-1) + \frac{1}{2!}(2)(x-1)^2 - \frac{1}{3!}(-6)(x-1)^3 + \frac{1}{4!}(24)(x-1)^4 + \dots$$

$$\implies \frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \dots$$

$$\implies \frac{1}{x} = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

Alternatively one can observe that $f^{(n)}(1) = (-1)^n n!$ directly and place that in the Taylor series formula. Apply the Ratio Test to find the interval of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-1)^{n+1}}{(-1)^n(x-1)^n}\right| = \lim_{n\to\infty}|x-1| = |x-1|$$

The Taylor series converges if

$$|x-1| < 1 \implies -1 < x-1 < 1 \implies 0 < x < 2\,.$$

At $x = 0$ the function $\frac{1}{x}$ is undefined and the series evaluates to the divergent series $\sum 1$. At $x = 2$ the series evaluates to $\sum(-1)^n$ which also diverges by the Term Test for Divergence. Therefore the interval of convergence of the Taylor series is $I = (0, 2)$.

Further Questions:

Find the Taylor Series of the given function at the specified value and determine the interval of convergence.

1. $f(x) = \sin x$ at $a = \frac{\pi}{2}$
2. $f(x) = \ln x$ at $a = 1$
3. $f(x) = e^x$ at $a = 2$

Assuming convergence to $f(x)$, Taylor series gives us a mechanism for calculating trigonometric and other functions, namely by evaluating the first n terms of the series at x. How many terms of the series are required for a good approximation will depend on the function, the value a about which it is expanded, and x.

Tayor series allows expansion of the function f about values other than $a = 0$ which is useful for functions that are not defined at 0. Also in general fewer terms of the expansion will be required for a good approximation if the Taylor series is generated about a value a near the x of interest. Indeed by truncating the Taylor series at the $k = 1$ term

$$f(x) \approx f(a) + f'(a)(x-a)\ ,$$

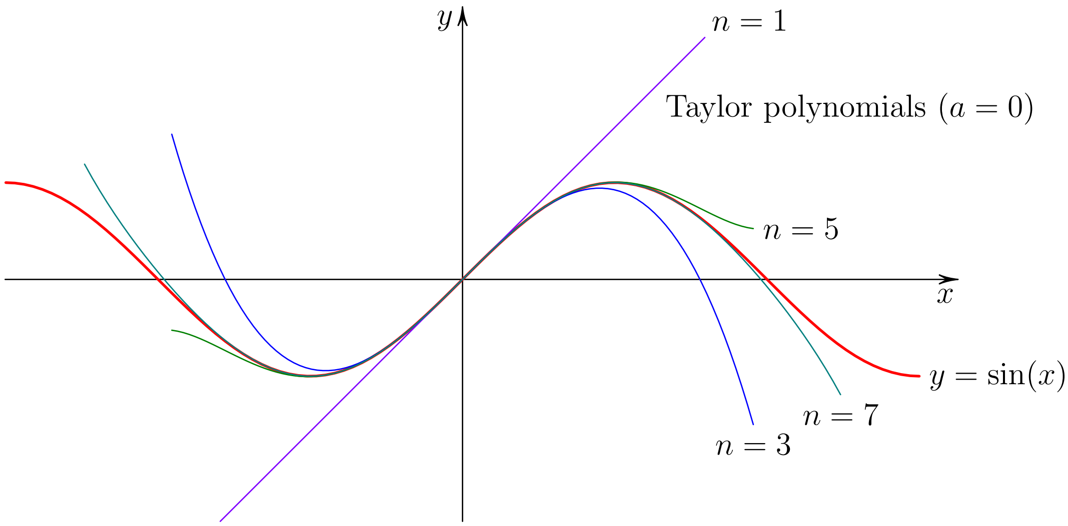
or at the $k = 2$ term,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2\ ,$$

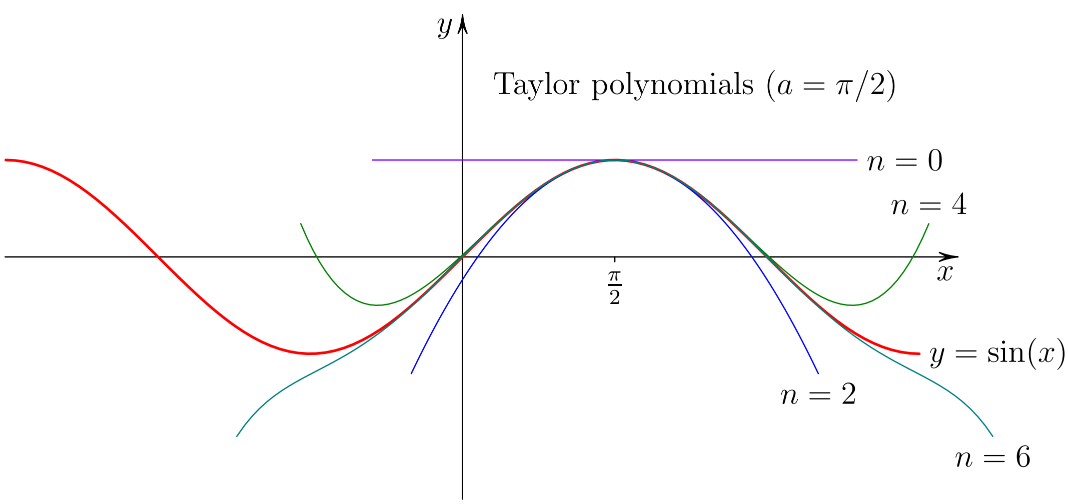
we are just reproducing the linear and quadratic approximations of a function at $x = a$ arrived at our previous course. In general truncating at the $k = n^{th}$ term,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

yields increasingly better approximations. The right hand side is called the n^{th} **Taylor polynomial** of the function $f(x)$ at a. The following diagram shows the first four Taylor polynomials of $\sin x$ at $a = 0$ (i.e. the Maclaurin series):



Here are the first four polynomials for the Taylor series of $\sin x$ at $a = \pi/2$:



One observes both the greater accuracy of the higher order approximations as well as the utility of expanding the Taylor series at a value a near the x at which you wish to approximate the function.

One useful result of Theorem 4-24 is that it justifies our original proof for the coefficients for the Maclaurin series where (recall) we required the power series to be differentiable. Generalizing this result to Taylor series we have the following result:

Theorem 4-25: If function $f(x)$ is represented by the power series

$$f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

on an open interval containing a then the coefficients are the Taylor series coefficients $c_k = f^{(k)}(a)/k!$, (i.e. $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$).

In other words, if a function $f(x)$ has a power series that converges to $f(x)$ it will be the Taylor series. The theorem does not say, however, that the Taylor series at a for a given function f will necessarily converge to f. For instance define the continuous piecewise function:

$$f(x) = \begin{cases} 0 & \text{if } x < -\pi/2 \\ \cos x & \text{if } -\pi/2 \le x \le \pi/2 \\ 0 & \text{if } x > \pi/2 \end{cases}$$

Then f will have the same Taylor series at $a = 0$ (i.e. Maclaurin series) as the function $\cos x$. However that series clearly cannot represent both functions. One must determine that the Taylor series for f at a really does converge to the function.

Theorem 4-26: If a function f has derivatives to all orders in an interval centred on a, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

will hold on the interval if and only if

$$\lim_{n\to\infty} R_n(x) = 0$$

for all x in the interval, where

$$R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \ .$$

Exercise 4-10

Answers: Page 269

1-5: Find the Taylor series for $f(x)$ centered at the given value of a.

1. $f(x) = e^{-x}, \ a = \ln 3$
2. $f(x) = \sin \pi x, \ a = \dfrac{1}{2}$
3. $f(x) = \ln x, \ a = e$
4. $f(x) = 2^x, \ a = 1$
5. $f(x) = \ln(3 + x), \ a = 1$

Answers: Page 269

Chapter 4 Review Exercises

1-3: Determine whether the given sequence is convergent or divergent.

1. $a_n = \dfrac{2n}{3n+5}$

2. $a_n = \ln\left(\dfrac{3}{2n+1}\right)$

3. $a_n = \sqrt{4n^2+5n} - 2n$

4-5: Determine whether the infinite sequence is increasing, decreasing or not monotonic.

4. $a_n = \dfrac{n+1}{5n+3}$

5. $a_n = n^2 e^{-6n}$

---

6. Consider the series $\displaystyle\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$.

   (a) Find the limit $\displaystyle\lim_{k\to\infty} a_k$. What does this say about the convergence of the series $\sum a_k$?

   (b) Find the n^{th} partial sum $S_n = \displaystyle\sum_{k=1}^{n} a_k$ of the series.
   (Hint: Expand the logarithm to get a telescoping sum.)

   (c) Evaluate $\displaystyle\lim_{n\to\infty} S_n$. Does the series $\displaystyle\sum_{k=1}^{\infty} a_k$ converge or diverge?

7-21: Determine whether the infinite series is convergent or divergent.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{4n}{n^4+7}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n^2+10}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{7+6^n}{4^n}$

10. $\displaystyle\sum_{n=1}^{\infty} \left[\frac{5}{n(n+1)} - 2^{-n}\right]$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{4n+3}$

12. $\displaystyle\sum_{n=1}^{\infty} n^3 e^{-2n^4}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2+6}{n^2\, e^n}$

14. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{2n+3}}{6n+7}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$

16. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{(n+1)(n+2)}$

17. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n^3+4}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{200-n}{n!}$

19. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3}{n^2\,(\ln n)^4}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)!}{e^{n^2}}$

21. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n-5}{5n+6}\right)^n$

22-25: Find the interval and the radius of convergence of the power series.

22. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(x-4)^{3n}}{(3n+1)!}$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}(x-7)^n$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt[3]{n}}(x-10)^n$

25. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+3}(2x+9)^n$

26-27: Find the Maclaurin series for $f(x)$ and state the radius of convergence.

26. $f(x) = \sin^2 x$

27. $f(x) = xe^{-5x}$

28-29: Find the Taylor series for $f(x)$ centered at the given value of a.

28. $f(x) = e^{-5x},\ a = 2$

29. $f(x) = \ln(x+2),\ a = 0$

Chapter 5: Integration Applications

5.1 Areas Between Curves

We have seen this topic in a previous course. We review it here for the purpose of reminding ourselves how to interpret differentials in integration formulae.

We have already calculated an area between curves. Namely, when $y = f(x) \geq 0$ on $[a, b]$ then

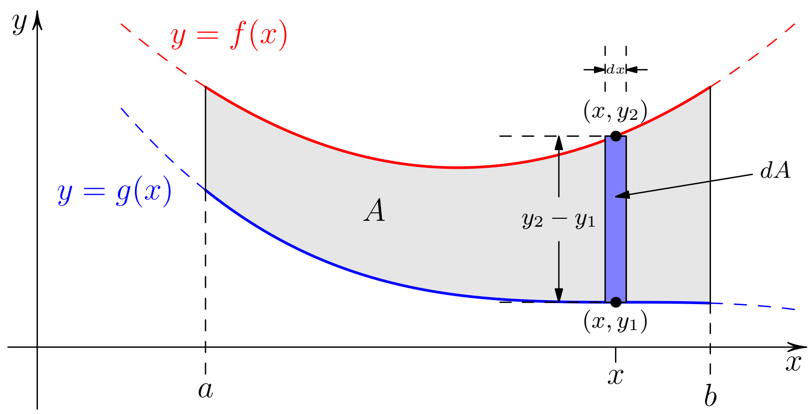
$$A = \int_a^b f(x)\,dx$$

is the area below the curve $y = f(x)$ and above the x-axis. But since the y-axis is just the curve $y = 0$, the above integral is also the area between the curve $y = f(x)$ and the curve $y = 0$. If the lower curve is $y = g(x)$ instead of $y = 0$ we get the following result.

Theorem 5-1: If $y = f(x)$ and $y = g(x)$ are integrable on the x-interval $[a, b]$ with $f(x) \geq g(x)$ on the interval then the **area between the curves** $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\ dx\ .$$

The situation is illustrated in the following diagram.



A simple way to remember the theorem is using differential notation. In the picture is shown a vertical differential area element at x. The endpoints have this same x coordinate but different y coordinates. The height of the area element is $y_2 - y_1$ where we subtract the lower value from the higher value to ensure a positive height. The width of the element is dx. The differential area dA is then

$$dA = \underbrace{(y_2 - y_1)}_{\text{height}} \cdot \underbrace{dx}_{\text{width}}\ .$$

We require the differential area to be written in terms of the differential variable, here x. Since (x, y_2) lies on the upper curve it satisfies that equation so $y_2 = f(x)$ and similarly (x, y_1) lies on the lower curve and satisfies that equation so $y_1 = g(x)$. The differential area element becomes

$$dA = [f(x) - g(x)]\,dx\,.$$

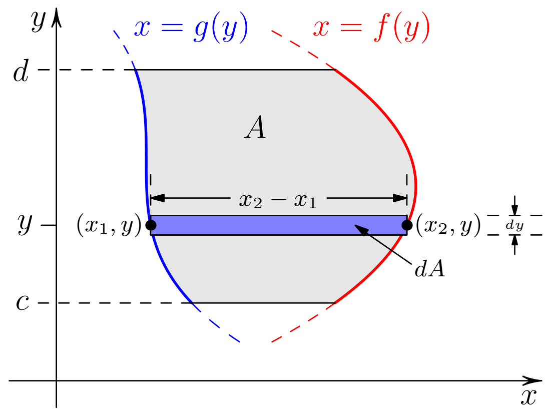
We are finding the sum $A = \int dA$ of these infinitesimal rectangular areas each of area dA starting at $x = a$ and ending at $x = b$ which returns us to our formula. Note that the limits must be in terms of the same variable as the differential, here x.

Sometimes the region for which one wants an area is better described by functions $x = f(y)$ and $x = g(y)$ denoting the right and left boundaries of the region respectively. One then has the following result.

Theorem 5-2: If $x = f(y)$ and $x = g(y)$ are integrable on the y-interval $[c, d]$ with $f(y) \geq g(y)$ on the interval then the **area between the curves** $x = f(y)$ and $x = g(y)$ and the lines $y = c$ and $y = d$ is

$$A = \int_c^d [f(y) - g(y)] \, dy \; .$$

The situation is depicted below.



Here we have a horizontal area element at y, so the y coordinate at its endpoints is this value, but the x coordinates must be labelled distinctly. The width of the area element is $x_2 - x_1$ where we subtract the leftmost value from the rightmost one to ensure a positive width. The height of the area element is dy and therefore the differential area element is

$$dA = \underbrace{(x_2 - x_1)}_{\text{width}} \cdot \underbrace{dy}_{\text{height}} \; .$$

We require the area element to be in terms of the differential variable, here y. Since (x_2, y) lies on the curve to the right it satisfies that equation and $x_2 = f(y)$. Similarly (x_1, y) lies on the curve on the left so $x_2 = g(y)$. The differential area element becomes

$$dA = [f(y) - g(y)] \, dy \, .$$

In this case the integral represents the addition $A = \int dA$ of horizontal area elements of area dA starting at $y = c$ and ending at $y = d$ which returns us to the formula in our theorem. Note that once again the limits must be in terms of the variable found in the differential, here y.

Sometimes no interval $[a, b]$ or $[c, d]$ is given. If one is asked to find the region bounded by $y = h_1(x)$ and $y = h_2(x)$ one uses vertical area elements (first theorem) to find the area. One must solve $h_1(x) = h_2(x)$ to find x-values x_i which enclose the bounded region(s). There may be multiple enclosed regions and these values determine the endpoints of the integration. One must ascertain which function is the higher one on each region.

If one is asked to find the region bounded by $x = h_1(y)$ and $x = h_2(y)$ then one uses horizontal area elements (second theorem) to find the area. One must solve $h_1(y) = h_2(y)$ to find y-values y_i which enclose the bounded region(s). Once again these determine the endpoints of integration. One must determine which function is greater (on the right) on each region.

As we proceed through this chapter we will try to construct our integrals using these intuitive steps to build the necessary differentials. This will limit the need to memorize many formulae.

Example 5-1

Find the area of the region bounded by the given curves.

1. $y = x^2 - 5x$ and $y = 10x - x^2$
2. $x = y^2 - 4$ and $x = -y^2$

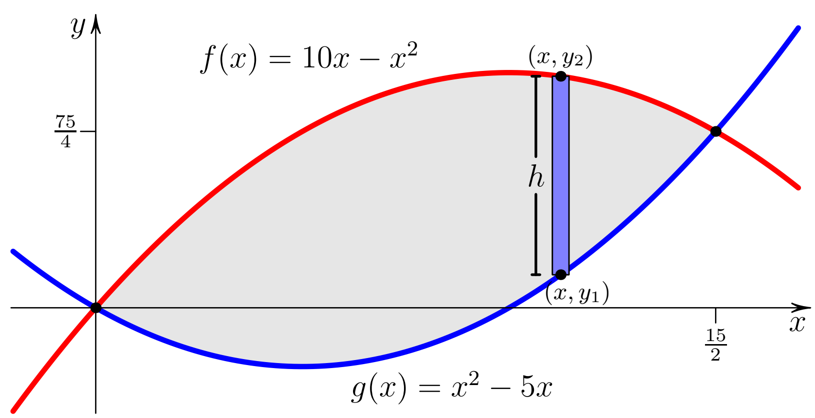
Solution:

1. $y = x^2 - 5x$ and $y = 10x - x^2$

   Here both curves are written as functions of x so we will use vertical area elements. The curves intersect when

$$x^2 - 5x = 10x - x^2 \implies 2x^2 - 15x = 0 \implies x(2x - 15) = 0 \implies x = 0 \text{ or } x = \frac{15}{2}\,.$$

Thus $x = 0$ and $x = \frac{15}{2}$ are the x-coordinates of the points of intersection. Evaluating the y-coordinates using either function gives the points of intersection to be $(0, 0)$ and $(15/2, 75/4)$. Consideration of the value $x = 3$ (between the endpoints) gives $y = 3^2 - 5(3) = -6$ for the first curve and $y = 10(3) - 3^2 = 21$ for the second curve showing that the second curve lies above the first over the interval of intersection. A graph of the situation follows.



Noting that the height $h = y_2 - y_1$, the differential area element is

$$dA = (y_2 - y_1)dx = [f(x) - g(x)]dx = \left[(10x - x^2) - (x^2 - 5x)\right] dx\,,$$

where we used the points on the diagram to write y_2 and y_1 as functions of the variable x found in the differential, dx. The rectangles are summed starting at $x = 0$ and ending at $x = \frac{15}{2}$. Once again these limits must be in terms of the variable x found in the differential. The area is therefore, using this argument or by formula $A = \int_a^b [f(x) - g(x)]\, dx$,

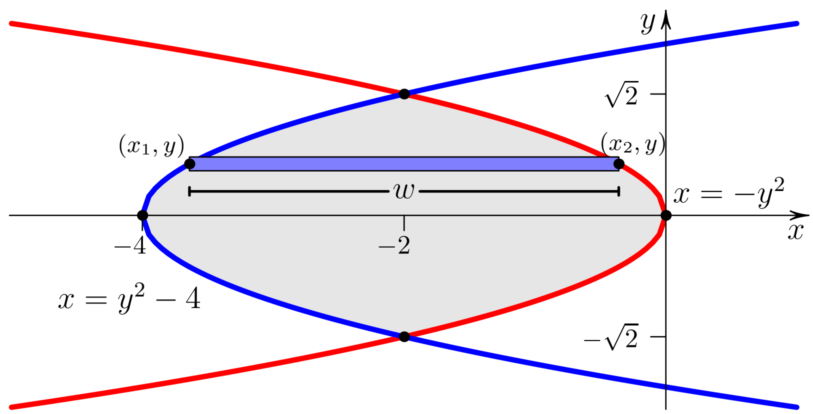
$$\begin{aligned} A &= \int_0^{\frac{15}{2}} \left[(10x - x^2) - (x^2 - 5x)\right]\, dx = \int_0^{\frac{15}{2}} (15x - 2x^2)\, dx \\ &= \left[\frac{15}{2}x^2 - \frac{2}{3}x^3\right|_0^{\frac{15}{2}} = \frac{15}{2}\left(\frac{15}{2}\right)^2 - \frac{2}{3}\left(\frac{15}{2}\right)^3 = \frac{15^3}{8}\left(1 - \frac{2}{3}\right) \\ &= \frac{15^3}{24} = \frac{3375}{24} = \frac{1125}{8} \text{ square units.} \end{aligned}$$

2. $x = y^2 - 4$ and $x = -y^2$

   Here both curves are written as functions of y so we will use horizontal area elements. The curves intersect when

$$y^2 - 4 = -y^2 \implies 2y^2 = 4 \implies y^2 = 2 \implies y = \pm\sqrt{2}\,.$$

   Evaluating x at these values shows the points of intersection are $(-2, -\sqrt{2})$ and $(-2, \sqrt{2})$. Evaluating x at the value $y = 0$ that lies within the interval $[-\sqrt{2}, \sqrt{2}]$ gives $x = -4$ for the first curve and $x = 0$ for the second curve showing that $f(y) = -y^2$ is the curve further to the right and $g(y) = y^2 - 4$ is the curve further to the left over the interval of interest. A graph of the situation follows.



   Noting that the width $w = x_2 - x_1$, the differential area element is

$$dA = (x_2 - x_1)dy = [f(y) - g(y)]dy = (-y^2 - (y^2 - 4))\,dy\,,$$

   where we used the points on the diagram to write x_2 and x_1 as functions of the variable y found in the differential, dy. The rectangles are summed starting at $y = -\sqrt{2}$ and ending at $y = \sqrt{2}$. Once again these limits must be in terms of the variable y found in the differential. The area is therefore, using this argument or by formula $A = \int_c^d [f(y) - g(y)]\,dy$,

$$\begin{aligned}
A &= \int_{-\sqrt{2}}^{\sqrt{2}} (-y^2 - (y^2 - 4))\,dy = \int_{-\sqrt{2}}^{\sqrt{2}} (-2y^2 + 4)\,dy = -\frac{2}{3}y^3 + 4y\bigg|_{-\sqrt{2}}^{\sqrt{2}} \\
&= -\frac{2}{3}(\sqrt{2})^3 + 4(\sqrt{2}) - \left[-\frac{2}{3}(-\sqrt{2})^3 + 4(-\sqrt{2})\right] \\
&= -\frac{4}{3}(2\sqrt{2}) + 8(\sqrt{2}) = \left(8 - \frac{8}{3}\right)\sqrt{2} = \frac{16\sqrt{2}}{3} \text{ square units.}
\end{aligned}$$

Further Questions:

Find the area of the region bounded by the following curves.

1. $y = 6 - x^2$ and $y = 3 - 2x$
2. $x = y^2$ and $x - y - 2 = 0$

Answers: Page 270

Exercise 5-1

1-5: Find the area of the region bounded by the given curves.

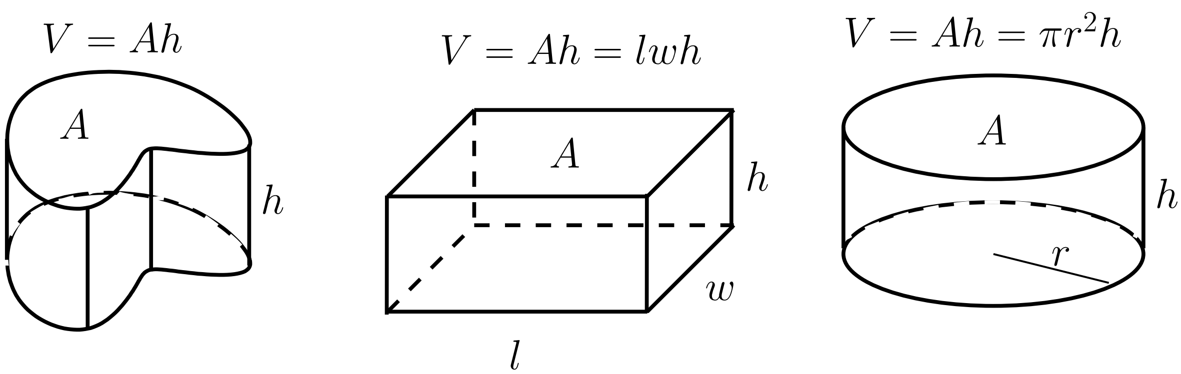
1. $y = -x^2 + 5, \;\; y = 2 - 2x$
2. $y = x + 5, \;\; y = x^2, \; x = 0, \; x = 1$
3. $x = y^2, \;\; y = x - 6$
4. $y = x^2, \;\; y = \sqrt{x}, \; x = 0, \; x = 4$
5. $y = x, \;\; y = 5x, \; x + y = 6$

5.2 Calculation of Volume

5.2.1 Volume as a Calculus Problem

The **volume** of a three-dimensional object is the amount of space it occupies (expressed in cubic units). If we had an odd-shaped jar and had a bag of caramels that were a cm on each side we could estimate the jar's volume by counting the number of caramels that would fit in the jar. This would be the volume in cubic centimetres (i.e. millilitres). In a more accurate estimate we would fill the jar with water and then measure the number of millilitres with a measuring cup.

We are interested in how we can use calculus to find such volumes. An object for which we have a formula for volume is a **right cylinder** . A cylinder is a three-dimensional geometric figure that has two congruent and parallel bases. When the bases are aligned one directly above the other one has a right cylinder. The following are three examples of right cylinders:



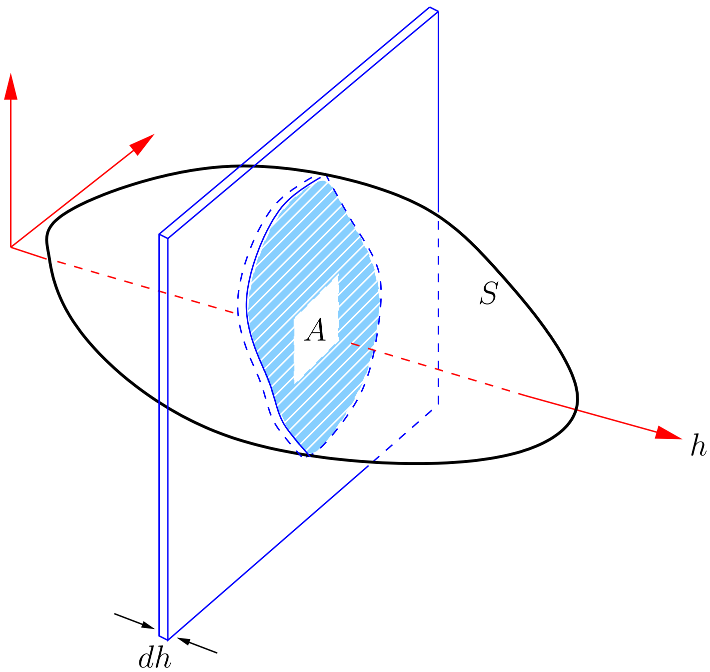
The first shows the most general case, while the second and third are special cases, a rectangular parallelepiped and a right circular cylinder, respectively. In general the area of a right cylinder is:

$$V = Ah$$

In the special case where the area is expressible in terms of other lengths we can further specialize the formula. So a right circular cylinder has volume:

$$V = \pi r^2 h$$

Returning to the general case of finding a volume of an arbitrary 3D object we may follow the same route we did for solving general areas. In the area case we sliced the figure into increasingly thinner rectangles ($\Delta x \to 0$) and then added those up. For a 3D object we can imagine slicing it into a large number of increasingly thinner (shorter) right cylinders (of height $\Delta h \to 0$) and adding them up. To do this we take an abitrary axis through our solid S and partition the axis into equal widths Δh. At each point h along the axis we will have a **cross-sectional area** A in the plane perpendicular to our axis h.



The area A will depend upon the position h along the axis and so it is a function $A(h)$ of h. The volume of our object will then be the limit of the Riemann Sum of the n cylinders with volumes V_i:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{A(h_i^*)\Delta h}_{V_i}$$

The position h_i^* is taken to be a point in the corresponding interval along h of the i^{th} cylinder. This limit is the definite integral:

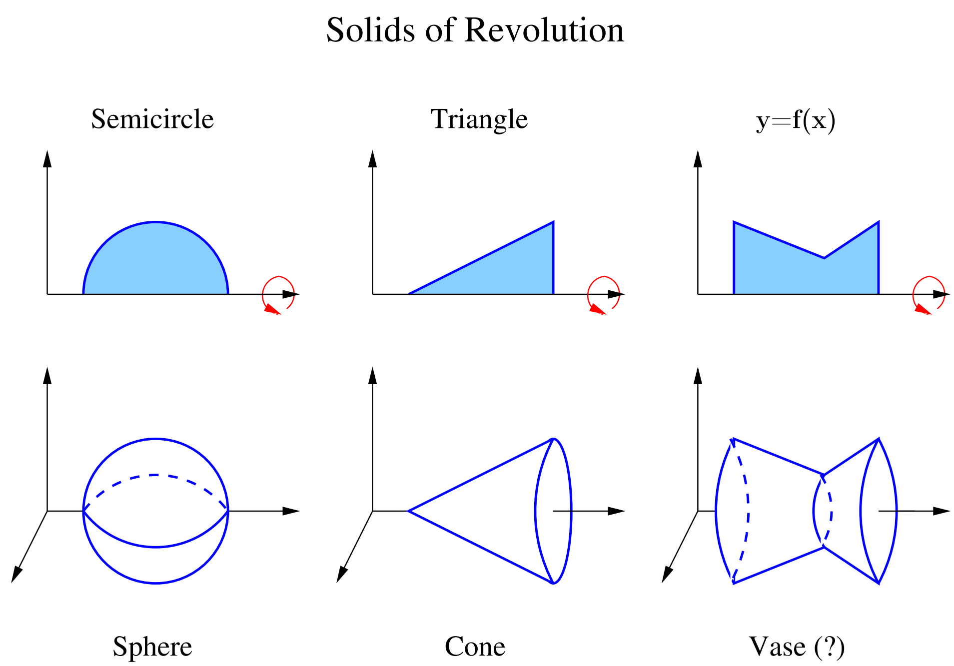
$$V = \int_a^b A(h)dh \ ,$$

where a and b denote the position of the first and last cylinders needed. Thinking in differential notation we are summing ($\int dV$) the volumes of infinitessimally thin cylinders given by:

$$dV = A\, dh$$

5.2.2 Solids of Revolution

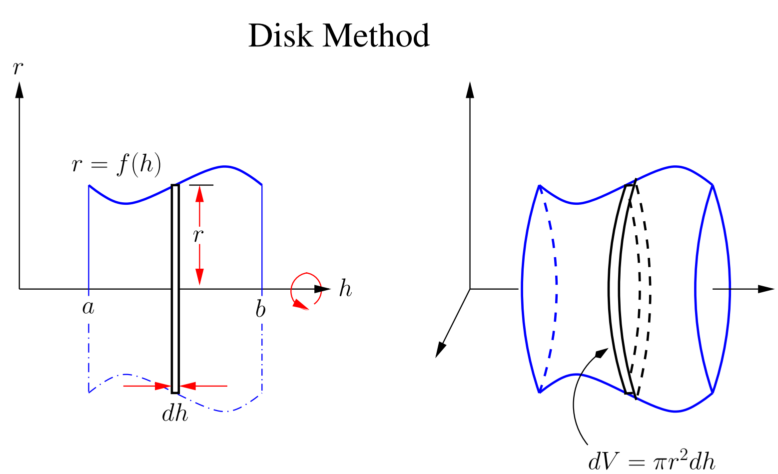
In general the function $A(h)$ for a particular solid S will be complicated. However if the solid has a symmetry and the axis h is directly along that axis of symmetry the function $A(h)$ may be calculated. A certain class of solids, called **Solids of Revolution** have such a symmetry because they are generated by sweeping out a two dimensional area around just such an axis:



As can be seen in the diagram some common shapes (spheres and cones) can be thought of as solids of revolution of simple areas (semicircles and triangles). Other interesting shapes may be constructed by choosing other functions for the upper boundary.

5.2.3 The Disk Method

For a solid of revolution for which the planar area of the revolution extends from the axis of symmetry to a boundary curve describable by a function, the volume is now calculable because the area function $A(h)$ can be found:



As shown in the diagram, the general right cylinders are now right **circular** cylinders with area $A = \pi r^2$. Our volume element is the **disk** of differential volume:

$$\boxed{dV = \pi r^2 dh}$$

where the radius r for the given disk is $r = f(h)$. The volume is then the sum $(V = \int dV)$ given by

$$V = \int_a^b \pi r^2 \, dh = \int_a^b \pi \left[f(h)\right]^2 dh$$

If the symmetry axis is the x-axis (so $h = x$ and $r = y = f(x)$) then the formula for a solid of revolution about the x-axis becomes

$$V = \int_a^b \pi \left[f(x)\right]^2 dx$$

This method of calculating volumes is called the **Disk Method** because we are approximating the volume with disks (i.e. short right cylinders).

Example 5-2

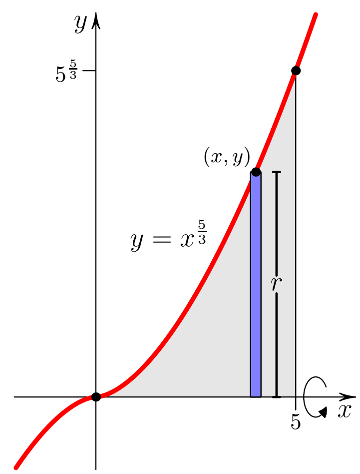
Find the volume of the solid obtained by revolving the following regions about the x-axis.

1. The region bounded by the curves $y = x^{\frac{5}{3}}, x = 5$, and $y = 0$.
2. The region bounded by the curves $y = 6x - x^2$ and $y = 0$.

Solution:

1. The region bounded by the curves $y = x^{\frac{5}{3}}, x = 5$, and $y = 0$.

   A graph of the region of rotation is the following.



The vertical area element at x being rotated has upper coordinate (x, y). The radius then is $r = y$ and the height of the disk $dh = dx$. The disk volume element dV is

$$dV = \pi r^2 dh = \pi y^2 dx = \pi \left(x^{\frac{5}{3}}\right)^2 dx\,.$$

Here the variable y must be written in terms of x, the variable found in the differential (dx). Since the upper point (x, y) sits on the curve $y = x^{\frac{5}{3}}$ this means it satisfies that equation allowing the replacement. In addition to replacing all variables in terms of those of the differential dx, the limits are also determined to be along the axis of the differential (again x), so $x = 0$ to $x = 5$. The Disk Method volume using the above derivation or by formula $V = \int_a^b \pi \left[f(x)\right]^2 dx$ is

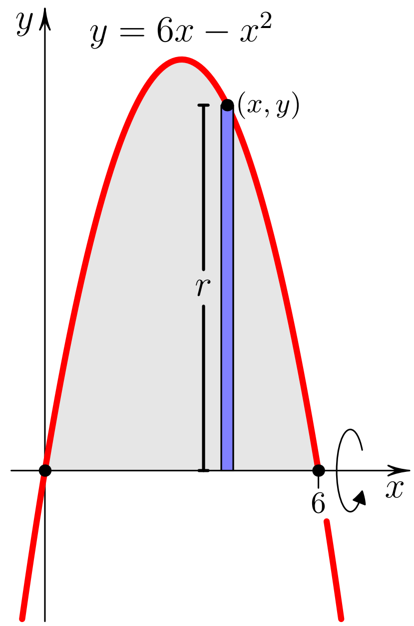
$$V = \int_0^5 \pi \left(x^{\frac{5}{3}}\right)^2 dx = \pi \int_0^5 x^{\frac{10}{3}} \, dx = \pi \left[\frac{3}{13} x^{\frac{13}{3}}\right]_0^5 = \pi \left[\frac{3}{13}(5)^{\frac{13}{3}} - 0\right] = \frac{3\pi}{13}(5)^{\frac{13}{3}} \text{ cubic units.}$$

2. The region bounded by the curves $y = 6x - x^2$ and $y = 0$.

   The curves intersect when

$$6x - x^2 = 0 \implies x(6-x) = 0 \implies x = 0 \text{ or } x = 6\,,$$

so at points $(0,0)$ and $(6,0)$. A graph of the region is the following.



Labelling the upper end of the area element (x, y) we see that

$$dV = \pi r^2 dh = \pi y^2 dx = \pi \left(6x - x^2\right)^2 dx\,.$$

with limits of integration along the axis of the differential (dx) being $x = 0$ and $x = 6$. The volume using the above differential or the formula $V = \int_a^b \pi\, [f(x)]^2\, dx$ is

$$V = \int_0^6 \pi \left(6x - x^2\right)^2 dx = \pi \int_0^6 (36x^2 - 12x^3 + x^4)\, dx = \pi \left[12x^3 - 30x^4 + \frac{1}{5}x^5\right|_0^6$$
$$= \pi \left(12(6)^3 - 3(6)^4 + \frac{1}{5}(6)^5 - [0 - 0 + 0]\right) = \frac{1296\pi}{5} \text{ cubic units.}$$

As this example illustrates, rather than memorizing a formula and going straight to the integral, one may simply remember the intuitive formula for the differential volume $dV = \pi r^2 dh$ and systematically replace the variables and differential for the given situation. Remember that the entire differential as well as the limits of the integral must ultimately be written in terms of the variable found in the differential dh.

Further Questions:

Find the volume of the solid obtained by revolving the following regions about the x-axis.

1. The region bounded by the curves $y = x^{\frac{3}{2}}$, $x = 4$, and $y = 0$.
2. The region bounded by the curve $y = 2x - x^2$ and the line $y = 0$.

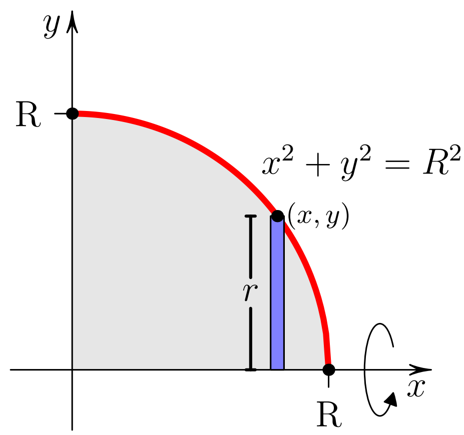
The constant π is defined to be the ratio of the circumferance to the diameter of any circle, $\pi = \frac{C}{D}$. Archimedes (c. 287-212 BCE) discovered (long before the invention of calculus) that the constant π surprisingly shows up in the formulae for the area of a circle ($A = \pi R^2$) as well as for a sphere's volume ($V = \frac{4}{3}\pi R^3$) and surface area ($S = 4\pi R^2$).

Example 5-3

Use calculus to show that the volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$.

Solution:

Consider a circle of radius R with centre at the origin. The region bounded by this circle and the lines $x = 0$ and $y = 0$ lying in the first quadrant is shown.



Here $y = f(x) = \sqrt{R^2 - x^2}$. If we rotate this region about the x-axis, we obtain half the volume of a sphere. Using the Disk Method, the differential volume element is

$$dV = \pi r^2 dh = \pi y^2 dx = \pi \left(R^2 - x^2\right) dx \,.$$

The limits along the differential axis (x) are $x = 0$ and $x = R$ and the integral is then, using the differential or formula $V = \int_a^b \pi \left[f(x)\right]^2 dx$,

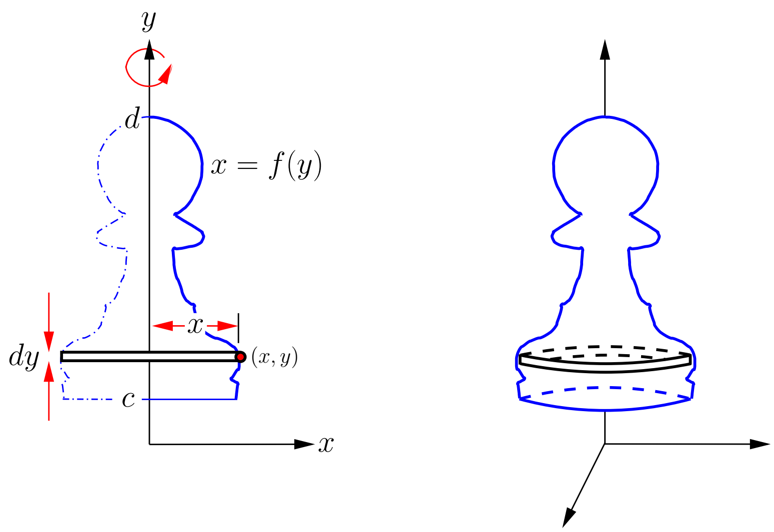
$$V = \pi \int_0^R (R^2 - x^2)\, dx = \pi \left[R^2 x - \frac{1}{3}x^3 \right|_0^R = \pi \left(R^3 - \frac{1}{3}R^3 - \left[R^2(0) - \frac{1}{3}(0)^3 \right] \right) = \frac{2}{3}\pi R^3 \,.$$

Therefore, the volume of the sphere is $2\left(\frac{2}{3}\pi R^3\right) = \frac{4}{3}\pi R^3$.

Further Question:

Use calculus to show that the volume of a right circular cone with base radius R and height H is $V = \frac{1}{3}\pi R^2 H$.

We sometimes wish to revolve a region about the y-axis. For a region bounded by the curve $x = f(y)$ and the lines $x = 0$ (i.e. the y-axis), $y = c$, and $y = d$ we can use horizontal disks. In this case $dh = dy$ (since the symmetry axis h is now y) and $r = x = f(y)$ as shown in the following diagram.



The formula for the volume of such a region revolved about the y-axis is therefore

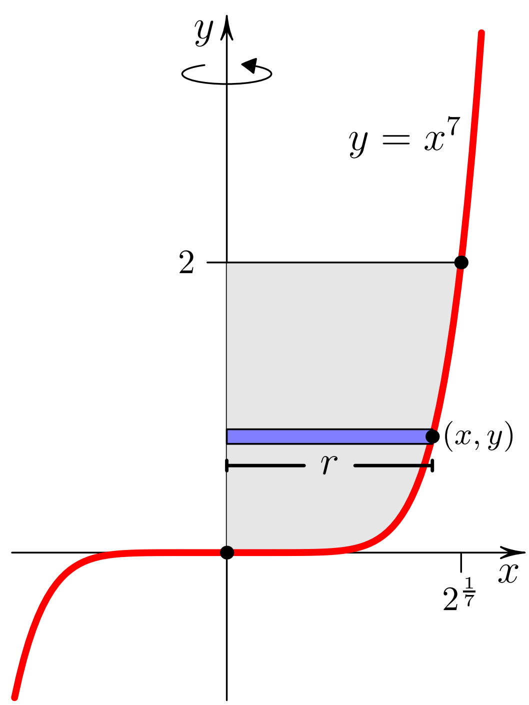
$$V = \int_c^d \pi \left[f(y)\right]^2 \, dy \, .$$

Example 5-4

Find the volume of the solid obtained by revolving the region bounded by $y = x^7$, $x = 0$, and $y = 2$ about the y-axis.

Solution:

The lines $y = x^7$ and $y = 2$ intersect when $x^7 = 2 \implies x = 2^{\frac{1}{7}}$, and so at the point $(2^{\frac{1}{7}}, 2)$. A graph of the region being rotated is as follows.



Consideration of the point (x, y) at the end of the horizontal differential element shows that $r = x$ and $dh = dy$, so the volume element is

$$dV = \pi r^2 dh = \pi x^2 dy = \pi \left(y^{\frac{1}{7}}\right)^2 dy\,,$$

where here we used that (x, y) lies on the curve $y = x^7 \implies x = f(y) = y^{\frac{1}{7}}$ to replace x in terms of the differential variable y. The limits in y are $y = 0$ and $y = 2$ so the Disk Method volume is therefore, using the differential or the formula $V = \int_c^d \pi\,[f(y)]^2\,dy$,

$$V = \int_0^2 \pi \left(y^{\frac{1}{7}}\right)^2 dy = \pi \int_0^2 y^{\frac{2}{7}}\,dy = \pi \left[\frac{7}{9} y^{\frac{9}{7}}\right]_0^2 = \frac{7\pi}{9}\left[(2)^{\frac{9}{7}} - (0)^{\frac{9}{7}}\right] = \frac{7\pi}{9}(2)^{\frac{9}{7}} \text{ cubic units.}$$

Further Question:

Find the volume of revolution of the solid obtained by revolving the region bounded by $y = x^2$, $x = 0$, and $y = 4$ about the y-axis.

In general we may be interested in revolving an area about a horizontal or vertical axis which is neither the x nor the y-axis. In this case we must work from the differential volume dV to determine the volume integral. Follow the following steps which will work for any volume problem:

1. Draw the region to be revolved labelling the curves appropriately as well as points of intersection. Draw the symmetry line about which the curve is to be revolved.

2. Decide on the differential volume element you will use and indicate a typical cross-section rectangle on your region.

3. Label the endpoints of rectangle with coordinates (x, y), etc. Note that for vertical elements the endpoint coordinates will have the same endpoint x values while you will need to distinguish the y values (i.e. y_1, y_2) as these values will differ. If a value is constant (e.g. 0) for all potential cross-section rectangles label it as such instead of using a variable. The opposite holds if you are using a horizontal element (i.e. same y values at endpoints, different x's.)

4. Label the differential width of the cross-section rectangle as dx or dy according to whether you are using a vertical or horizontal elements respectively.

5. Write down your integral, $V = \int dV$, and substitute your volume element for dV.

6. Next replace each variable in the integral in terms of the variables of your cross-section rectangle endpoint coordinate variables. Some calculation may be required here, especially if you are not revolving about a coordinate axis.

7. From your diagram determine the limits of integration. These must be along the same axis as found in your differential, i.e. x for dx or y for dy. Note the limits are for the rectangles only within the region being revolved.

8. All variables of the integrand must be in terms of the integration variable. Replace any variables in your integral that are not the same as the differential variable in terms of the differential variable by considering the curve that the endpoint involving the variable lies upon. You may need to solve for the variable using the curve equation to do so.

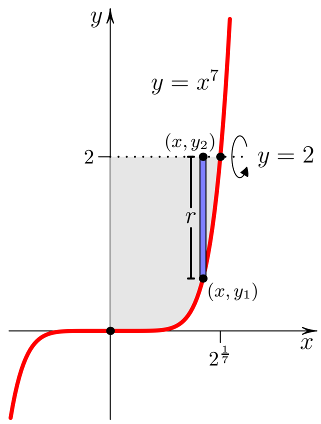
9. Calculate the integral.

Example 5-5

Find the volume of the solid generated by revolving the region bounded by $y = x^7$, $x = 0$, and $y = 2$ about the line $y = 2$.

Solution:

The region being rotated is the same as in Example 5-4, but now we are rotating about the line $y = 2$ and so require a vertical area element as shown in the following diagram.



Because the rotation is not about the Cartesian axes there is no formula that can be used directly. Instead we build up the volume element dV by first labelling the endpoints of the vertical element (x, y_1) and (x, y_2). Note these points have the same x coordinate but different y coordinates which must be distinguished. (The opposite would be true for a horizontal area element.) From the diagram we see that the radius $r = y_2 - y_1$, where here we have subtracted the lower y coordinate from the upper one to ensure a positive radius. (For horizontal elements we would subtract the value on the left from that on the right to ensure positivity.) Finally $dh = dx$ and we have for the volume element

$$dV = \pi r^2 dh = \pi (y_2 - y_1)^2 dx = \pi \left(2 - x^7\right)^2 dx\,.$$

Here we replaced y_2 and y_1 to get them in terms of the differential variable x by consideration of which curve the points lie on in the diagram. The limits in x are $x = 0$ on the left to $x = 2^{\frac{1}{7}}$ on the right. The volume is then

$$\begin{aligned} V &= \int_0^{2^{\frac{1}{7}}} \pi(2 - x^7)^2\,dx = \pi \int_0^{2^{\frac{1}{7}}} (4 - 4x^7 + x^{14})\,dx = \pi \left[4x - \frac{1}{2}x^8 + \frac{1}{15}x^{15} \right|_0^{2^{\frac{1}{7}}} \\ &= \pi \left(4(2^{\frac{1}{7}}) - \frac{1}{2}(2^{\frac{1}{7}})^8 + \frac{1}{15}(2^{\frac{1}{7}})^{15} - [0 - 0 + 0] \right) \\ &= \pi \left(4(2^{\frac{1}{7}}) - \frac{1}{2}(2^{\frac{8}{7}}) + \frac{1}{15}(2^{\frac{15}{7}}) \right) = \pi(2^{\frac{1}{7}}) \left(4 - \frac{1}{2}(2)^{\frac{7}{7}} + \frac{1}{15}(2)^{\frac{14}{7}} \right) \\ &= \pi(2^{\frac{1}{7}}) \left(4 - 1 + \frac{4}{15} \right) = \frac{49\pi(2^{\frac{1}{7}})}{15} \text{ cubic units.} \end{aligned}$$

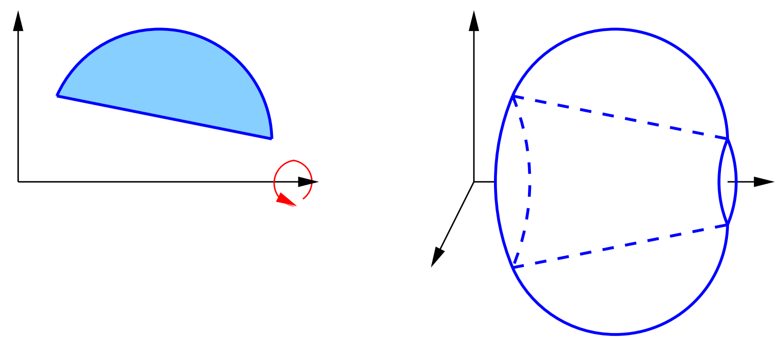
The reader is encouraged to visualize the volume generated in this problem compared to that of Example 5-4. The axis about which one rotates matters!

Further Question:

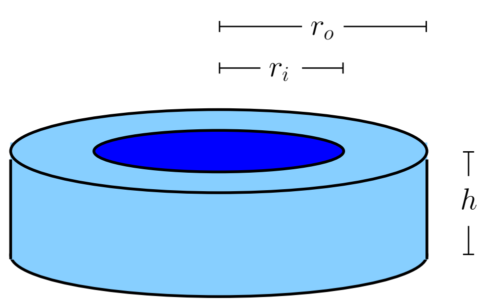
Find the volume of the solid generated by revolving the region bounded by $y = x^2$ and $y = 4$ about the line $y = 4$.

5.2.4 The Washer Method

Suppose we generate a solid of revolution in which the lower boundary is not the axis of revolution but, more generally the curve $g(h)$ above it. These regions are what we considered when we looked at areas between curves having an upper boundary determined by a function f and a lower boundary determined by a function g. If such a region is revolved about a chosen axis of symmetry h then the solid of revolution generated will have a hole in it and the disk method will not, at first glance, work.



The disk method can be used to calculate the volume of the outer solid and then again the inner solid with the actual volume being the difference of the two, but instead we introduce a new method which solves the problem with a single integral. Consider a right circular cylinder of height h and outer radius r_o with an interior cylinder of inner radius r_i removed:



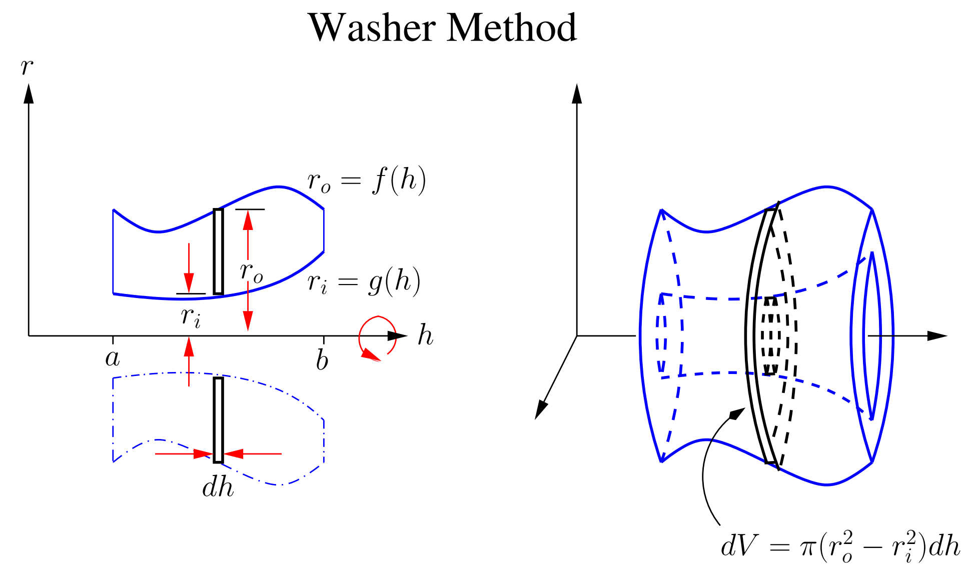
The volume of the solid is just the difference of the volumes of the cylinders:

$$V = V_o - V_i = \pi r_o^2 h - \pi r_i^2 h$$

Factoring out the π and h gives:

$$V = \pi(r_o^2 - r_i^2)h$$

If we imagine shrinking the height h to a small height Δh the shape looks like a metal *washer*. In the **Washer Method** we will slice solids of revolution with symmetrical holes in them into such washer-shaped volumes and sum them up to find the total volume of the solid.



In the Washer Method the height of the washer Δh is replaced with the differential dh so the differential element of volume is:

$$\boxed{dV = \pi(r_o^2 - r_i^2)dh}$$

The total volume is just the sum ($V = \int dV$) and we have:

$$V = \int_a^b \pi(r_o^2 - r_i^2)dh = \int_a^b \pi\left([f(h)]^2 - [g(h)]^2\right) dh$$

where we have replaced the outer and inner radii by their functional values at h (see diagram). If the symmetry axis is the x-axis (so $h = x$, $r_o = y_2 = f(x)$, and $r_i = y_1 = g(x)$) then the formula for a solid of revolution about the x-axis becomes

$$V = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right) dx$$

This may always be found starting with the differential volume element $dV = \pi(r_o^2 - r_i^2)dh$ and following the general steps.

Example 5-6

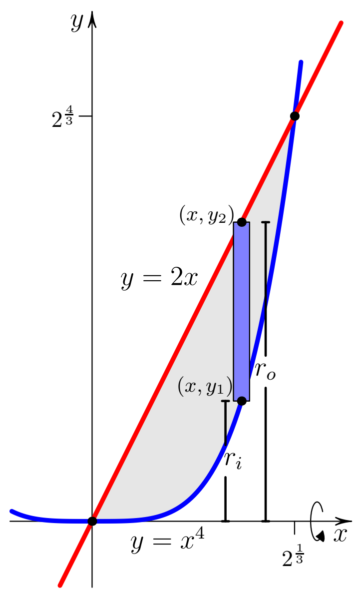
Find the volume of the solid obtained by revolving the region bounded by the curves $y = 2x$ and $y = x^4$ about the x-axis.

Solution:

The curves $y = 2x$ and $y = x^4$ intersect when

$$2x = x^4 \implies 0 = x^4 - 2x \implies 0 = x(x^3 - 2) \implies x = 0 \text{ or } x = 2^{\frac{1}{3}}\,,$$

and so at the points $(0, 0)$ and $(0, 2^{\frac{4}{3}})$. A graph of the region is



Because the area to be revolved is not flush with the axis of revolution (the x-axis), we use the Washer Method that can accommodate the hole. As with the Disk Method, the area element for the Washer Method is perpendicular to the axis of symmetry as shown. From the diagram the outer and inner radii are $r_o = y_2$ and $r_i = y_1$, and the differential height is $dh = dx$. The volume element is

$$dV = \pi(r_o^2 - r_i^2)dh = \pi(y_2^2 - y_1^2)dx = \pi\left((2x)^2 - (x^4)^2\right)dx\,,$$

where we write the y coordinates in terms of the differential variable x by consideration of the curves upon which their points lie. The limits of that variable are $x = 0$ and $x = 2^{\frac{1}{3}}$. The volume is therefore, using the above reasoning or directly by formula $V = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right)dx$,

$$\begin{aligned} V &= \int_0^{2^{\frac{1}{3}}} \pi\left((2x)^2 - (x^4)^2\right)dx = \pi\int_0^{2^{\frac{1}{3}}}\left(4x^2 - x^8\right)dx = \pi\left[\frac{4}{3}x^3 - \frac{1}{9}x^9\right|_0^{2^{\frac{1}{3}}} \\ &= \pi\left(\frac{4}{3}(2^{\frac{1}{3}})^3 - \frac{1}{9}(2^{\frac{1}{3}})^9 - [0-0]\right) = \pi\left(\frac{4}{3}(2) - \frac{1}{9}(2)^3\right) = \pi\left(\frac{8}{3} - \frac{8}{9}\right) \\ &= \frac{16\pi}{9} \text{ cubic units.} \end{aligned}$$

Further Question:

Find the volume of the solid obtained by revolving the region bounded by the curves $y = x^2 + 1$ and $y = 3 - x^2$ about the x-axis.

If the symmetry axis is the y-axis then $h = y$ and we can write the outer radii as $r_o = x_2 = f(y)$ and the inner radii as $r_i = x_1 = g(y)$. The general formula for a solid of revolution about the y-axis becomes:

$$V = \int_c^d \pi\left([f(y)]^2 - [g(y)]^2\right) dy$$

Once again we use our convention of using constants c and d for limits along the y-axis.

Example 5-7

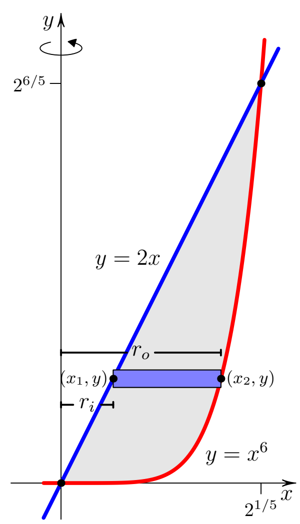
Find the volume of the solid obtained by revolving the region bounded by $y = 2x$ and $y = x^6$ about the y-axis.

Solution:

The curves intersect when

$$2x = x^6 \implies 2x - x^6 = 0 \implies x(2 - x^5) = 0 \implies x = 0 \text{ or } x = 2^{\frac{1}{5}}\,.$$

Evaluating the y coordinates for each give $y = 0$ and $y = 2^{\frac{6}{5}}$ respectively, so the intersection points are $(0,0)$ and $(2^{\frac{1}{5}}, 2^{\frac{6}{5}})$. A graph of the region follows.



Since we are rotating about the y-axis we require a horizontal area element perpendicular to the axis to use the Washer Method. We label the endpoints (x_1, y) and (x_2, y) since they share the same y coordinate but different x coordinates. The differential volume element is

$$dV = \pi(r_o^2 - r_i^2)dh = \pi(x_2^2 - x_1^2)dy = \pi\left(\left(y^{\frac{1}{6}}\right)^2 - \left(\frac{1}{2}y\right)^2\right) dy\,,$$

where here we had to solve our curves in terms of y as

$$y = x^6 \implies x = f(y) = y^{\frac{1}{6}}$$
$$y = 2x \implies x = g(y) = \frac{y}{2}\,,$$

in order to write x_1 and x_2 in terms of the differential variable y. The limits in that variable are $y = 0$ and $y = 2^{\frac{6}{5}}$.

The volume is, using the previous reasoning or by formula $V = \int_c^d \pi \left([f(y)]^2 - [g(y)]^2\right) dy$,

$$V = \pi \int_0^{2^{\frac{6}{5}}} \left(\left(y^{\frac{1}{6}}\right)^2 - \left(\frac{1}{2}y\right)^2\right) dy = \pi \int_0^{2^{\frac{6}{5}}} \left(y^{\frac{1}{3}} - \frac{1}{4}y^2\right) dy$$

$$= \pi \left[\frac{3}{4}y^{\frac{4}{3}} - \frac{1}{12}y^3\right|_0^{2^{\frac{6}{5}}} = \pi \left(\frac{3}{4}(2^{\frac{6}{5}})^{\frac{4}{3}} - \frac{1}{12}(2^{\frac{6}{5}})^3 - [0-0]\right)$$

$$= \pi \left(\frac{3}{4}(2^{\frac{8}{5}}) - \frac{1}{12}(2^{\frac{18}{5}})\right) = \pi(2^{\frac{3}{5}}) \left(\frac{3}{4}(2^{\frac{5}{5}}) - \frac{1}{12}(2^{\frac{15}{5}})\right)$$

$$= \pi(2^{\frac{3}{5}}) \left(\frac{3}{2} - \frac{2}{3}\right) = \frac{5\pi(2^{\frac{3}{5}})}{6} \text{ cubic units.}$$

Further Question:

Find the volume of the solid obtained by revolving the region bounded by $y = 2x$ and $y = x^2$ about the y-axis.

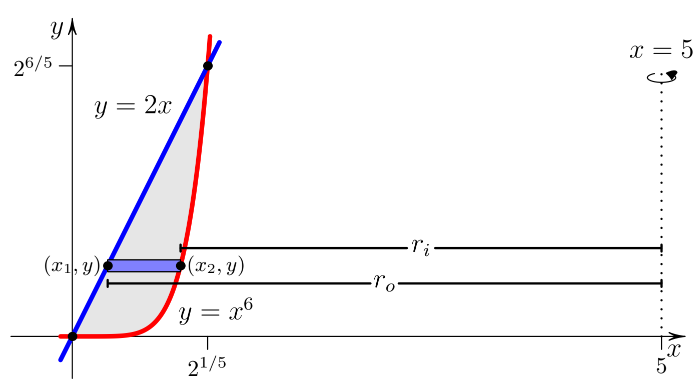
If one is revolving about an axis that is not a coordinate axis there is no immediate formula since the curve functions no longer represent distances from the axes of revolution. Follow the steps outlined previously starting with the volume element and identifying the variables from your diagram. In this case the radii will need to be calculated in terms of the variables.

Example 5-8

Find the volume of the solid obtained by revolving the region bounded by the curves $y = 2x$ and $y = x^6$ about the line $x = 5$.

Solution:

This is the same region as in Example 5-6 but now rotated about the line $y = 5$. A graph of the situation is



As we are not rotating about one of the Cartesian axes a formula will not work. Consideration of the diagram shows that the inner and outer radii of the washer are

$$r_i = 5 - x_2 = 5 - y^{\frac{1}{6}}$$

$$r_o = 5 - x_1 = 5 - \frac{1}{2}y\,.$$

Here we ensured the radii were positive by subtracting the left endpoint from the right endpoint. We then replaced the x coordinates in terms of the differential variable y using the curves that the points lie upon as in Example 5-6. The volume element is therefore

$$dV = \pi(r_o^2 - r_i^2)dh = \pi[(5-x_1)^2 - (5-x_2)^2]dy = \pi\left[\left(5 - \frac{1}{2}y\right)^2 - \left(5 - y^{\frac{1}{6}}\right)^2\right]dy\,,$$

and the volume is

$$\begin{aligned}
V &= \pi\int_0^{2^{\frac{6}{5}}}\left[\left(5-\frac{1}{2}y\right)^2 - \left(5-y^{\frac{1}{6}}\right)^2\right]dy = \pi\int_0^{2^{\frac{6}{5}}}\left[25 - 5y + \frac{1}{4}y^2 - \left(25 - 10y^{\frac{1}{6}} + \left(y^{\frac{1}{6}}\right)^2\right)\right]dy\\
&= \pi\int_0^{2^{\frac{6}{5}}}\left[-5y + \frac{1}{4}y^2 + 10y^{\frac{1}{6}} - y^{\frac{1}{3}}\right]dy = \pi\left[-\frac{5}{2}y^2 + \frac{1}{12}y^3 + \frac{60}{7}y^{\frac{7}{6}} - \frac{3}{4}y^{\frac{4}{3}}\right|_0^{2^{\frac{6}{5}}}\\
&= \pi\left(-\frac{5}{2}(2^{\frac{6}{5}})^2 + \frac{1}{12}(2^{\frac{6}{5}})^3 + \frac{60}{7}(2^{\frac{6}{5}})^{\frac{7}{6}} - \frac{3}{4}(2^{\frac{6}{5}})^{\frac{4}{3}}\right) = \pi\left(-\frac{5}{2}(2^{\frac{12}{5}}) + \frac{1}{12}(2^{\frac{18}{5}}) + \frac{60}{7}(2^{\frac{7}{5}}) - \frac{3}{4}(2^{\frac{8}{5}})\right)\\
&= \pi\left(-\frac{5}{2}(2)^{\frac{10}{5}}(2^{\frac{2}{5}}) + \frac{1}{12}(2^{\frac{15}{5}})(2^{\frac{3}{5}}) + \frac{60}{7}(2^{\frac{5}{5}})(2^{\frac{2}{5}}) - \frac{3}{4}(2^{\frac{5}{5}})(2^{\frac{3}{5}})\right)\\
&= \pi\left(\left[-10 + \frac{120}{7}\right](2^{\frac{2}{5}}) + \left[\frac{2}{3} - \frac{3}{2}\right](2^{\frac{3}{5}})\right) = \frac{50\pi}{7}(2^{\frac{2}{5}}) - \frac{5\pi}{6}(2^{\frac{3}{5}}) \approx 25.6 \text{ cubic units.}
\end{aligned}$$

The reader should compare the volume generated in this example with that of Example 5-6.

Further Question:

Find the volume of the solid obtained by revolving the region bounded by the curves $y = \sqrt[3]{x}$ and $y = \frac{1}{4}x$ and lying in the first quadrant ($x \geq 0$ and $y \geq 0$) about the line $x = 10$.

Exercise 5-2

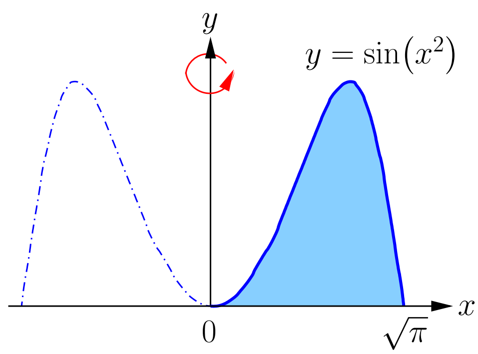
Answers: Page 271

1-10: Find the volume of the solid obtained by revolving the given region about the given axis.

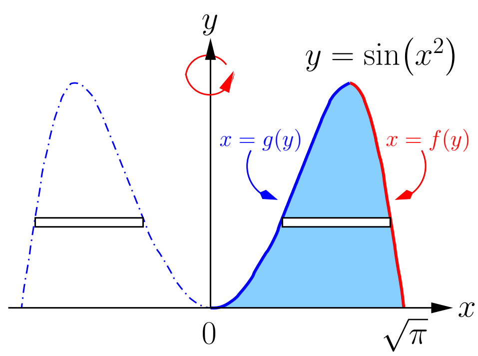
1. $y = \sqrt{x},\ x = 1,\ x = 9;$ about the x-axis
2. $y = (x-2)^2,\ x = 0,\ x = 2;$ about the x-axis
3. $x = y^2 + 4,\ y = 0,\ y = 2;$ about the y-axis
4. $y = \sqrt{x-4},\ x = 4,\ x = 8$ about the x-axis
5. $y = x^{1/3},\ x = 0,\ x = 8;$ about the x-axis
6. $y = 1 + x^2,\ y = x,\ x = 0,\ x = 2;$ about the x-axis
7. $x = y^2,\ x = 2y + 3;$ about the y-axis
8. $y = \sqrt{x},\ y = \frac{1}{2}x;$ about the y-axis
9. $y = \frac{1}{4}x^2,\ x = 2y^2;$ about the line $x = 3$
10. $y = x^3,\ y = x,\ y \geq 0;$ about the line $y = 2$

5.2.5 The Shell Method

Suppose one wished to calculate the volume of the solid generated by revolving the region bounded by the curve $y = \sin(x^2)$ and the line $y = 0$ about the y-axis as shown.

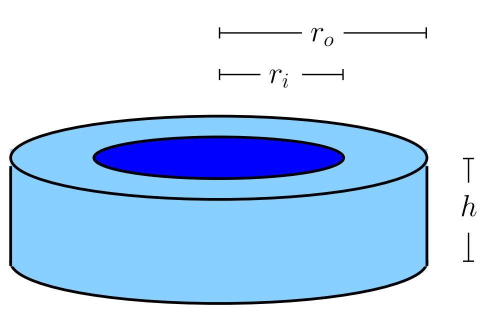


While one can imagine dividing the solid into washer-shaped regiond perpendicular to the y-axis, there would be difficulties.



Ultimately we would need to solve for x in terms of y to find the inner and outer radii of the washer in terms of the position y of the disk. This could be tricky in part because of the trigonometric function. Moreover in inverting the function we would somehow need to generate both an upper and lower function $x(y)$ corresponding to the two radii of the washer!

We now consider a different approach by the introduction of a new volume element. Recall the cylinder with the inner cylinder removed.



We found before the formula for the volume of this solid was just the difference of the volume of the cylinders:

$$V = \pi\left(r_o^2 - r_i^2\right)h$$

Notice that we can factor the difference of squares:

$$V = \pi\left(r_o + r_i\right)\left(r_o - r_i\right)h$$

Defining the average radius r to be:

$$r = \frac{r_o + r_i}{2} \ ,$$

and the difference of the radii to be

$$\Delta r = r_o - r_i \ ,$$

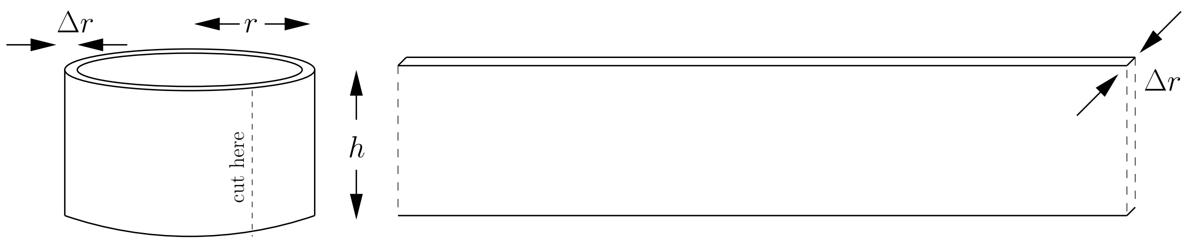
we can rewrite the volume as

$$V = \pi(2r)(\Delta r)h \ ,$$

or equivalently

$$V = 2\pi r h\, \Delta r \ .$$

Now suppose we make the two radii approximately equal. Then the average radius r approximately equals either of them with little error, while their difference Δr will be small. The solid will be like a cylindrical tin can (with the ends cut off) which we will refer to as a "shell". The volume formula for the shell now becomes quite intuitive. If we imagine cutting the shell along its length and unrolling it we would effectively have a rectangular sheet with top length equalling the circumference of the circle $(2\pi r)$ and the other side length equalling the height. The thin thickness will be Δr.

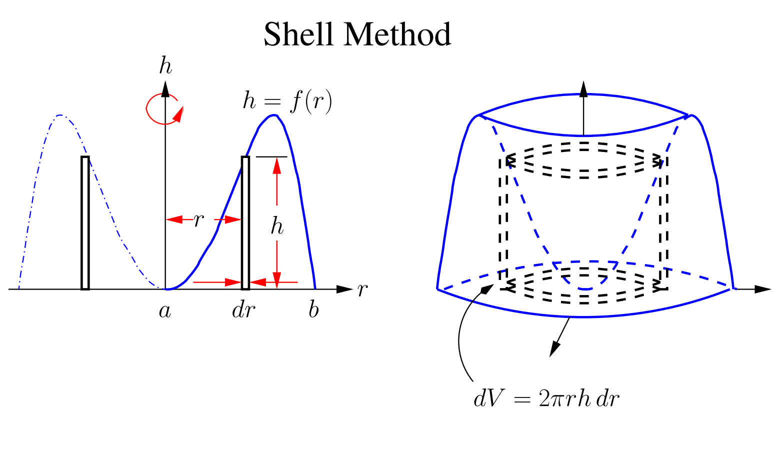


So the volume of the rectangular sheet (the shell) is then:

$$V = (\text{circumference})(\text{height})(\text{depth}) = (2\pi r)(h)(\Delta r) = 2\pi r h\, \Delta r$$

as found before.

Now in the Shell Method we imagine slicing the solid of revolution into concentric cylindrical shells and adding up the volumes.



In the limit that the number of shells goes to infinity, so $\Delta r \to 0$, the limit of the Riemann Sum is an integral ($\int dV$) of differential volume

$$\boxed{dV = 2\pi r h \, dr}$$

with, as usual the finite V and Δr replaced with dV and dr in our shell volume formula. Specifically, for the region depicted above we have for the volume of the solid of revolution

$$V = \int_a^b 2\pi r h \, dr = \int_a^b 2\pi r f(r) \, dr \, .$$

Notice that in the Shell Method the differential dr is now *perpendicular* to the axis of symmetry.

If the axis of symmetry is the y-axis then, setting $h = y$ and $r = x$ in the above formula we get the following result for an a solid of revolution generated by revolving the region bounded by the curve $y = f(x)$, $y = 0$, $x = a$, and $x = b$ about the y-axis

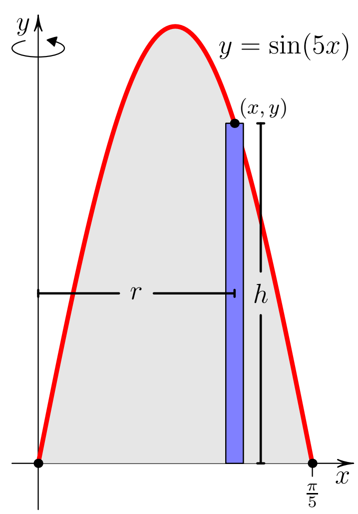
$$V = \int_a^b 2\pi x f(x) \, dx \, .$$

Example 5-9

Find the volume of the solid generated by revolving the region bounded by the curves $y = \sin(5x)$ and $y = 0$ from $x = 0$ to $x = \frac{\pi}{5}$ about the y-axis.

Solution:

A graph of the region of interest is as follows.



We note that the area is not flush with the axis of symmetry so the Disk Method will not work. To use the Shell Method we draw an area element that is parallel to the axis of symmetry as shown. From the diagram it follows that $dr = dx$, $r = x$, and $h = y$. The differential volume element is therefore

$$dV = 2\pi r h \, dr = 2\pi x y \, dx = 2\pi x \sin(5x) dx \, ,$$

where here used that (x, y) lies on the curve $y = \sin(5x)$ to write y in terms of the differential variable x. The limits in that variable are $x = 0$ and $x = \frac{\pi}{5}$ so the volume becomes, using the above differential or formula $V = \int_a^b 2\pi x f(x)\,dx$,

$$V = \int_0^{\frac{\pi}{5}} 2\pi x \sin(5x)\,dx = 2\pi \int_0^{\frac{\pi}{5}} x\sin(5x)\,dx = 2\pi\left[x\left(-\frac{1}{5}\cos(5x)\right)\Big|_0^{\frac{\pi}{5}} - \int_0^{\frac{\pi}{5}}\left(-\frac{1}{5}\cos(5x)\right)dx\right]$$

$$= 2\pi\left[-\frac{1}{5}x\cos(5x) + \frac{1}{25}\sin(5x)\Big|_0^{\frac{\pi}{5}}\right] = 2\pi\left(-\frac{\pi}{25}\cos\pi + \frac{1}{25}\sin\pi - \left[-\frac{1}{5}(0)(1) + \frac{1}{25}(0)\right]\right)$$

$$= 2\pi\left(\frac{\pi}{25}\right) = \frac{2\pi^2}{25} \text{ cubic units.}$$

This problem could also be done using horizontal area elements and the Washer Method but it becomes much more difficult. One needs first to solve for $y = \sin(5x)$ for x to determine $x = g(y) = \frac{1}{5}\sin^{-1}y$ for the left of the hill and $x = f(y) = \frac{\pi}{5} - \frac{1}{5}\sin^{-1}y$ for the right of the hill. The resulting Washer Method integral also is considerably more challenging. As such when confronted with a volume of rotation it is worth considering which of the Shell or Washer Methods is easier to apply.

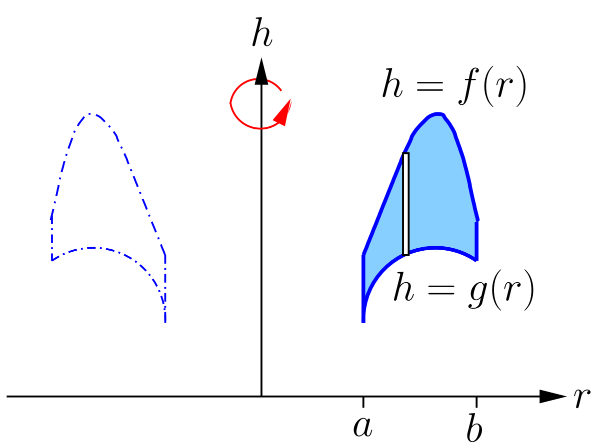
Further Question:

Find the volume of the solid generated by revolving the region bounded by the curves $y = \sin(x^2)$ and $y = 0$ between $x = 0$ and $x = \sqrt{\pi}$ about the y-axis.

If the axis of symmetry is the x-axis then, setting $h = x$ and $r = y$ in the above formula we get the following formula for a solid of revolution generated by the region bounded by the curves $x = f(y)$, $x = 0$, $y = c$, and $y = d$.

$$V = \int_c^d 2\pi y f(y)\,dy$$

More generally the lower bound of the region need not be along the radial axis:



If the upper bound of the figure remains described by the function $h = f(r)$ while the lower bound is now the function $h = g(r)$ the the height of the shell is $h = f(r) - g(r)$. The volume for the solid generated by revolving about the h-axis is then

$$V = \int_a^b 2\pi r h\,dr = \int_a^b 2\pi r\,[f(r) - g(r)]\,dr$$

One observes that, in the example depicted, the washer method would now be impossible in a single integral. For even should we be able to solve for $r(h)$ on the curves, there is no single washer generated toward the bottom of the region. One also notes in this example that the region is generally bounded by four curves, two are vertical lines. In general the shell method works well for solids with straight cylindrical boundaries parallel to the axis of revolution.

In the event we are revolving about the y-axis, so $h = y$ and $r = x$ in the above diagram, the solid generated by revolving the region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ is

$$V = \int_a^b 2\pi x\,[f(x) - g(x)]\,dx$$

On the other hand the volume of the solid generated by revolving about the x-axis the region bounded by $x = f(y)$, $x = g(y)$, and the horizontal lines $y = c$ and $y = d$ is, setting $h = x$, and $r = y$,

$$V = \int_c^d 2\pi y\,[f(y) - g(y)]\,dy$$

These formulae may all, as usual be derived following the general steps starting with the differential volume element $dV = 2\pi r h\,dr$.

Example 5-10

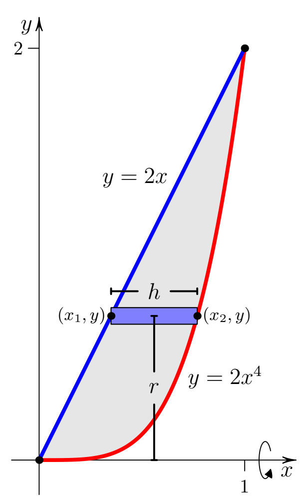
Find the volume of the solid generated by revolving the region bounded by the curves $y = 2x^4$ and $y = 2x$ about the x-axis.

Solution:

The curves intersect when

$$2x^4 = 2x \implies x^4 - x = 0 \implies x(x^3 - 1) = 0 \implies x = 0 \text{ or } x = 1\,.$$

Evaluating y at these values shows the intersection points are $(0, 0)$ and $(2, 0)$. A graph of the region follows.



Here we have drawn a horizontal area element parallel to the axis of symmetry as required by the Shell Method. The volume element is

$$dV = 2\pi r h\,dr = 2\pi y(x_2 - x_1)dy = 2\pi y\left(\left(\frac{1}{2}y\right)^{\frac{1}{4}} - \left(\frac{1}{2}y\right)\right)dy\,,$$

where here we needed to solve

$$y = 2x^4 \implies x = f(y) = \left(\frac{1}{2}y\right)^{\frac{1}{4}}$$
$$y = 2x \implies x = g(y) = \frac{1}{2}y\,,$$

in order to write x_1 and x_2 in terms of the differential variable y. The shells start at $y = 0$ and end at $y = 2$ which become the limits of the desired integral. The volume is therefore, using this reasoning or by formula $V = \int_c^d 2\pi y\,[f(y) - g(y)]\,dy$,

$$\begin{aligned} V &= \int_0^2 2\pi y\left(\left(\frac{1}{2}y\right)^{\frac{1}{4}} - \left(\frac{1}{2}y\right)\right)dy = 2\pi\int_0^2\left(\left(\frac{1}{2}\right)^{\frac{1}{4}}y^{\frac{5}{4}} - \left(\frac{1}{2}\right)y^2\right)dy \\ &= 2\pi\left[\frac{1}{(2)^{\frac{1}{4}}}\left(\frac{4}{9}\right)y^{\frac{9}{4}} - \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)y^3\right|_0^2 = 2\pi\left(\left(\frac{4}{9}\right)\frac{(2)^{\frac{9}{4}}}{(2)^{\frac{1}{4}}} - \left(\frac{1}{6}\right)(2)^3 - [0-0]\right) \\ &= \frac{32}{9}\pi - \frac{8}{3}\pi = \frac{8\pi}{9}\text{ cubic units.} \end{aligned}$$

Further Question:

Find the volume of the solid generated by revolving the region bounded by the curves $x^2 = 4y$ and $y = x$ about the x-axis.

If one wishes to use the shell method to evaluate a solid generated by revolution of a region about a non-coordinate axis the differential volume element $dV = 2\pi rh\,dh$ and the general steps must be used.

Example 5-11

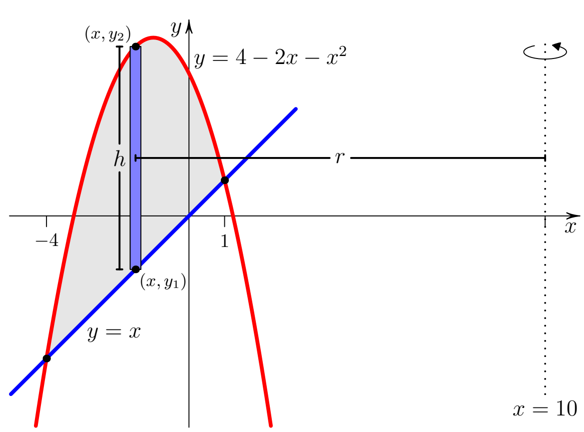
Find the volume of the solid generated by revolving the region bounded by the curves $y = 4 - 2x - x^2$ and $y = x$ about the line $x = 10$.

Solution:

The curves intersect when

$$4 - 2x - x^2 = x \implies 0 = x^2 + 3x - 4 \implies 0 = (x-1)(x+4) \implies x = 1 \text{ or } x = -4\,,$$

and so at the points $(1, 1)$ and $(-4, -4)$. A graph of the region is the following.



Since we are using the Shell Method a vertical area element parallel to the axis of symmetry has been drawn with endpoints (x, y_1) and (x, y_2) as shown. No formula can be used in this case as we are not rotating about the x or y-axis. From the diagram we see the radius is, subtracting the left x value from the right one, $r = 10 - x$. The differential $dr = dx$ and the differential volume is

$$dV = 2\pi r h\, dr = 2\pi(10-x)(y_2 - y_1)dx = 2\pi(10-x)\left(\left(4-2x-x^2\right)-x\right)dx\,,$$

where we used the curves on which our area endpoints lie to replace y_1 and y_2 in terms of the differential variable x. The limits, also in terms of x are $x = -4$ to $x = 1$. The volume of the solid is

$$\begin{aligned} V &= \int_{-4}^{1} 2\pi(10-x)\left(4-2x-x^2-x\right)dx = \int_{-4}^{1} 2\pi(10-x)(4-3x-x^2)dx \\ &= 2\pi\int_{-4}^{1}(40-34x-7x^2+x^3)dx = 2\pi\left[40x - \frac{34}{2}x^2 - \frac{7}{3}x^3 + \frac{1}{4}x^4\right|_{-4}^{1} \\ &= 2\pi\left(40 - 17 - \frac{7}{3} + \frac{1}{4} - \left[-160 - 272 + \frac{448}{3} + 64\right]\right) \\ &= 2\pi\left(391 - \frac{455}{3} + \frac{1}{4}\right) = \pi\left(782 - \frac{910}{3} + \frac{1}{2}\right) = \frac{2875\pi}{6} \text{ cubic units.} \end{aligned}$$

The reader may consider finding this volume using horizontal area elements and the Washer Method. In that case the volume would need to be broken into two integrals as the right-most curve changes at $y = 1$. Additionally one would need to solve the quadratic $y = 4 - 2x - x^2$ for x to get the two functions of y needed.

Further Question:

Find the volume of the solid generated by revolving the region bounded by the curves $y = x^2$ and $x + y = 2$ about the line $x = 3$.

Answers: Page 271

Exercise 5-3

1-10: Find the volume of the solid obtained by revolving the given region about the given axis.

1. $y = 8 - x^2,\ \ y = 8 - 3x\,;$ about the y-axis
2. $y = 4 - x^2,\ \ y = 0,\ \ x = 0\,;$ about the x-axis
3. $y = x^{1/3},\ \ y = x^2\,;$ about the y-axis
4. $y = x^{1/2},\ \ y = x^2;$ about the x-axis
5. $y + x = 6,\ \ y = x,\ \ y = 1;$ about the x-axis
6. $y = x^3,\ \ y = 2x^2;$ about the line $x = 3$
7. $x = y^2,\ \ x - y = 2;$ about the line $y = 4$
8. $y = x^3 + 2,\ \ y = 4 - x^2,\ \ x = 2;$ about the line $x = 3$
9. $y = x^2,\ \ y = 2x + 3;$ about the line $x = 4$
10. $y = 2\sqrt{x},\ \ y = x;$ about the line $y = -1$

5.2.6 Determining the Volume Method

We have considered several methods for calculating volumes of solids of revolution. Which to use in a particular instance can be determined by the following set of steps:

1. Draw the region to be revolved and identify the axis of revolution.

2. Next imagine a line *perpendicular* to the axis of revolution. Where this intersects the region represents (the cross-section of) a particular volume element. Slide the line through the region keeping it perpendicular to the axis of revolution. If all such lines intersect the boundary of the region two or fewer times then the *Disk/Washer Method* will potentially work. An exception to the intersection rule is when you first enter and leave the region; here one can intersect the boundary in a straight line.

   As you pass the line through the region consider the endpoints of the intersection with the region. If these do not always lie on the same curves then the region will need to be partitioned into two (or more) regions each requiring a separate integral to use this method.

3. Repeat the previous step but now with lines *parallel* to the axis of revolution. If the region passes this test then it is a candidate for the *Shell Method*.

4. For any region that passes the *Disk/Washer* test, if one of the boundary curves of the region is the axis of revolution then use the *Disk Method* for that interval, otherwise the *Washer Method* will be used.

5. To distinguish multiple candidate methods, next look at the curve equations. Identify the variable of integration (dx or dy). It will be the variable of the coordinate axis *parallel* to the axis of revolution in the case of the *Disk/Washer Method* and *perpendicular* to the axis of revolution for the *Shell Method*. The curves in the boundary region (aside from any straight lines at the beginning or end of the region for the method in question) will have to have the non-differential variable written as a function of the differential variable. If these curves cannot be solved for the non-differential variable, reject the method.

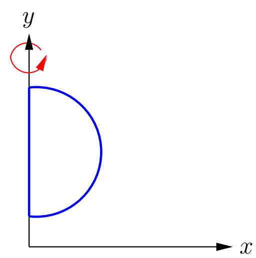
6. If several candidate methods still remain, set up the volume integral for one candidate. If it is not easily solved, try the other candidate.

7. If no candidates lead to a solution, then the solid cannot be calculated with a single volume method. Try partitioning the region into several areas and calculate the volume for each region revolved separately. If these can be calculated the volume of the entire solid is their sum.

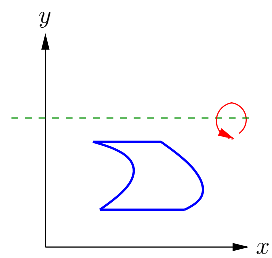
Example 5-12

For each of the following regions identify the simplest volume method(s) to use to calculate the solid of revolution found by rotating the region about the indicated axis. For the method state whether to use vertical or horizontal differential volume elements.

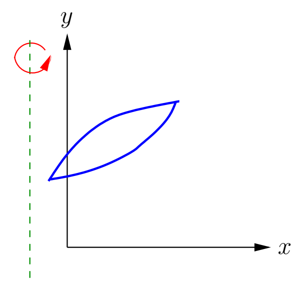
1.



2.



3.



Solution:

1. Passing a line perpendicular to the axis of the rotation (so horizontal) through the region shows the right and left endpoints remain on the same curves throughout the entire region. This suggests using the Disk Method or Washer Method. The Disk Method is sufficient because the region is flush with the axis of rotation. The volume element will be horizontal. If we try passing a line parallel to the axis of rotation (so vertical) we see that the Shell Method could also be used with vertical volume elements. This, however, would require solving the circular arc into an upper and lower boundary function for the shells.

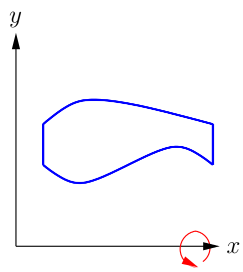
2. Passing a line perpendicular to the axis of rotation (so vertical) beginning at the left of the region shows it intersects the region in two distinct intervals. Thus the region would need to be broken into several pieces to use the Washer Method. Passing a line parallel to the axis of rotation (so horizontal) from top to bottom has the same curve on the left and right so the Shell Method with horizontal volume elements should be used instead. Note that the horizontal lines at the top and bottom of the region also suggest that using a horizontal volume element is desirable here.

3. Passing a line perpendicular to the axis of rotation (so horizontal) from top to bottom of the region shows the left and right endpoints of the intersection lie on the same curve over the entire region. Since the region is not flush with the axis of rotation the Washer Method with horizontal volume elements could be used. Passing a line parallel to the axis of rotation (so vertical) from left to right of the region shows the top and bottom endpoints of the intersection also lie on the same curves over the entire region. Thus the Shell Method with vertical volume elements could also be used. Which of the Washer or Shell Methods to use would likely be determined by whether the curves were written as functions of x (use Shell Method) or y (use Washer Method). If the given functions were easily inverted both volume integrals could be written down and the one easier to integrate chosen.

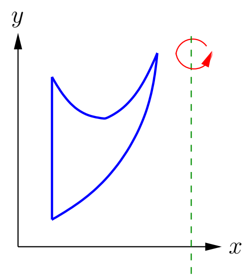
Further Questions:

For each of the following regions identify the simplest volume method(s) to use to calculate the solid of revolution found by rotating the region about the indicated axis. For the method state whether to use vertical or horizontal differential volume elements.

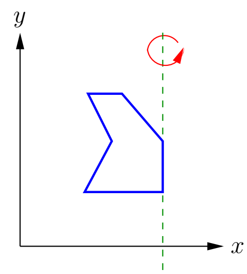
1.



2.



3.



Exercise 5-4

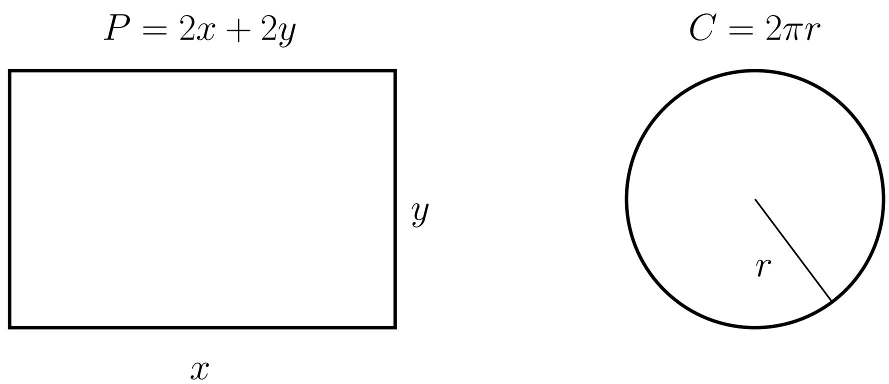
Answers: Page 271

1-10: Find the volume of the solid obtained by revolving the given region about the given line.

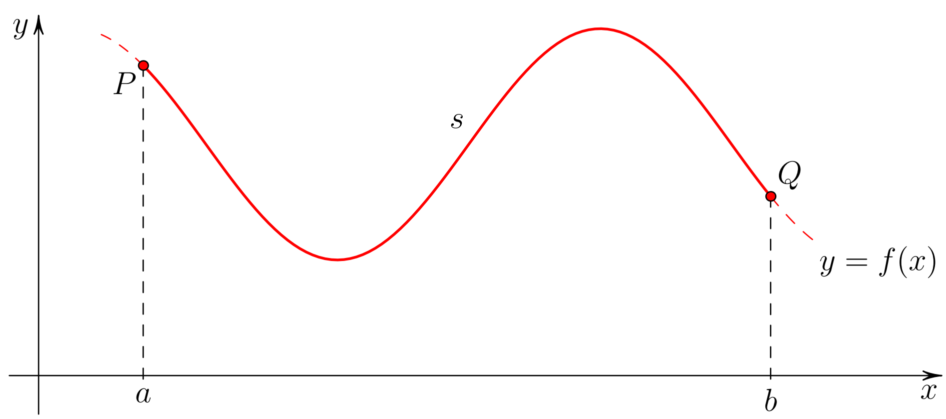
1. $y = x^2$, $y = 2 - x$; about the line $y = -1$
2. $y = -x^2 + 6x$, $y = x^2 - 2x$; about the line $x = 5$
3. $y = x^2$, $y = -x^2 + 8$; about the line $y = -1$
4. $y = x^2$, $y = 2x + 3$; about the line $y = -1$
5. $y = -x^2 + 4$, $y = -3x$; about the line $x = -2$
6. $x = -y^2 + 8$, $x = 2y$; about the line $y = 4$
7. $x = -y^2$, $x = -2y^2 + 4$; about the line $x = 6$
8. $y = x^3$, $y = \sqrt{x}$; about the line $y = -1$
9. $y = 4x - x^2$, $y = 0$; about the y-axis
10. $y = x^2 - 2x + 4$, $y = 2x^2 - 3x + 2$; about the line $x = 4$

5.3 Arc Length

From basic geometry the perimeter P of a polygon or the circumference C of a circle are the distance around the given object. They are the length one would measure if the boundary of the object were cut, straightened out, and measured with a ruler.



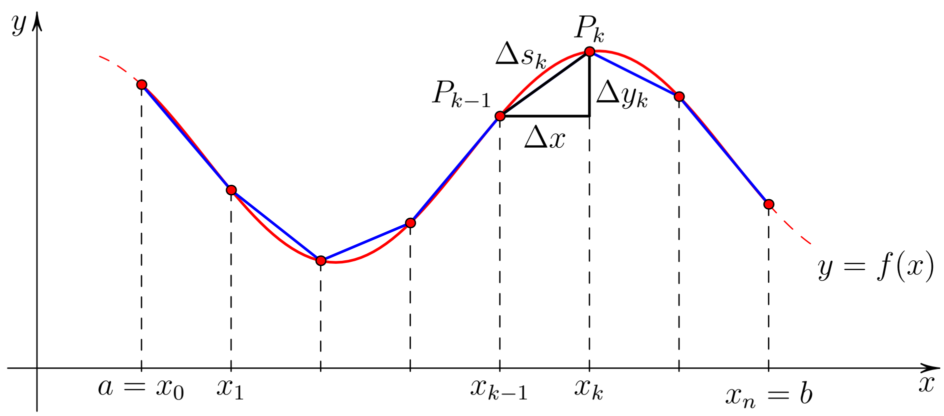
Intuitively the **arc length** $\boldsymbol{s}$ of a curve between points P and Q is similarly the "length of a piece of string" that follows the curve joining those two points.



Consider a curve defined by a function $y = f(x)$ with a continuous first derivative $f'(x)$ defined on interval $[a, b]$. Such a function is called a **smooth function** and its graph $y = f(x)$ is said to be **smooth**. An approximation to the arc length can be found by breaking the interval into n subintervals of equal length $\Delta x = \frac{b-a}{n}$ with endpoints

$$a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$$

The points $P_k = (x_k, f(x_k))$, $k = 0, 1, \ldots, n$, lie along the curve. The length of the chord (straight line segment) joining P_{k-1} to its neighbour P_k will approximate the arc length between those two points. Call that chord length Δs_k. It equals, by the Pythagorean Theorem, $\Delta s_k = \sqrt{(\Delta x)^2 + (\Delta y_k)^2}$ where we define $\Delta y_k = y_k - y_{k-1} = f(x_k) - f(x_{k-1})$.



By the Mean Value Theorem applied to the interval $[x_{k-1}, x_k]$ there exists an x_k^* in (x_{k-1}, x_k) satisfying

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*)$$

Equivalently $\Delta y_k = f'(x_k^*)\Delta x$. From this it follows that we can write Δs_k as:

$$\begin{aligned}
\Delta s_k &= \sqrt{(\Delta x)^2 + (\Delta y_k)^2} \\
&= \sqrt{(\Delta x)^2 + [f'(x_k^*)\Delta x]^2} \\
&= \sqrt{(\Delta x)^2 + [f'(x_k^*)]^2\,(\Delta x)^2} \\
&= \sqrt{\left(1 + [f'(x_k^*)]^2\right)\cdot(\Delta x)^2} \\
&= \sqrt{1 + [f'(x_k^*)]^2}\cdot\sqrt{(\Delta x)^2} \\
&= \sqrt{1 + [f'(x_k^*)]^2}\,\Delta x
\end{aligned}$$

An approximation to the arc length between the points $P(a, f(a))$ and $Q(b, f(b))$ is therefore the length of the polygonal curve we have constructed, namely the Riemann sum

$$\sum_{k=1}^{n} \Delta s_k = \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\,\Delta x$$

Taking the limit as $n \to \infty$ (so $\Delta x \to 0$) we have the following:

Definition: If function f has a continuous derivative f' over the x-interval $[a, b]$ then the length of the graph over the interval (the **arc length**) is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx \quad \text{or} \quad s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Note that if a curve is described by the equation $x = g(y)$ with continuous derivative $g'(y)$ over the y-interval $[c, d]$ then the arc length between the endpoints equals

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy \quad \text{or} \quad s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

Example 5-13

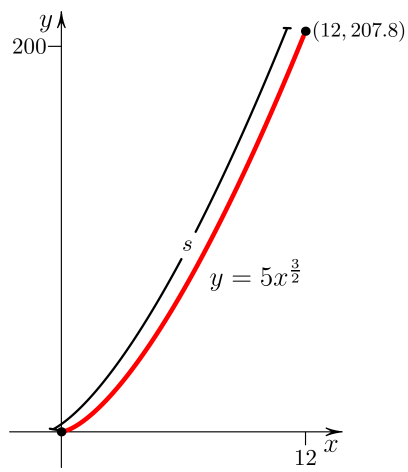
Find the length of the following curves over the given intervals.

1. $y = 5x^{\frac{3}{2}} + 2;\ 0 \le x \le 12$
2. $x = \frac{2}{3}y^{\frac{3}{2}};\ 1 \le y \le 9$

Solution:

1. $y = 5x^{\frac{3}{2}} + 2;\ 0 \le x \le 12$

   The curve is between the points $(0, 0)$ and $(0, 5(12)^{\frac{3}{2}}) \approx (12, 207.8)$.



The derivative is given by $\dfrac{dy}{dx} = \dfrac{15}{2}x^{\frac{1}{2}}$ and is continuous on $[0, 12]$. The arc length of the curve over that interval is

$$s = \int_0^{12} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{12} \sqrt{1 + \left(\frac{15}{2}x^{\frac{1}{2}}\right)^2}\, dx = \int_0^{12} \sqrt{1 + \frac{225}{4}x}\ dx\,.$$

Substitute $u = 1 + \frac{225}{4}x$, so $du = \frac{225}{4}dx \implies dx = \frac{4}{225}du$. The limits become

$$\begin{aligned} x = 12 &\implies u = 676 \\ x = 0 &\implies u = 1, \end{aligned}$$

and so

$$\begin{aligned} s &= \frac{4}{225}\int_1^{676} u^{\frac{1}{2}} du = \frac{4}{225}\left[\frac{2}{3}u^{\frac{3}{2}}\right|_1^{676} = \frac{8}{675}\left[(676)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] \\ &= \frac{8(17575)}{675} = \frac{8(703)}{27} = \frac{5624}{27} \approx 208.296 \text{ units.} \end{aligned}$$

As a check note that the straight line distance between the endpoints is

$$\sqrt{(12)^2 + \left(5(12)^{\frac{3}{2}}\right)^2} = \sqrt{43344} \approx 208.192 \text{ units,}$$

which is just below the distance found, showing our curve is almost straight between the two points.

2. $x = \frac{2}{3}y^{\frac{3}{2}}$; $1 \le y \le 9$

   Since x is written as a function of y and our limits are in terms of y we use the arc length formula with differential dy. The derivative is $\frac{dx}{dy} = y^{\frac{1}{2}}$ and is continuous on $[1, 9]$. The arc length is

$$s = \int_1^9 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^9 \sqrt{1 + \left(y^{\frac{1}{2}}\right)^2}\, dy = \int_1^9 \sqrt{1 + y}\, dy = \frac{2}{3}(1 + y)^{\frac{3}{2}}\Big|_1^9$$
$$= \frac{2}{3}\left[(1 + 9)^{\frac{3}{2}} - (1 + 1)^{\frac{3}{2}}\right] = \frac{2}{3}\left[(10)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right] \approx 19.2 \text{ units.}$$

Note that the integrand of the arc length formula typically results in integrals which are not integrable. In that case numerical integration is used to find an approximate result.

Further Questions:

Find the lengths of the following curves over the given intervals.

1. $f(x) = 3x^{\frac{2}{3}} - 10$; $8 \le x \le 27$
2. $y = \frac{x^4}{16} + \frac{1}{2x^2}$; $1 \le x \le 2$

If we have a smooth function $f(x)$ on interval $[a, b]$ we can consider the arc length from the starting point $(a, f(a))$ up to the point $(x, f(x))$ for some value x in $[a, b]$. This length is clearly dependent on the choice of x and hence is a function $s(x)$.

Definition: Given a smooth curve described by the function $y = f(x)$ over the x-interval $[a, b]$, the arc length between the initial point $(a, f(a))$ and the point $(x, f(x))$ for x in $[a, b]$ is given by the **arc length function**

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt\,.$$

Notes:

1. We changed the variable of integration to t to distinguish it from the limit of integration x.
2. The derivative of the arc length function $s(x)$ is, by the Fundamental Theorem of Calculus, $\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2}$. This in turn implies, since $ds = \frac{ds}{dx}\, dx$, that the **differential of the arc length** is

$$ds = \sqrt{1 + [f'(x)]^2} dx\,,$$

   in agreement with our previous formula for arc length. Noting that $dy = f'(x)dx$ the differential of arc length is sometimes written

$$ds = \sqrt{dx^2 + dy^2}\ ,$$

   reflecting its Pythagorean origins. The latter differential formula, along with the relation $dy = f'(x)dx$ for x-intervals $[a, b]$ or $dx = g'(y)dy$ for y-intervals $[c, d]$, and the idea of the integral as a sum ($s = \int ds$) make it easy to remember/derive the arc length formulae.

Example 5-14

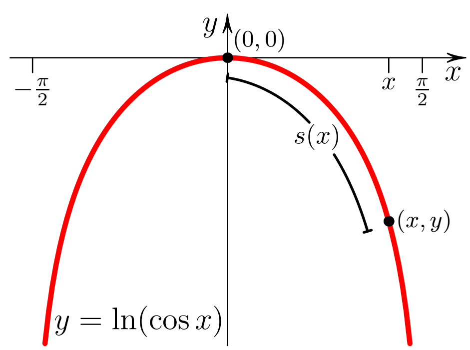
Find the arc length function $s(x)$ of

1. $y = \ln(\cos x)$ with starting point $P(0,0)$.
2. $y = 1 - x^2$ with starting point $P(0,1)$.

Solution:

1. $y = \ln(\cos x)$ with starting point $P(0,0)$.

   Since the argument of the logarithm must be positive we require $\cos x > 0$. Also the function must be continuous over the interval containing our starting point which has $x = 0$ so our arc length function is restricted to x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. A graph of the curve $y = f(x) = \ln(\cos x)$ with the arc length $s(x)$ from $(0,0)$ to $(x, f(x))$ is shown.



   The needed derivative is given by

$$f'(x) = \frac{1}{\cos x}(-\sin x) = -\frac{\sin x}{\cos x} = -\tan x\,.$$

   Therefore the arc length function is

$$\begin{aligned} s(x) &= \int_0^x \sqrt{1 + (-\tan t)^2}\,dt = \int_0^x \sqrt{1 + \tan^2 t}\,dt = \int_0^x \sqrt{\sec^2 t}\,dt = \int_0^x \sec t\,dt \\ &= \ln|\sec t + \tan t|\Big|_0^x = \ln|\sec x + \tan x| - \ln|\sec(0) + \tan(0)| \\ &= \ln|\sec x + \tan x| - \ln|1 + 0| \\ &= \ln|\sec x + \tan x|\,. \end{aligned}$$

   The restriction of x and therefore t to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ was used when simplifying $\sqrt{\sec^2 t} = |\sec t| = \sec t$ in the integration since secant is positive over that interval.

   As an aside, note that our arc length formula is valid for x in $[a, b) = [0, \frac{\pi}{2})$. If we consider a point to the left of our starting point with $x < 0$ then the function $s(x)$ will actually be the negative of the arc length between the two points because then the differential dt is effectively negative due to the upper limit being less than the lower limit. From the symmetry of the original curve ($y = \ln(\cos x)$ is an even function) this means that $s(x) = \ln|\sec x + \tan x|$ is actually an odd function of x. This can be demonstrated by plotting the function. This is somewhat surprising since secant is even and tangent is odd! The reader is encouraged to prove the result. See Exercise 5-5 Problem 11 .

2. $y = 1 - x^2$ with starting point $P(0, 1)$.

   The derivative of $y = f(x) = 1 - x^2$ is given by $f'(x) = -2x$. The arc length function is

$$s(x) = \int_0^x \sqrt{1 + (-2t)^2}dt = \int_0^x \sqrt{1 + 4t^2}\,dt = \int_0^x \sqrt{1 + (2t)^2}\,dt\,.$$

   Using the trigonometric substitution $2t = \tan\theta \implies 2dt = \sec^2\theta\,d\theta$ gives

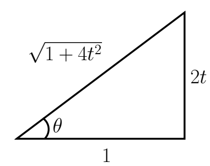
$$\int \sqrt{1 + (2t)^2}\,dt = \frac{1}{2}\int \sqrt{1 + \tan^2\theta}\sec^2\theta\,d\theta = \frac{1}{2}\int \sqrt{\sec^2\theta}\sec^2\theta\,d\theta = \frac{1}{2}\int \sec^3\theta\,d\theta\,.$$

   Using Integration by Parts one has

$$\begin{aligned}\int \sec^3\theta\,d\theta &= \int \sec\theta\left(\sec^2\theta\,d\theta\right) = \sec\theta\tan\theta - \int \tan\theta(\sec\theta\tan\theta\,d\theta)\\ &= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta\\ &= \sec\theta\tan\theta - \int \sec\theta\left(\sec^2\theta - 1\right)d\theta = \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta\end{aligned}$$

$$\implies \int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C\,.$$

   To return to variable t we draw a triangle with $\tan\theta = \dfrac{2t}{1}$ and hence hypotenuse $\sqrt{1 + (2t)^2} = \sqrt{1 + 4t^2}$. The indefinite integral is then



$$\begin{aligned}\int \sqrt{1 + (2t)^2}\,dt &= \frac{1}{4}\left(\frac{\sqrt{1 + 4t^2}}{1}\right)\left(\frac{2t}{1}\right) + \frac{1}{4}\ln\left|\frac{\sqrt{1 + 4t^2}}{1} + \frac{2t}{1}\right| + C\\ &= \frac{1}{2}t\sqrt{1 + 4t^2} + \frac{1}{4}\ln\left|2t + \sqrt{1 + 4t^2}\right| + C\,,\end{aligned}$$

   and so the arc length function is

$$\begin{aligned}s(x) &= \left[\frac{1}{2}t\sqrt{1 + 4t^2} + \frac{1}{4}\ln(2t + \sqrt{1 + 4t^2})\right|_0^x\\ &= \frac{1}{2}x\sqrt{1 + 4x^2} + \frac{1}{4}\ln\left(2x + \sqrt{1 + 4x^2}\right) - \left[0 + \frac{1}{4}\ln(1)\right]\\ &= \frac{1}{2}x\sqrt{1 + 4x^2} + \frac{1}{4}\ln\left(2x + \sqrt{1 + 4x^2}\right).\end{aligned}$$

Further Questions:

Find the arc length function $s(x)$ of

1. $y = \sqrt{x - x^2} - \cos^{-1}\left(\sqrt{x}\right)$ with starting point $P(0, -\pi/2)$.
2. $y = e^x$ with starting point $P(0, 1)$.

Answers: Page 271

Exercise 5-5

1-6: Find the length of the given curve over the given interval.

1. $y = x^{3/2} + 1$ over $1 \leq x \leq 4$
2. $y = \dfrac{2}{3}(x-2)^{3/2}$ over $2 \leq x \leq 5$
3. $y = \ln(\sin x)$ over $\frac{\pi}{6} \leq x \leq \frac{\pi}{4}$
4. $x = \dfrac{1}{32}y^4 + y^{-2}$ over $1 \leq x \leq 3$
5. $x = \dfrac{1}{5}y^{5/2} + y^{-1/2}$ over $1 \leq x \leq 4$
6. $y = e^{x/2} + e^{-x/2}$ over $0 \leq x \leq 1$

7-10: Find the arc length function $s(x)$ of the given curve.

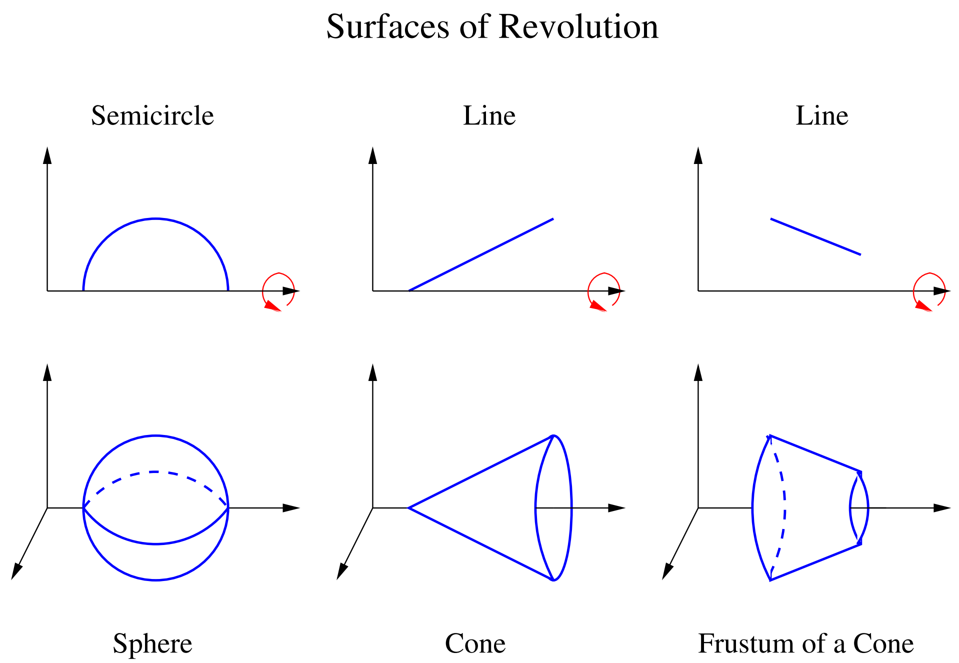
7. $y = 1 + x^2$ with starting point $P(0,1)$
8. $y = 2x^{1/2}$ with starting point $P(1,2)$
9. $y = \dfrac{1}{3}(x-2)^{3/2} - (x-2)^{1/2}$ with starting point $P(3, -\frac{2}{3})$
10. $y = \dfrac{4}{3}(x+4)^{3/2} + 5$ with starting point $P\left(0, \frac{47}{3}\right)$

---

11. Show that the arc length function $s(x) = \ln|\sec x + \tan x|$ from Example 5-14 is an odd function of x.

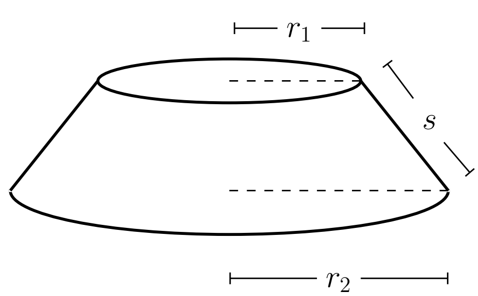
5.4 Areas of Surfaces of Revolution

Just as volumes of revolution are constructed by revolving an area about a line, a **surface of revolution** is generated by revolving the graph of a continuous function about a line.



Note that the surface generated in this manner does not produce any end surfaces such as the circular bottom of the cone or ends of the frustum of the cone. (The spherical surface however is fully generated.)

We turn ourselves to the problem of calculating the area of such a surface of revolution. Consider a simple object like the frustum of the cone above, with end radii of r_1 and r_2 and **slant height** s.



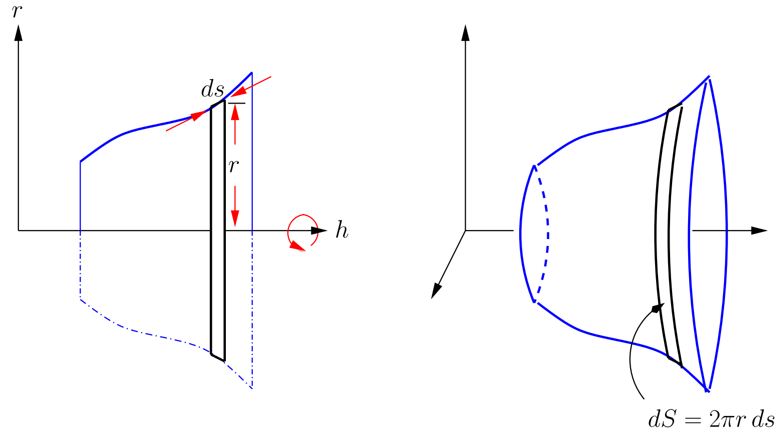
The **lateral** surface area of the frustum (i.e. the area of the side but not including the circular ends), can be shown to be

$$S = 2\pi r s \; ,$$

where $r = \frac{1}{2}(r_1 + r_2)$ is the average of the two radii.

The approximate area of a surface of revolution may be found by slicing it perpendicular to its axis of revolution (which we will call h). The intersection of these planes with the surface form the circular ends of conical frusta the net lateral surface area of which will approximate the area of the surface of revolution.

Differential Surface Area



A Riemann sum for the area is thereby formed:

$$S \approx \sum_{k=1}^{n} \Delta S_k \ ,$$

where ΔS_k is the lateral surface area of the k^{th} conical frustum. Upon taking the limit of an infinite number ($n \to \infty$) of conical frusta one arrives at an exact expression for the surface area S in terms of an integral $S = \int dS$ where the differential area of the conical frustum is given, analagously to the finite surface area:

$$dS = 2\pi r \, ds$$

where here the finite slant height s is replaced by the **differential arc length** $ds = \sqrt{dx^2 + dy^2}$ of the curve generating the surface. Four different formulae are possible depending on whether the axis of symmetry h is the x or the y-axis and whether the curve-generating function is written as $y = f(x)$ over the x-interval $[a, b]$ or $x = g(y)$ over the y-interval $[c, d]$. In all cases we assume the function is positive and has continuous first derivative (i.e. is smooth).

Considering first the case where the axis of symmetry h is the x-axis and the curve revolved is described by the function $y = f(x)$, $a \leq x \leq b$, then our radius is $r = y = f(x)$ and we write the differential arc length in terms of the independent variable x as $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ so that $S = \int dS = \int 2\pi r \, ds$ is given by the formula:

$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx \quad \text{or} \quad S = \int_a^b 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

If we still consider the case where we are rotating about the x-axis but the curve is instead described by the function $x = g(y)$, $c \leq y \leq d$ then the differential arc length must be written in terms of the independent variable y as $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$. The radius $r = y$ still but as that is already expressed in terms of the variable of the differential (dy) variable our formula becomes (still using $S = \int dS = \int 2\pi r \, ds$):

$$S = \int_c^d 2\pi y \sqrt{1 + [g'(y)]^2} \, dy \quad \text{or} \quad S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

Consider next the cases where the surface is generated by revolving the curve around the y-axis. In this case the radius $r = x$. If the curve is described by the function $y = f(x)$, $a \leq x \leq b$ then the independent variable is x so we use the differential ds written in terms of dx to get, since $S = \int dS = \int 2\pi r\, ds$,

$$S = \int_a^b 2\pi x \sqrt{1 + [f'(x)]^2}\, dx \quad \text{or} \quad S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If instead the curve is written $x = g(y)$, $c \leq y \leq d$, then we must write the radius x in terms of the differential (dy) as $g(y)$ and also select the appropriate form of the differential arc length in terms of dy:

$$S = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2}\, dy \quad \text{or} \quad S = \int_c^d 2\pi g(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

None of these formulae need to be memorized. Just remember that we are integrating ($S = \int dS$) the differential surface area $dS = 2\pi r\, ds$. Then look at the curve's function; the independent variable determines the form of ds and the limits on the integral. The radius r is the variable perpendicular to the axis of revolution. If that is not the independent variable then use the function for the curve to express it in terms of the independent (differential) variable.

Example 5-15

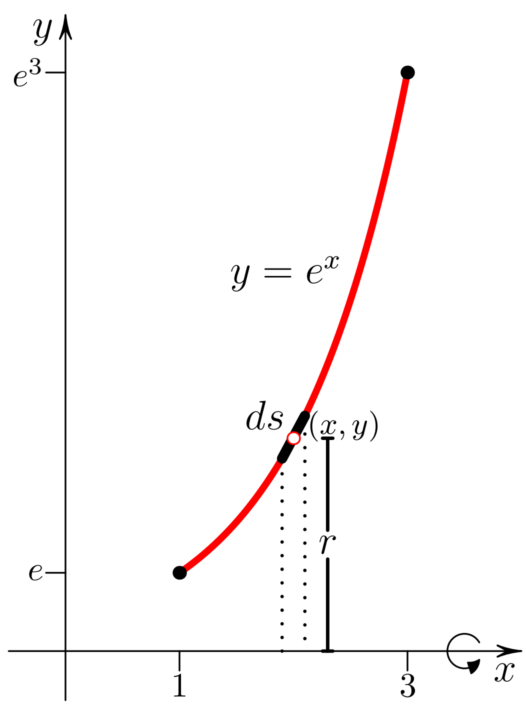
Find the area of the following surfaces of revolution generated by the given curves.

1. $y = e^x$ from $x = 1$ to $x = 3$ revolved about the x-axis.
2. $y = \sqrt[3]{3x}$ from $x = 9$ to $x = 72$ revolved about the y-axis.

Solution:

1. $y = e^x$ from $x = 1$ to $x = 3$ revolved about the x-axis.

   A graph of the situation follows.



Since we have y written as a function of x it is easiest to write our differential area element in terms of dx. From the diagram

$$dS = 2\pi r\, ds = 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = 2\pi e^x \sqrt{1+(e^x)^2}\, dx$$

where here the radius $r = y$ and the differential ds were were written in terms of the differential variable x. The limits in that variable are $x = 1$ and $x = 10$ so the total area is therefore, using this derivation or by formula $S = \int_a^b 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$,

$$S = \int_1^3 2\pi e^x \sqrt{1+(e^x)^2}\, dx$$

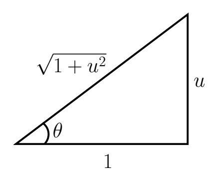
Using substitution $u = e^x \implies du = e^x\, dx$ the indefinite integral is

$$2\pi \int e^x \sqrt{1+(e^x)^2}\, dx = 2\pi \int \sqrt{1+u^2}\, du\,.$$

Using the trigonometric substitution $u = \tan\theta \implies du = \sec^2\theta\, d\theta$ this equals

$$\begin{aligned} 2\pi \int \sqrt{1+\tan^2\theta}\, d\theta &= 2\pi \int \sqrt{\sec^2\theta}\sec^2\theta\, d\theta = 2\pi \int \sec^3\theta\, d\theta \\ &= \pi \sec\theta\tan\theta + \pi \ln|\sec\theta + \tan\theta| + C\,, \end{aligned}$$

where we used the integral for $\sec^3\theta$ derived in Example 5-14. Draw a triangle with $\tan\theta = \dfrac{u}{1}$ and thus hypotenuse $\sqrt{1+u^2}$. The indefinite integral is then



$$\begin{aligned} 2\pi \int e^x \sqrt{1+e^{2x}}\, dx &= \pi\left(\frac{\sqrt{1+u^2}}{1}\right)\left(\frac{u}{1}\right) + \pi \ln\left|\frac{\sqrt{1+u^2}}{1} + \frac{u}{1}\right| + C \\ &= \pi u\sqrt{1+u^2} + \pi\ln\left|u + \sqrt{1+u^2}\right| + C \\ &= \pi e^x\sqrt{1+e^{2x}} + \pi \ln|e^x + \sqrt{1+e^{2x}}| + C\,. \end{aligned}$$

Therefore the surface area is

$$\begin{aligned} S &= \pi\left[e^x\sqrt{1+e^{2x}} + \ln|e^x + \sqrt{1+e^{2x}}|\,\right|_1^3 \\ &= \pi\left[e^3\sqrt{1+e^6} + \ln(e^3 + \sqrt{1+e^6}\,) - e\sqrt{1+e^2} - \ln(e + \sqrt{1+e^2}\,)\right] \\ &\approx 1250.4 \text{ square units.} \end{aligned}$$

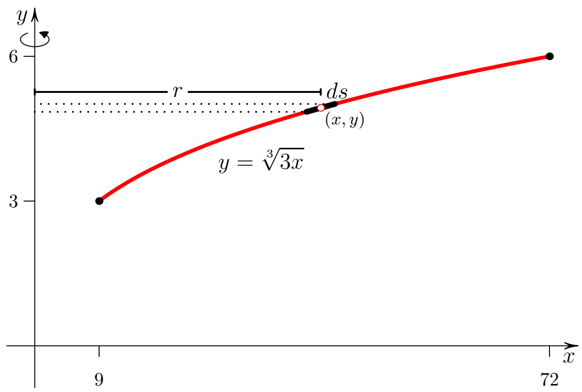
Finally note that the area could also have been found using differential dx. Since $y = f(x) = e^x$ it follows that $x = g(y) = \ln y$ and the differential area is

$$dS = 2\pi r\, ds = 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = 2\pi y\sqrt{1+\left(\frac{1}{y}\right)^2}\, dy\,,$$

which would need to be integrated from $y = f(1) = e$ to $y = f(3) = e^3$. The reader is encouraged to solve this integral and confirm the above area.

2. $y=\sqrt[3]{3x}$ from $x=9$ to $x=72$ revolved about the y-axis.

   A graph of the situation follows.



Since $y=f(x)=\sqrt[3]{3x}=(3x)^{\frac{1}{3}} \implies \frac{dy}{dx}=\frac{1}{3}(3x)^{-\frac{2}{3}}(3)=(3x)^{-\frac{2}{3}}$ the differential area element written in terms of dx is, from the diagram,

$$dS=2\pi r\,ds=2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=2\pi x\sqrt{1+\left((3x)^{-\frac{2}{3}}\right)^2}\,dx=2\pi x\sqrt{1+\frac{1}{(3x)^{\frac{4}{3}}}}\,dx\,,$$

which would need to be integrated from $x=8$ to $x=72$. This integral looks challenging so instead consider writing dS in terms of dy. Solving $y=\sqrt[3]{3x}$ for x gives $x=g(y)=\frac{1}{3}y^3$ with simple derivative $\frac{dy}{dx}=y^2$. The differential area element in this case becomes

$$dS=2\pi r\,ds=2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=2\pi\left(\frac{1}{3}y^3\right)\sqrt{1+(y^2)^2}\,dy\,,$$

with the limits going between $y=f(9)=\sqrt[3]{3(9)}=3$ and $y=f(72)=\sqrt[3]{3(72)}=6$. The surface area is, using this differential or by formula $S=\int_c^d 2\pi g(y)\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$,

$$S=2\pi\int_3^6\left(\frac{1}{3}y^3\right)\sqrt{1+(y^2)^2}\,dy=\frac{2\pi}{3}\int_3^6 y^3\sqrt{1+y^4}\,dy\quad(\text{Let } u=1+y^4 \implies du=4y^3dy)$$

$$=\frac{2\pi}{3}\left(\frac{1}{4}\right)\int_{82}^{1297}\sqrt{u}\,du=\frac{\pi}{6}\left[\frac{2}{3}u^{\frac{3}{2}}\Big|_{82}^{1297}\right]=\frac{\pi}{9}\left[(1297)^{\frac{3}{2}}-(82)^{\frac{3}{2}}\right]\approx 5107.5 \text{ square units.}$$

While this integral was straightforward the one in terms of dx can also be done and the reader is encouraged to try it. (Hint: Write the term under the square root as a fraction and then pull the denominator $(3x)^{\frac{4}{3}}$ outside from under the root. Next simplify and use a substitution.)

Further Questions:

Find the areas of the following surfaces of revolution generated by the given curves.

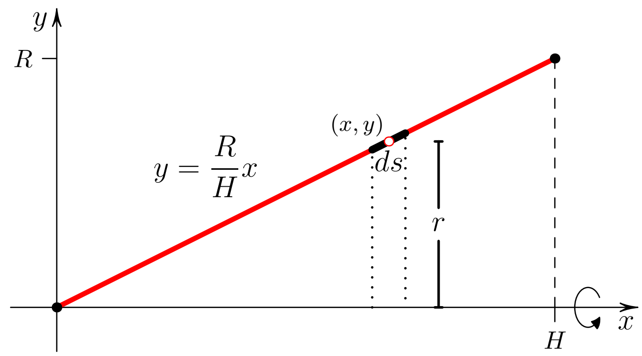
1. $y=x^3$ from $x=1$ to $x=2$ revolved about the x-axis.
2. $x=\frac{1}{8}y^3$ from $x=1$ to $x=8$ revolved about the y-axis.
3. $x=1+2y^2$, $1\le y\le 2$ revolved about the x-axis.
4. $y=\sqrt[3]{x}$, $1\le x\le 8$ revolved about the y-axis.
5. $y=e^{-x}$ from $x=0$ to $x=1$ revolved about the x-axis.

Example 5-16

Use calculus to show that the lateral (side) surface area of a right circular cone of height H and base radius R is $S = \pi R\sqrt{R^2 + H^2}$.

Solution:

Represent the cone by a line of rise R and run H going through the origin rotated about the x-axis. Then $y = f(x) = \frac{R}{H}x$ and the situation is as shown in the following diagram.



The differential area element is

$$dS = 2\pi r\, ds = 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi\left(\frac{R}{H}x\right)\sqrt{1 + \left(\frac{R}{H}\right)^2}\, dx\,.$$

The limits in terms of the differential variable x are $x = 0$ and $x = H$. The lateral surface area is, using this differential or by formula $S = \int_a^b 2\pi f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$,

$$\begin{aligned} S &= \int_0^H 2\pi\left(\frac{R}{H}x\right)\sqrt{1 + \left(\frac{R}{H}\right)^2}\, dx = \frac{2\pi R}{H}\sqrt{1 + \frac{R^2}{H^2}}\int_0^H x\, dx = \frac{2\pi R}{H}\sqrt{1 + \frac{R^2}{H^2}}\left[\frac{1}{2}x^2\right|_0^H \\ &= \frac{\pi R}{H}\sqrt{1 + \frac{R^2}{H^2}}\left(H^2 - (0)^2\right) = \pi R H\sqrt{1 + \frac{R^2}{H^2}} = \pi R\sqrt{H^2\left(1 + \frac{R^2}{H^2}\right)} \\ &= \pi R\sqrt{R^2 + H^2} \end{aligned}$$

For the total surface area of a cone one would need to add the area of the base (πR^2) to this result.

Note that our original derivation of the surface area integration formulae used the lateral area of a frustum as its starting point. Since a cone is just a frustum with upper radius of $r_1 = 0$, bottom radius of $r_2 = R$, and slant height $s = \sqrt{R^2 + H^2}$ our formula above follows directly from the frustum lateral area formula $S = 2\pi\left(\frac{r_1 + r_2}{2}\right)s$. The above derivation for the cone shows the surface area integral formula is self-consistent with this starting point.

Further Question:

Use calculus to show that the surface area of a sphere of radius R is $S = 4\pi R^2$.
(This result was originally discovered by Archimedes.)

Exercise 5-6

Answers: Page 272

1-8: Find the area of the surfaces of revolution generated by the given curves.

1. $y = 2x + 5, \quad 0 \leq x \leq 2$ revolved about the x-axis
2. $y = (9 - x^{2/3})^{3/2}, \quad 1 \leq x \leq 8$ revolved about the x-axis
3. $y = \dfrac{1}{20}x^5 + \dfrac{1}{3}x^{-3}, \quad 1 \leq x \leq 2$ revolved about the x-axis
4. $y = \sqrt{6 - x}, \quad 1 \leq x \leq 2$ revolved about the x-axis
5. $x = \dfrac{1}{6}y^3, \quad 0 \leq y \leq 2$ revolved about the y-axis
6. $x = \sqrt{y - y^2}, \quad 0 \leq y \leq 2$ revolved about the y-axis
7. $x = \dfrac{1}{3}(y^2 + 2)^{3/2}, \quad 0 \leq y \leq 1$ revolved about the x-axis
8. $y = \dfrac{1}{3}x^3, \quad 1 \leq x \leq 3$ revolved about the y-axis

Answers: Page 272

Chapter 5 Review Exercises

1-4: Find the area of the region bounded by the given curves.

1. $y = x^2 - 2x + 4,\;\; y = 2x - x^2,\;\; x = 1,\;\; x = 2$
2. $y = x^3,\;\; y = x,\;\; y = 4x,\;\; y \ge 0$
3. $x = y^2 - 4,\;\; x = 0$
4. $y = \cos x,\;\; y = \sin x,\;\; x = 0,\;\; x = \dfrac{\pi}{2}$

5-10: Find the volume of the solid obtained by revolving the given region about the given axis.

5. $y = x^2 + 2,\;\; y = 10 - x^2;$ about the x-axis
6. $y = x^2 + 1,\;\; y = 3 - x^2$ about the line $y = -2$
7. $y = 4 - x^2,\;\; x = 0,\;\; y = 0;$ about the line $x = -2$
8. $y = x^2 - 2x,\;\; y = 0;$ about the y-axis
9. $y = 6 - x,\;\; y = 3x + 2,\;\; x = 0;$ about the line $x = 8$
10. $y^2 = 2x,\;\; y = x;$ about the line $y = 4$

11-12: Find the length of the given curve over the given interval.

11. $y = (x-1)^{3/2} + 2$ over $1 \le x \le 4$
12. $x = \dfrac{1}{6}y^3 + \dfrac{1}{2y} + 3$ over $1 \le x \le 2$

---

13. Find the arc length function $s(x)$ of the curve $y = \ln x$ with starting point $P(1,0)$.

14-15: Find the area of the surfaces of revolution generated by the given curves.

14. $y = \sqrt{5 - x^2},\quad 0 \le x \le 4$ revolved about the x-axis
15. $x = \sqrt{1 + 3y^2},\quad 0 \le y \le 2$ revolved about the y-axis

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Chapter 6: Parametric Equations

6.1 Parametric Equations

We have seen how a function $F(x)$ can define a curve via the equation $y = F(x)$. Similarly a function $G(y)$ of the y-coordinate can define a curve via $x = G(y)$. Finally a relation $H(x, y) = 0$ such as $x^2 + y^2 - 25 = 0$ can define a curve in the coordinate plane.

Another way in which we can define a planar curve is to consider writing both x and y as functions of a further variable.

Definition: Let $f(t)$ and $g(t)$ be continuous functions of a real variable t with common domain D. Then

$$\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}$$

are called **parametric equations** . The variable $\boldsymbol{t}$ is called the **parameter** .

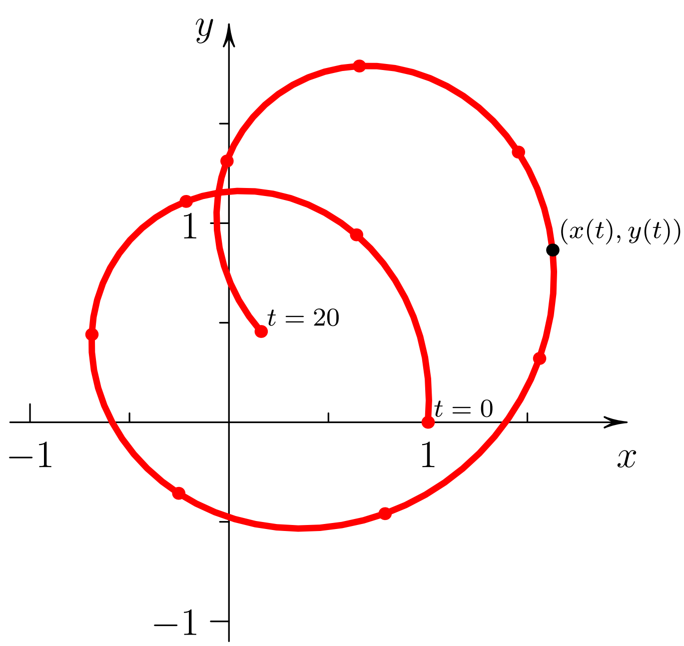
Geometrically such a parametric equation represents a curve. Given a value for t the point $(x, y) = (f(t), g(t))$ is defined. As t ranges over the domain D a curve $\mathcal{C}$ is defined.

Example 6-1

The following shows the curve $\mathcal{C}$ generated by the parametric equations

$$\begin{aligned} x(t) &= \cos(t/2) + \frac{t}{20} \\ y(t) &= \sin(t/2) + \frac{t}{20} \end{aligned}$$

from $t = 0$ to $t = 20$. The dots highlight the points at $t = 0, 2, 4, \ldots, 20$.



The parameter t can, as the choice of variable name suggests, represent a time. In this case $(x, y) = (f(t), g(t))$ can represent the position of an object in the plane at a given time. The curve, in this case, is the path that the object takes in time. However the parameter need not be time. The parameter could, for instance, represent a physical variable such as an angle θ. Another common physical parameter is s, the arc length from some fixed point on the curve to the point (x, y). Finally the parameter may have no physical interpretation whatsoever.

The curve $\mathcal{C}$ in the last example could not be represented either by a function $y = F(x)$ or $x = G(y)$ as vertical and horizontal lines intersect the curve at more than one place. However, as a parametric curve the curve is represented entirely by functions; one just requires two of them.

Example 6-2

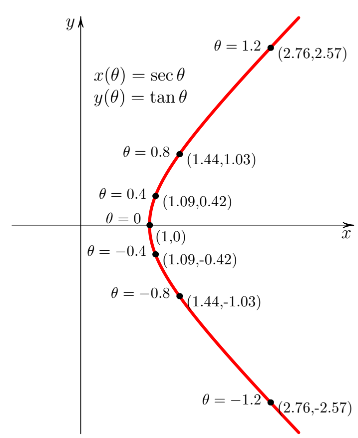
Sketch the curve given by the following parametric equations.

1. $x = \sec\theta,\ y = \tan\theta;\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
2. $x = 4t^2 - 5,\ y = 2t + 3;\ t$ in $\mathbb{R}$

Solution:

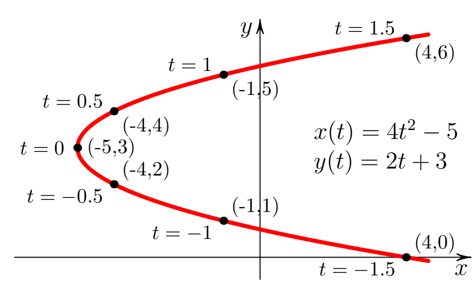
1. $x = \sec\theta,\ y = \tan\theta;\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

   Here the parametric equations are written in terms of parameter θ rather than t. We note that as $\theta \to \pm\frac{\pi}{2}$ that $x \to +\infty$ and $y \to \pm\infty$. Choosing values of θ as 0, ± 0.4, ± 0.8, and ± 1.2 we can evaluate $x(\theta)$ and $y(\theta)$ to get the points (x, y) on the curve. These are shown on the following graph of the parametric equation.



2. $x = 4t^2 - 5,\ y = 2t + 3;\ t$ in $\mathbb{R}$

   Note that as parameter $t \to \pm\infty$ that $x \to +\infty$ and $y \to \pm\infty$. Choosing values of t as in increments of $\Delta t = 0.5$ about $t = 0$ gives the labelled points (x, y) on the curve.



Further Question:

Sketch the curve given by the parametric equations $x = 3\cos\theta$, $y = 2\sin\theta$; $0 \le \theta \le 2\pi$.

One way to identify a curve from its parametric definition is to eliminate the parameter between the two equations to get a single equation involving only x and y.

Example 6-3

Identify the curves from Example 6-2 by writing each as a single equation involving only x and y.

Solution:

1. $x = \sec\theta,\ y = \tan\theta;\ -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

   Identity $\sec^2\theta = \tan^2\theta + 1$ implies $x^2 = y^2 + 1$ and thus

   $$x^2 - y^2 = 1\,.$$

   We recognize the standard form of a hyperbola oriented along the x-axis with vertex $(1, 0)$. Note that the graph of the latter equation admits solutions that are a reflection of our curve about the y-axis. These extraneous solutions have been introduced by squaring. We must restrict the equation of the hyperbola to $x > 0$ since secant is positive on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

2. $x = 4t^2 - 5,\ y = 2t + 3;\ \ t$ in $\mathbb{R}$

   As a general strategy we can sometimes solve for the parameter in terms of one of the variables and then substitute that result into the equation for the other variable. In this case we solve for t in terms of y since that provides a single solution,

   $$y = 2t + 3 \implies t = \frac{y-3}{2}\,.$$

   Substitution into $x(t) = 4t^2 - 5$ gives

   $$x = 4\left(\frac{y-3}{2}\right)^2 - 5 \implies x = (y-3)^2 - 5$$

   We recognize the equation of a horizontal parabola with vertex $(-5, 3)$ opening to the right.

Further Question:

Identify the curve from Example 6-2 given by $x = 3\cos\theta$, $y = 2\sin\theta$; $0 \le \theta \le 2\pi$ by writing it as a single equation involving only x and y.

Example 6-4

Eliminate the parameter t from the given parametric equations to find an equation for the curve in terms of x and y only.

1. $x = 3t^3 - 4,\ y = 4t + 1$
2. $x = e^{3t} - 1,\ y = e^{-3t}$

Solution:

1. $x = 3t^3 - 4,\ y = 4t + 1$

   From $y = 4t + 1$, we have

   $$4t = y - 1 \implies t = \frac{1}{4}(y-1)\,.$$

Substitute this into $x = 3t^3 - 4$, we obtain

$$x = 3t^3 - 4 \implies x = 3\left[\frac{1}{4}(y-1)\right]^3 - 4 \implies x = \frac{3}{16}(y-1)^3 - 4\,.$$

We note in passing that we could have solved for t in terms of x to get $t = \sqrt[3]{(x+4)/3}$ and substituted that into the equation for y, but solving for t in terms of y here was easier.

2. $x = e^{3t} - 1,\ y = e^{-3t}$
   From $y = e^{-3t}$ we have

$$y = e^{-3t} \implies y = \frac{1}{e^{3t}} \implies e^{3t} = \frac{1}{y}\,.$$

Substitute this into $x = e^{3t} - 1$ to obtain

$$x = e^{3t} - 1 \implies x = \frac{1}{y} - 1\,.$$

Further Question:

Eliminate the parameter t from the parametric equations $x = e^{2t}$, $y = t + 1$ to find an equation for the curve in terms of x and y only.

One may also want to go in the other direction, namely to take a curve written in terms of an equation involving x and y and turn that into parametric equations. We note that a function $y = F(x)$ can always be written in parametric form by associating the independent variable with the parameter

$$\begin{aligned} x(t) &= t \\ y(t) &= F(t) \end{aligned}$$

or, equivalently, $\mathcal{C}(t) = (t, F(t))$. Similarly $x = G(y)$ can be generated by the parametrization $\mathcal{C}(t) = (G(t), t)$. Where x and y are defined implicitly in a relation it may yet be possible to write one variable as a function of the other and proceed from there.

Example 6-5

Write the given curves in parametric form.

1. $y = \sqrt{x^2 - 1}$
2. $x = 3y + 5$
3. $x^2 = e^y$

Solution:

1. $y = \sqrt{x^2 - 1}$
   Let $x(t) = t$, then

$$y = \sqrt{x^2 - 1} \implies y = \sqrt{t^2 - 1}\,.$$

Thus the parametric equations are $x(t) = t,\ \ y(t) = \sqrt{t^2 - 1}\,.$

2. $x = 3y + 5$
   Let $y(t) = t$, then

$$x = 3y + 5 \implies x = 3t + 5\,.$$

Thus the parametric equations are $x(t) = 3t + 5, \quad y(t) = t\,.$

3. $x^2 = e^y$

   Here neither x nor y are explicitly defined as functions of the other variable. However solving for y in terms of x gives

$$y = \ln(x^2)$$

   Let $x(t) = t$, then

$$y = \ln(x^2) \implies y = \ln(t^2)\,.$$

   Thus the parametric equations are $x(t) = t,\ \ y(t) = \ln(t^2)$.

   Note that since x^2 is not invertible we cannot write x as a function of y. Indeed, solving we have $x = \pm\sqrt{e^y} = \pm e^{\frac{y}{2}}$. We would need to use two sets of parametric equations $x(t) = e^{\frac{t}{2}}$, $y(t) = t$, and $x(t) = -e^{\frac{t}{2}}$, $y(t) = t$ to generate the full curve.

Further Questions:

Write the given curves in parametric form.

1. $y = 1 + x^2$
2. $x = e^y + \sin y$

Answers: Page 273

Exercise 6-1

1-5: In each of the following,

(a) Sketch the graph of the curve having the indicated parametric equations.

(b) Eliminate the parameter to find a Cartesian equation of the curve.

1. $x = t^3 + 10, \quad y = t^3 + 7; \quad -1 \le t \le 2$
2. $x = t + 2, \quad y = t^2 + 4; \quad t \in \mathbb{R}$
3. $x = 3\cos(2t), \quad y = 3\sin(2t); \quad 0 \le t \le 2\pi$
4. $x = 5\cos t, \quad y = 4\sin t; \quad 0 \le t \le 2\pi$
5. $x = 2\sec t + 1, \quad y = 3\tan t + 2; \quad 0 \le t \le 10$

6.2 Calculus of Parametric Curves

6.2.1 Tangent Slope and Concavity

For a curve defined by $y = F(x)$ we know the slope of the tangent at the point $(x, y) = (x, F(x))$ is given by the derivative $F'(x)$. What then is the slope of the tangent to a curve defined parametrically by $x = f(t)$, $y = g(t)$? Suppose we can eliminate the parameter t as we have done in the previous section to obtain $y = F(x)$ for some function F.[1] Then it follows that

$$g(t) = F(f(t))\,.$$

Differentiating both sides of the latter equation with respect to t one obtains, by the Chain Rule,

$$g'(t) = F'(f(t)) \cdot f'(t)\,.$$

If $f'(t) \neq 0$ then $F'(f(t)) = \frac{g'(t)}{f'(t)}$ and hence the tangent slope at the point $(x(t), y(t))$ is given by

$$F'(x) = \frac{g'(t)}{f'(t)} \qquad (f'(t) \neq 0).$$

In Leibniz derivative notation this is easily remembered; the slope of the tangent line $x = f(t)$, $y = g(t)$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \left(\frac{dx}{dt} \neq 0\right).$$

Note that:

1. The curve has a horizontal tangent line when $\frac{dy}{dt} = 0$ (provided $\frac{dx}{dt} \neq 0$).
2. The curve has a vertical tangent line when $\frac{dx}{dt} = 0$ (provided $\frac{dy}{dt} \neq 0$).

To find the second derivative $\frac{d^2y}{dx^2}$, one may continue as before noting that this would be $F''(x)$. Differentiating $F'(f(t)) = \frac{g'(t)}{f'(t)}$ with respect to t on both sides gives, using the Chain Rule,

$$F''(f(t)) \cdot f'(t) = \frac{d}{dt}\left[\frac{g'(t)}{f'(t)}\right].$$

Thus, for the second derivative with respect to x we have

$$F''(x) = \frac{\frac{d}{dt}\left[\frac{g'(t)}{f'(t)}\right]}{f'(t)} \qquad (f'(t) \neq 0)\ ,$$

or in Leibniz notation,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \qquad \left(\frac{dx}{dt} \neq 0\right).$$

Here we must find $\frac{dy}{dx}$ as a function of t as above and then differentiate.[2]

[1] While this may not be possible for the entire curve, this may plausibly be done for a portion of the curve near the point at which we desire the tangent slope.

[2] This latter formula may be remembered by imagining $\frac{dy}{dx}$ as a function of t and then differentiating this with respect to x. Assuming we can find an inverse of f (at least locally around the point of interest), so that $t = f^{-1}(x)$ and $\frac{dt}{dx}$ is defined, then, by the Chain Rule,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

Noting that our inverse functions satisfy $\frac{dt}{dx} \cdot \frac{dx}{dt} = 1$ we can bring $\frac{dt}{dx}$ into the denominator as $\frac{dx}{dt}$ to recover our previously derived formula.

Note that $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are of interest as these are, within the given the coordinate system, quantities that will be independent of the particular parameterization used to generate the curve.

Example 6-6

Find the equation of the tangent line to the curve $x(t) = t^3$, $y(t) = t^2 - 1$ at the point corresponding to $t = -4$.

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2} = \frac{2}{3t}$$
$$\left.\frac{dy}{dx}\right|_{t=-4} = \frac{2}{3(-4)} = \frac{2}{-12} = -\frac{1}{6}$$

When $t = -4$ we have

$$x = (-4)^3 = -64,\ y = (-4)^2 - 1 = 16 - 1 = 15\,.$$

The equation of the tangent line is, using the point-slope formula $y = m(x - x_0) + y_0$,

$$y = -\frac{1}{6}[x - (-64)] + 15 \implies y = -\frac{1}{6}(x + 64) + 15$$
$$\implies y = -\frac{1}{6}x - \frac{64}{6} + 15 \implies y = -\frac{1}{6}x - \frac{32}{3} + 15 \implies y = -\frac{1}{6}x + \frac{13}{3}\,.$$

Further Question:

Find the equation of the tangent line to the curve $x(t) = 1 - t^3$, $y(t) = t^2 - 3t + 1$ at the point corresponding to $t = 1$.

Example 6-7

Find the equation(s) of the tangent(s) to the curve $x(t) = 2\sin t$, $y(t) = 3\cos t$, t in $[0, 2\pi)$ at the point $\left(\sqrt{2}, \frac{3\sqrt{2}}{2}\right)$.

Solution:

Solving for t for the given point $(x, y) = \left(\sqrt{2}, \frac{3\sqrt{2}}{2}\right)$ gives

$$\sqrt{2} = 2\sin t \implies \sin t = \frac{\sqrt{2}}{2} \implies t = \begin{cases} \frac{\pi}{4} + 2n\pi \\ \pi - \frac{\pi}{4} + 2n\pi \end{cases} = \begin{cases} \frac{\pi}{4} + 2n\pi \\ \frac{3\pi}{4} + 2n\pi \end{cases} \quad (n \text{ an integer})$$
$$\frac{3\sqrt{2}}{2} = 3\cos t \implies \cos t = \frac{\sqrt{2}}{2} \implies t = \begin{cases} \frac{\pi}{4} + 2n\pi \\ 2\pi - \frac{\pi}{4} + 2n\pi \end{cases} = \begin{cases} \frac{\pi}{4} + 2n\pi \\ \frac{7\pi}{4} + 2n\pi \end{cases} \quad (n \text{ an integer})$$

Since both equations must hold and t also lie in $[0, 2\pi)$ we conclude $t = \dfrac{\pi}{4}$. Thus the curve only passes through the point at this value of the parameter and hence only one tangent line can occur.

We now evaluate the derivative at the value $t = \dfrac{\pi}{4}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3\sin t}{2\cos t}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -\frac{3\sin\frac{\pi}{4}}{2\cos\frac{\pi}{4}} = -\frac{3\left(\frac{\sqrt{2}}{2}\right)}{2\left(\frac{\sqrt{2}}{2}\right)} = -\frac{3}{2}$$

Thus the equation of the tangent line is

$$y = -\frac{3}{2}(x - \sqrt{2}) + \frac{3\sqrt{2}}{2}$$
$$\implies y = -\frac{3}{2}x + \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} \implies y = -\frac{3}{2}x + 3\sqrt{2}\,.$$

Further Question:

Find the equation(s) of the tangent(s) to the curve $x(t) = 1 - 2\cos^2 t$, $y(t) = (\tan t)(1 - 2\cos^2 t)$, t in $(-\pi/2, \pi/2)$ at the point $(0, 0)$.

Example 6-8

For the following curves

1. $x(t) = 4t^2,\ y(t) = t^3 - t$
2. $x(t) = 3(t - \sin t),\ y(t) = 3(1 - \cos t)$

(a) Find the points at which the tangent line is horizontal or vertical.

(b) Find $\dfrac{d^2y}{dx^2}$ and discuss the concavity.

Solution:

1. $x(t) = 4t^2,\ y(t) = t^3 - t$

   Note the curve is defined for all t in $\mathbb{R}$. Taking derivatives gives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 1}{8t} = \frac{3}{8}t - \frac{1}{8t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{8} + \frac{1}{8t^2}}{8t} = \frac{3}{64t} + \frac{1}{64t^3} = \frac{3t^2 + 1}{64t^3}\,.$$

   (a) $\dfrac{dy}{dt} = 0 \implies 3t^2 - 1 = 0 \implies t = \pm\sqrt{\dfrac{1}{3}} = \pm\dfrac{1}{\sqrt{3}} = \pm\dfrac{\sqrt{3}}{3}$. At these values of t, $\dfrac{dx}{dt} \neq 0$ and thus the following are points with horizontal tangents.

$$t = \frac{\sqrt{3}}{3} \implies (x, y) = \left(4\left(\frac{\sqrt{3}}{3}\right)^2, \left(\frac{\sqrt{3}}{3}\right)^3 - \frac{\sqrt{3}}{3}\right) = \left(\frac{4}{3}, -\frac{2\sqrt{3}}{9}\right)$$

$$t = -\frac{\sqrt{3}}{3} \implies (x, y) = \left(4\left(-\frac{\sqrt{3}}{3}\right)^2, \left(\frac{-\sqrt{3}}{3}\right)^3 + \frac{\sqrt{3}}{3}\right) = \left(\frac{4}{3}, \frac{2\sqrt{3}}{9}\right)$$

Solving $\dfrac{dx}{dt} = 0 \implies 8t = 0 \implies t = 0$. At this value of t, $\dfrac{dy}{dt} = -1 \neq 0$ and thus the point $(x, y) = (4(0)^2, 0^3 - 0) = (0, 0)$ has a vertical tangent.

(b) $\dfrac{d^2y}{dx^2}$ is not defined if $t = 0$. It vanishes nowhere since $3t^2 + 1 > 0$. Thus we need to consider two intervals for t, $(-\infty, 0)$ and $(0, \infty)$. On the interval $(-\infty, 0)$ we have

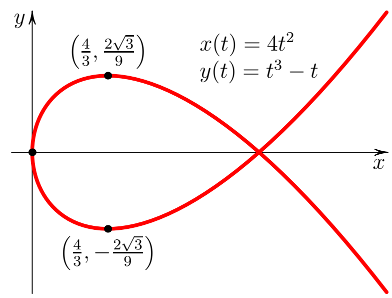
$$\frac{d^2y}{dx^2} = \frac{(+)}{(+)(-)^3} < 0\,.$$

Thus on $(-\infty, 0)$ the curve is concave down. On the interval $(0, \infty)$ we have

$$\frac{d^2y}{dx^2} = \frac{(+)}{(+)(+)^3} > 0\,.$$

Thus on $(0, \infty)$ the curve is concave up.

These properties can be identified on the following plot of the curve.



2. $x(t) = 3(t - \sin t),\ y(t) = 3(1 - \cos t)$

Once again the curve is defined for all t in $\mathbb{R}$. Calculating derivatives gives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\sin t}{3(1-\cos t)} = \frac{\sin t}{1-\cos t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}} = \frac{\frac{\cos t(1-\cos t) - \sin t(\sin t)}{(1-\cos t)^2}}{3(1-\cos t)} = \frac{\cos t - \cos^2 t - \sin^2 t}{3(1-\cos t)^3}$$

$$= \frac{\cos t - 1}{3(1-\cos t)^3} = -\frac{1-\cos t}{3(1-\cos t)^3} = -\frac{1}{3(1-\cos t)^2}\,.$$

(a) We solve $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$,

$$\frac{dy}{dt} = 0 \implies \sin t = 0 \implies t = \begin{cases} 0 + 2n\pi \\ \pi + 2n\pi \end{cases} = \begin{cases} 2n\pi \\ (2n+1)\pi \end{cases} \quad (n \text{ an integer})$$

$$\frac{dx}{dt} = 0 \implies 3(1-\cos t) = 0 \implies \cos t = 1 \implies t = 0 + 2n\pi = 2n\pi \quad (n \text{ an integer})\,.$$

Now $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$ when $t = (2n+1)\pi$ so there are horizontal asymptotes at the points having coordinates

$$x = 3\left[(2n+1)\pi - \sin\left((2n+1)\pi\right)\right] = 3[(2n+1)\pi + 0] = (6n+3)\pi$$
$$y = 3\left(1 - \cos\left((2n+1)\pi\right)\right] = 3(1-(-1)) = 6\,,$$

so $(x, y) = (6n\pi + 3\pi, 6)$, n an integer.

There are no values of t for which $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$ so we cannot discern vertical asymptotes from that criterion. When $t = 2n\pi$ both derivatives vanish and we must consider the limit

$$\lim_{t\to 2n\pi} \frac{dy}{dx} = \lim_{t\to 2n\pi} \frac{\sin t}{1-\cos t} = \lim_{t\to 2n\pi} \frac{\cos t}{\sin t} = \lim_{t\to 2n\pi} \cot t\,.$$

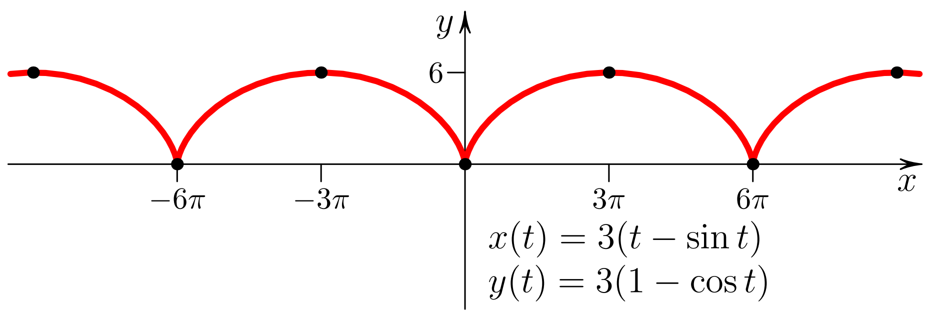
Noting that $\lim\limits_{t\to 2n\pi^-} \cot t = -\infty$ and $\lim\limits_{t\to 2n\pi^+} \cot t = \infty$ we observe that vertical asymptotes occur when $t = 2n\pi$. The coordinates there are

$$\begin{aligned} x &= 3\,[2n\pi - \sin(2n\pi)] = 3[2n\pi + 0] = 6n\pi \\ y &= 3\,(1 - \cos(2n\pi)] = 3(1-1) = 0\,, \end{aligned}$$

so $(x, y) = (6n\pi, 0)$, n an integer.

(b) The second derivative $\frac{d^2y}{dt^2} = -\frac{1}{3(1-\cos t)^2} < 0$ where it is defined. That occurs except when $\cos t = 1 \implies t = 2n\pi$. Thus the curve is concave downward everywhere except at points $(x, y) = (6n\pi, 0)$, n an integer, where concavity is not defined.

A plot of the curve illustrates these findings.



Further Questions:

For the curve given by $x(t) = 3t^2 - 6t$, $y(t) = \sqrt{t}$, with $t \geq 0$,

1. Find the points at which the tangent line is horizontal or vertical.

2. Find $\frac{d^2y}{dx^2}$ and discuss the concavity.

6.2.2 Area Under the Curve

For non-negative function $F(x)$, the area under the curve $y = F(x)$ and above the x-axis for $a \leq x \leq b$ is found using vertical area elements and is given by $A = \int_a^b F(x)\,dx = \int_a^b y\,dx$. If the curve is parametrically defined by $x = f(t)$, $y = g(t)$, for t in $[t_a, t_b]$, so that $a = f(t_a)$ and $b = f(t_b)$, the area becomes, upon change of variables via $x = f(t)$ and noting $dx = \frac{dx}{dt}\,dt$,

$$A = \int_a^b y\,dx = \int_{t_a}^{t_b} y\frac{dx}{dt}\,dt = \int_{t_a}^{t_b} g(t)f'(t)\,dt$$

Example 6-9

Find the area enclosed by the x-axis and the given curve.

1. $x(t) = e^{2t} + 3,\ y(t) = t - t^2$
2. $x(t) = t + 2,\ y(t) = -t^2 + 4t + 5$

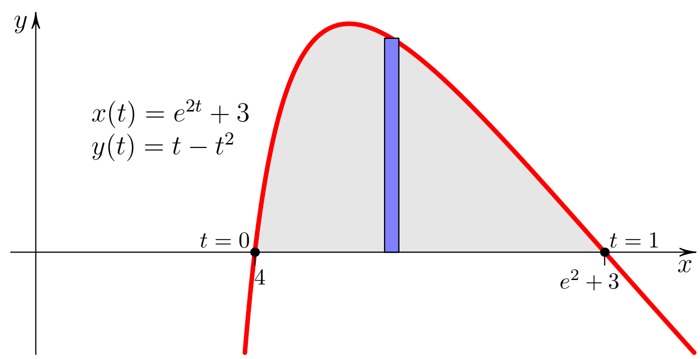
Solution:

1. $x(t) = e^{2t} + 3,\ y(t) = t - t^2$

   The intersection points of the curve with the x-axis occur when $y = 0$.

   $$y(t) = 0 \implies t - t^2 = 0 \implies t(t-1) = 0 \implies t = 0 \text{ or } t = 1$$

   Evaluating $x(t)$ gives the points $(4, 0)$ and $(e^2 + 3, 0)$ respectively. A graph of the curve follows.



   The area enclosed by the given curve and the x-axis is therefore

$$\begin{aligned}
A = \int_0^1 y\frac{dx}{dt}dt &= \int_0^1 (t-t^2)(2e^{2t})\,dt = (t-t^2)e^{2t}\Big|_0^1 - \int_0^1 e^{2t}(1-2t)\,dt \\
&= \left[(t-t^2)e^{2t} - (1-2t)\left(\frac{e^{2t}}{2}\right)\right]_0^1 + \int_0^1 \frac{e^{2t}}{2}(-2)\,dt \\
&= \left[(t-t^2)e^{2t} - \frac{1}{2}(1-2t)e^{2t} - \frac{1}{2}e^{2t}\right]_0^1 \\
&= (1-1)e^2 - \frac{1}{2}(1-2)e^2 - \frac{1}{2}e^2 - \left[(0-0)e^0 - \frac{1}{2}(1-0)e^0 - \frac{1}{2}e^0\right] \\
&= \frac{1}{2}e^2 - \frac{1}{2}e^2 + \frac{1}{2} + \frac{1}{2} = 1 \text{ square unit}\,,
\end{aligned}$$

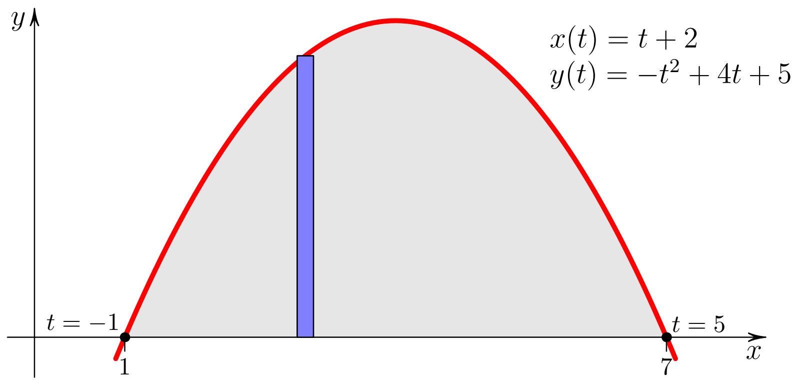
   where here we used repeated Integration by Parts.

2. $x(t) = t + 2,\ y(t) = -t^2 + 4t + 5$

First find the intersection points of the given curve and the x-axis.

$$y(t) = 0 \implies -t^2 + 4t + 5 = 0 \implies t^2 - 4t - 5 = 0 \implies (t+1)(t-5) = 0$$
$$\implies t = -1 \text{ or } t = 5$$

Substituting these values for $x(t)$ gives the intersection with the x-axist to be the points $(1, 0)$ and $(7, 0)$. A graph of the curve is as follows.



Thus the area enclosed by the given curve and the x-axis is

$$A = \int_0^1 y\frac{dx}{dt}dt = \int_{-1}^5 (-t^2 + 4t + 5)(1)\,dt = -\frac{1}{3}t^3 + 2t^2 + 5t\Big|_{-1}^5$$
$$= -\frac{1}{3}(5)^3 + 2(5)^2 + 5(5) - \left[-\frac{1}{3}(-1)^3 + 2(-1)^2 + 5(-1)\right]$$
$$= -\frac{125}{3} + 50 + 25 - \frac{1}{3} - 2 + 5$$
$$= -\frac{126}{3} + 78 = \frac{-126 + 234}{3} = \frac{108}{3} \text{ square units.}$$

Further Question:

Find the area enclosed by the x-axis and the curve $x = 1 + \sin t$, $y = \frac{\pi}{2}t - t^2$.

Similarly if a curve is defined by the non-negative function $x = G(y)$ for $c \le y \le d$ then the area between that curve and the y-axis is found using horizontal area elements and is given by $A = \int_c^d G(y)\,dy = \int_c^d x\,dy$. If the curve is parametrically defined by $x = f(t)$, $y = g(t)$, for t in $[t_c, t_d]$, such that $c = g(t_c)$ and $d = g(t_d)$, the area becomes, upon changinge variable via $y = g(t)$ and noting $dy = \frac{dy}{dt}\,dt$,

$$A = \int_c^d x\,dy = \int_{t_c}^{t_d} x\frac{dy}{dt}\,dt = \int_{t_c}^{t_d} f(t)g'(t)\,dt$$

These formulae can be generalized for areas between parametric curves by consideration of the coordinates of the endpoints of the area elements.

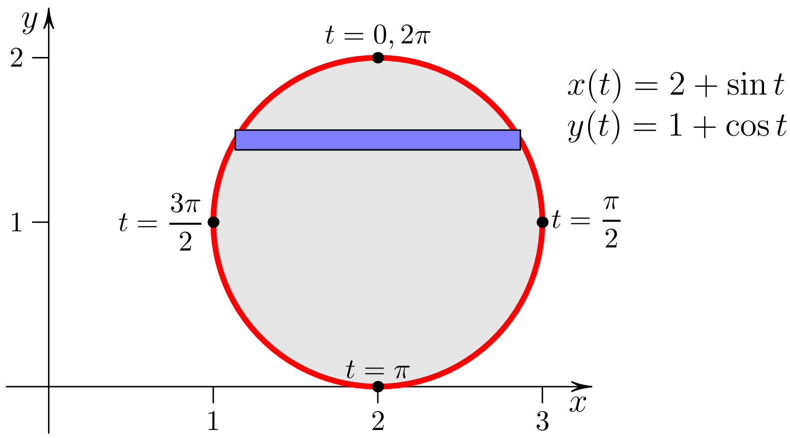
Example 6-10

Find the area enclosed by the curve described by the parametric equations.

$$x(t) = 2 + \sin t,\; y(t) = 1 + \cos t;\; 0 \le t \le 2\pi\,.$$

Solution:

A graph of the parametric curve follows.



In this case we can use vertical or horizontal area elements. We choose the latter (as shown) because the rightmost curve corresponds to t in $[0, \pi]$ while the leftmost curve corresponds to $[\pi, 2\pi]$. Considering the right endpoint of the element to be (x_2, y) and the left endpoint to be (x_1, y) the differential area element is $dA = (x_2 - x_1)dy$. The area is given by the integral

$$A = \int_0^2 (x_2 - x_1)dy = \int_0^2 x_2 dy - \int_0^2 x_1 dy = \int_\pi^0 x\frac{dy}{dt}dt - \int_\pi^{2\pi} x\frac{dy}{dt}dt\,.$$

Note here that for the first integral the limits of $y = 0$ and $y = 2$ correspond to $t = \pi$ and $t = 0$ respectively since we are following the curve on the right while for the second integral these values of y correspond to $t = \pi$ and $t = 2\pi$ as we are then traversing the curve on the left. Exchanging limits simplifies the integral which can then be evaluated to be

$$\begin{aligned}
A &= -\int_0^\pi x\frac{dy}{dt}dt - \int_\pi^{2\pi} x\frac{dy}{dt}dt = -\int_0^{2\pi} x\frac{dy}{dt}dt \\
&= -\int_0^{2\pi} (2 + \sin t)(-\sin t)\,dt = \int_0^{2\pi} (2\sin t + \sin^2 t)\,dt = \int_0^{2\pi} \left(2\sin t + \frac{1}{2}\left[1 - \cos(2t)\right]\right) dt \\
&= \int_0^{2\pi} \left(2\sin t + \frac{1}{2} - \frac{1}{2}\cos(2t)\right) dt = \left[-2\cos t + \frac{1}{2}t - \frac{1}{4}\sin(2t)\right|_0^{2\pi} \\
&= -2\cos(2\pi) + \frac{1}{2}(2\pi) - \frac{1}{4}\sin(4\pi) - \left[-2\cos(0) + 0 - \frac{1}{4}\sin(0)\right] \\
&= -2 + \pi + 2 = \pi \text{ square units.}
\end{aligned}$$

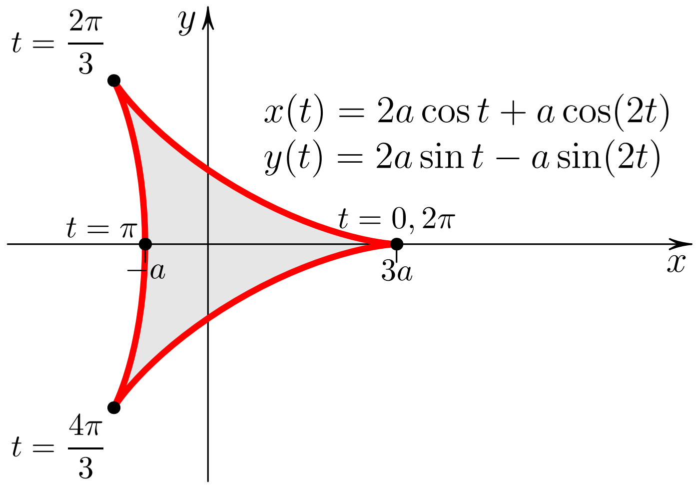
As expected from the diagram this is just the area of a circle of radius $r = 1$.

Further Question:

Find the area inside the **deltoid curve** described by the parametric equations

$$x(t) = 2a\cos t + a\cos(2t), \quad y(t) = 2a\sin t - a\sin(2t); \quad 0 \le t \le 2\pi\,.$$

Here a is a positive constant. A graph of the curve is the following.



6.2.3 Arc Length

We have seen that the arc length of a curve $\mathcal{C}$ described by $y = F(x)$ for $a \le x \le b$ is given by

$$s = \int_a^b \sqrt{1 + [F'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If curve $\mathcal{C}$ is also parameterized by $x = f(t)$, $y = g(t)$, for $t_a \le t \le t_b$ where $t_a = f(a)$ and $t_b = f(b)$ one has, by the change of variable $x = f(t)$, that

$$s = \int_{t_a}^{t_b} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt\ ,$$

where here we used our expression for the derivative $\frac{dy}{dx}$ in terms of t found before. If we further assume that our function $x = f(t)$ is an increasing function of x, so $dx/dt > 0$ and $\sqrt{(dx/dt)^2} = dx/dt$, we may rewrite this:

$$s = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\ .$$

The previous result can be shown to be be true quite generally for any parameterized curve $\mathcal{C}$, not simply for curves that may be described by a function of x.

Theorem 6-1: Suppose curve $\mathcal{C}$ is traced exactly once by the parameterization $x = f(t)$, $y = g(t)$ as t increases from t_i to t_f. If derivatives f' and g' exist and are continuous on $[t_i, t_f]$ then the arc length of $\mathcal{C}$ is

$$s = \int_{t_i}^{t_f} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\ .$$

One can recover special case formulae from Chapter 5 for $y = F(x)$ using parameterization $x = t$, $y = F(t)$ and for $x = G(y)$ using parameterization $x = G(t)$, $y = t$.

Note that the formula is easily remembered by $s = \int ds$, where the differential arc length element $ds = \sqrt{dx^2 + dy^2}$. This formula requires some form of parameterization of the curve for its application. However, the differential $ds = \sqrt{dx^2 + dy^2}$ reflects the fact that the arc length of a curve is independent of the parameterization of the curve chosen to calculate it.

Example 6-11

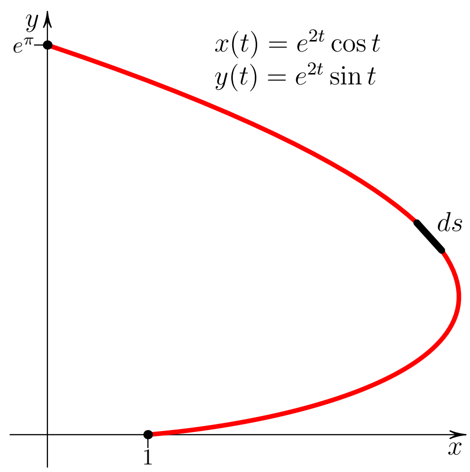
Find the length of the given curve.

1. $x(t) = e^{2t}\cos t,\ y(t) = e^{2t}\sin t;\ 0 \le t \le \dfrac{\pi}{2}$
2. $x(t) = 2 + \sin t,\ y(t) = 1 + \cos t;\ 0 \le t \le 2\pi$

Solution:

1. $x(t) = e^{2t}\cos t,\ y(t) = e^{2t}\sin t;\ 0 \le t \le \dfrac{\pi}{2}$

   A graph showing the parametric curve over the given interval is the following.



After taking the derivatives

$$\frac{dx}{dt} = 2e^{2t}\cos t - e^{2t}\sin t$$
$$\frac{dy}{dt} = 2e^{2t}\sin t + e^{2t}\cos t\,,$$

we evaluate

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (2e^{2t}\cos t - e^{2t}\sin t)^2 + (2e^{2t}\sin t + e^{2t}\cos t)^2 \\ &= 4e^{4t}\cos^2 t - 4e^{4t}\cos t\sin t + e^{4t}\sin^2 t \\ &\quad + 4e^{4t}\sin^2 t + 4e^{4t}\cos t\sin t + e^{4t}\cos^2 t \\ &= 4e^{4t} + e^{4t} = 5e^{4t}\,.\end{aligned}$$

Thus the arc length is

$$\begin{aligned}s &= \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{\frac{\pi}{2}} \sqrt{5e^{4t}}\,dt = \sqrt{5}\int_0^{\frac{\pi}{2}} e^{2t}\,dt = \frac{\sqrt{5}}{2}e^{2t}\Big|_0^{\frac{\pi}{2}} \\ &= \frac{\sqrt{5}}{2}e^{\pi} - \frac{\sqrt{5}}{2}e^{0} = \frac{\sqrt{5}}{2}(e^{\pi} - 1) \text{ units.}\end{aligned}$$

2. $x(t) = 2 + \sin t,\ y(t) = 1 + \cos t;\ 0 \le t \le 2\pi$

$$\frac{dx}{dt} = \cos t,\ \ \frac{dy}{dt} = -\sin t \implies \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \cos^2 t + \sin^2 t = 1$$

Thus the arc length is given by

$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} (1)\, dt = t\Big|_0^{2\pi} = 2\pi - 0 = 2\pi \text{ units.}$$

Note that a graph of the parametric curve is the same as in Example 6-10, namely a circle of radius $r = 1$ centred at $(2, 1)$. As such its arc length is just the circumference $2\pi r = 2\pi$ as found.

Further Question:

Find the length s of the arc of the circle $x = R\cos\theta$, $y = R\sin\theta$ for $0 \leq \theta \leq \frac{2\pi}{3}$ where R is the constant radius.

Exercise 6-2

Answers: Page 273

1-4: Find the equation of the tangent line to the curve at the point corresponding to the given value of the parameter.

1. $x = 3t^2 + 4, \quad y = 2t + 1; \quad t = 1$
2. $x = e^{2t} + 3, \quad y = e^{-t} + 1; \quad t = 0$
3. $x = \sec t, \quad y = 2\tan t; \quad t = \frac{\pi}{6}$
4. $x = 2\cos t, \quad y = 6\sin t; \quad t = \frac{\pi}{3}$

5-6: Find the points on the curve at which the tangent line is horizontal or vertical.

5. $x = t^3 + 4, \quad y = t^3 - 12t + 1$
6. $x = e^t - 3e^{-t} - 4t, \quad y = e^{2t} - 8t$

7-8: Find $\dfrac{d^2y}{dx^2}$ and analyze the concavity of the following curves.

7. $x = \sqrt{t}, \quad y = t^2 - 3\sqrt{t}; \quad t \geq 0$
8. $x = t + 2, \quad y = e^t - t^2; \quad t \geq 0$

---

9. Find the area enclosed by the x-axis and the curve $x = t^2 + 5, \quad y = 8t - t^2; \; t \in \mathbb{R}$.
10. Find the area enclosed by the curve $x = \sin t, \quad y = 1 + \cos t; \; 0 \leq t \leq 2\pi$.

11-12: Find the length of the given curve.

11. $x = t + \frac{1}{t}, \quad y = 2\ln t; \; 1 \leq t \leq 10$
12. $x = t^2 + 3, \quad y = 2t + 1; \; 0 \leq t \leq 1$

Answers: Page 273

Chapter 6 Review Exercises

1-2: Find the equation of the tangent line to the curve at the point corresponding to the given value of the parameter.

1. $x = 2t + e^{-t}, \quad y = 3e^t; \quad t = 0$
2. $x = t + \sin t, \quad y = \cos t; \quad t = \frac{\pi}{2}$

3-4: Find the points on the curve at which the tangent line is horizontal or vertical.

3. $x = e^t - 3t, \quad y = e^{-2t} + 4t$
4. $x = t - 2\sin t, \quad y = t + \cos t; \;\; 0 \le t \le 2\pi$

5-6: Find the area enclosed by the given curves.

5. $x = t + 5$, $y = \ln t$; the x-axis, and the line $x = 15$.
6. $x = \frac{1}{2}t^2 + 3$, $y = 3t + e^{-t} - 1$; the x-axis, and the line $x = 11$.

7-8: Find the length of the given curve.

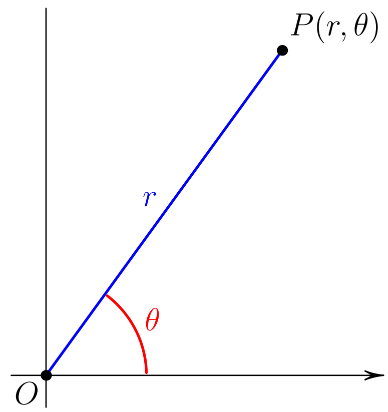
7. $x = \frac{1}{2}t^2 - \ln t, \quad y = 2t; \;\; 1 \le t \le 5$
8. $x = e^t \cos t, \quad y = e^t \sin t; \;\; 0 \le t \le \pi$

Chapter 7: Polar Coordinates

7.1 Polar Coordinates

A coordinate system identifies a point P in space by an ordered list of one or more numbers called coordinates. We are familiar with the **Cartesian Coordinate System** in which we identify a point P in two-dimensional space by the ordered pair (x, y) where x is the x-coordinate of the point and y is the y-coordinate of the point.

In polar coordinates one selects a point to O to be the origin or pole of the coordinate system. The first coordinate of an arbitrary point P is the distance r from O to P. Next one draws a **polar axis** as a ray emanating from O which, for planar coordinates, is aligned with the positive x-axis. The second coordinate of P is the angle θ made between this axis and the ray OP. The point P is then represented by the ordered pair (r, θ) which are the **polar coordinates** of P.



Notes:

1. Angles θ are positive if measured in the usual counter-clockwise sense from the polar axis and are negative if measured clockwise.
2. By convention polar coordinates (r, θ) with a negative r represent the point P at $(|r|, \theta + \pi)$. In other words, they are at the positive distance $|r|$ directed along the ray opposite the ray subtending the angle θ.

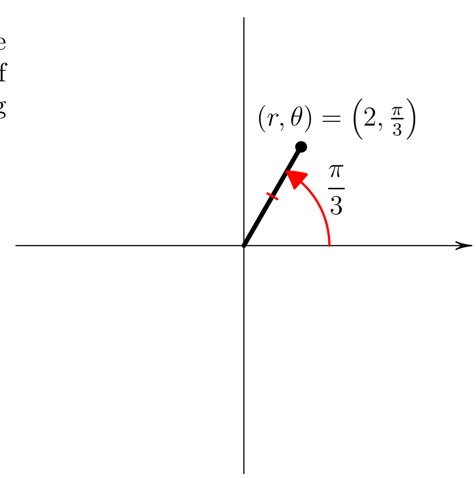
Example 7-1

Plot the point with each of the following polar coordinates.

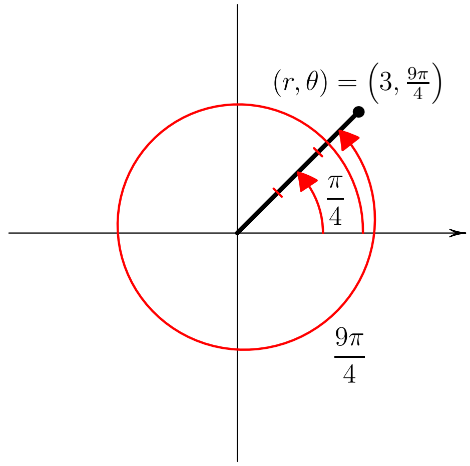
1. $\left(3, \dfrac{\pi}{3}\right)$
2. $\left(2, \dfrac{9\pi}{4}\right)$
3. $\left(4, \dfrac{3\pi}{4}\right)$

Solution:

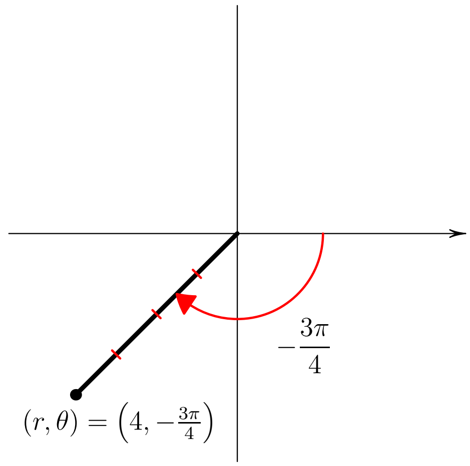
1. Measure an angle of $\pi/3 = 60°$ counterclockwise from the x-axis to determine the terminal ray of the angle followed by a distance of 2 units along the ray to find the point.



2. Since $9\pi/4 - 2\pi = \pi/4$ the point is 3 units along the ray with terminal axis of $\pi/4 = 45°$ degrees measured counterclockwise from the polar axis.



3. A negative angle of $-3\pi/4$ is measured clockwise from the polar axis. We then measure 4 units along the terminal ray.



Further Questions:

Plot the point with each of the following polar coordinates.

1. $(2, 3\pi/4)$
2. $(2, -2\pi/3)$
3. $(-2, 3\pi/4)$

Unlike in Cartesian Coordinates, the polar coordinates for point P are not unique since the addition of a multiple of 2π to the angle returns one to the same physical point.

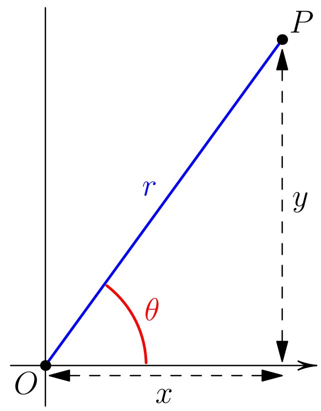
Example 7-2

The following polar coordinates all represent the same point P.

$$(2, \pi/4), \qquad (2, 9\pi/4), \qquad (2, -7\pi/4), \qquad (-2, 5\pi/4)$$

7.2 Converting Between Polar and Cartesian Coordinates

Suppose P has Cartesian coordinates (x, y) and polar coordinates (r, θ).



As suggested by the diagram, one has

$$x = r\cos\theta \qquad\qquad y = r\sin\theta\ .$$

These formulae allow conversion from polar to Cartesian coordinates.

Example 7-3

Find the Cartesian coordinates of the given point.

1. $(r, \theta) = \left(2, \frac{\pi}{3}\right)$
2. $(r, \theta) = \left(4, \frac{3\pi}{4}\right)$

Solution:

1. For $(r, \theta) = \left(2, \frac{\pi}{3}\right)$:

$$x = r\cos\theta \implies x = 2\cos\frac{\pi}{3} \implies x = 2\left(\frac{1}{2}\right) \implies x = 1$$

$$y = r\sin\theta \implies y = 2\sin\frac{\pi}{3} \implies y = 2\left(\frac{\sqrt{3}}{2}\right) \implies y = \sqrt{3}$$

Thus the Cartesian coordinates of the point are $(x, y) = \left(1, \sqrt{3}\right)$.

2. For $(r, \theta) = \left(4, \frac{3\pi}{4}\right)$:

$$x = 4\cos\frac{3\pi}{4} \implies x = 4\left(-\frac{\sqrt{2}}{2}\right) \implies x = -2\sqrt{2}$$

$$y = 4\sin\frac{3\pi}{4} \implies y = 4\left(\frac{\sqrt{2}}{2}\right) \implies y = 2\sqrt{2}$$

Thus the Cartesian coordinates of the points are $(x, y) = \left(-2\sqrt{2}, 2\sqrt{2}\right)$.

Further Question:

Find the Cartesian coordinates of $(r, \theta) = (3, \pi/4)$.

To convert from Cartesian to polar coordinates we solve for r and θ. Squaring and adding both sides of the two equations gives:

$$x^2 + y^2 = r^2 \cos^2\theta + r^2 \sin^2\theta = r^2$$

Dividing both sides of the second equation by the corresponding sides of the first yields:

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

In summary we have, to convert from Cartesian to polar coordinates,

$$r = \sqrt{x^2 + y^2} \qquad\qquad \tan\theta = \frac{y}{x}$$

Note that when solving $\tan\theta = y/x$ for θ:

1. There will typically be two solutions in $[0, 2\pi)$. The selection of the correct value is made by looking at the quadrant determined by the original x and y values.
2. If $x = 0$ (so y/x is undefined) consider the y-value to determine if $\theta = \pi/2$ or $\theta = 3\pi/2$.

Example 7-4

Find the polar coordinates of the given point.

1. $(x, y) = \left(-2\sqrt{3},\, 2\right)$
2. $(x, y) = \left(-3\sqrt{2},\, -3\sqrt{2}\right)$

Solution:

1. For $(x, y) = \left(-2\sqrt{3},\, 2\right)$:

$$r = \sqrt{x^2 + y^2} \implies r = \sqrt{12 + 4} \implies r = \sqrt{16} \implies r = 4$$
$$\tan\theta = \frac{y}{x} \implies \tan\theta = \frac{2}{-2\sqrt{3}} \implies \tan\theta = \frac{-1}{\sqrt{3}}$$

To solve the trigonometric equation for θ we note the (acute) reference angle (the angle of the terminal ray with respect to the x-axis) satisfies $\tan\theta' = \left|\frac{-1}{\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$. Thus $\theta' = \frac{\pi}{6}$ using a $30° - 60° - 90°$ triangle or the arctangent function. Since $\tan\theta < 0$ we have two possible angles in $[0, 2\pi)$:

$$\text{Quadrant II} : \theta = \pi - \theta' = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$
$$\text{Quadrant IV} : \theta = 2\pi - \theta' = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

Since $x < 0$ and $y > 0$ our point lies in Quadrant II and hence the polar coordinates are $(r, \theta) = \left(4, \frac{5\pi}{6}\right)$.

2. For $(x, y) = \left(-3\sqrt{2},\, -3\sqrt{2}\right)$:

$$r = \sqrt{x^2 + y^2} = \sqrt{18 + 18} = \sqrt{36} = 6$$
$$\tan\theta = \frac{y}{x} \implies \tan\theta = \frac{-3\sqrt{2}}{-3\sqrt{2}} \implies \tan\theta = 1$$

Reference angle θ' satisfies $\tan\theta' = |1| = 1$. Thus $\theta' = \frac{\pi}{4}$ using a $45° - 45° - 90°$ triangle or the arctangent function. Since $\tan\theta > 0$ we have two possible angles in $[0, 2\pi)$:

$$\text{Quadrant I} : \theta = \theta' = \frac{\pi}{4}$$

$$\text{Quadrant III} : \theta = \pi + \theta' = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

Since $x < 0$ and $y < 0$ our point lies in Quadrant III and hence the polar coordinates are $(r, \theta) = \left(6, \frac{5\pi}{4}\right)$.

Note that in both these examples $\theta \neq \tan^{-1}\left(\frac{y}{x}\right)$. Using that formula is only correct if your point happens to lie in Quadrant I or IV.

Further Question:

Find the polar coordinates of $(x, y) = \left(1, -\sqrt{3}\right)$.

Answers: Page 274

Exercise 7-1

1-4: Find Cartesian coordinates of the given point.

1. $(r, \theta) = \left(2, \frac{\pi}{3}\right)$
2. $(r, \theta) = \left(3, \frac{3\pi}{2}\right)$
3. $(r, \theta) = \left(4, \frac{5\pi}{4}\right)$
4. $(r, \theta) = \left(1, \frac{5\pi}{6}\right)$

5-8: Find polar coordinates of the given point.

5. $(x, y) = (1, -1)$
6. $(x, y) = \left(2\sqrt{3}, 2\right)$
7. $(x, y) = (-5, 0)$
8. $(x, y) = \left(-4, -4\sqrt{3}\right)$

7.3 Curves in Polar Coordinates

If we consider the coordinate θ as an independent variable and r a dependent variable determined by function $f(\theta)$ then $r = f(\theta)$ describes a curve in polar coordinates. More generally the solutions (r, θ) of the relation $g(r, \theta) = 0$ describe a curve.

Example 7-5

For each of the following,

(a) Sketch the curve represented by the given polar equation.

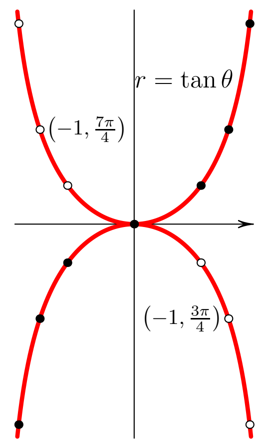
(b) Find a Cartesian equation for this curve.

1. $r = \tan\theta$
2. $r^2 \sin(2\theta) = 16$

Solution:

1. $r = \tan\theta$

   (a) Here the curve is explicitly defined as a function $r = f(\theta) = \tan\theta$. In $[0, 2\pi)$ the function is defined everywhere except $\pi/2$ and $3\pi/2$ at which values $r = f(\theta)$ approaches $\pm\infty$ from right and left. To find a few points (r, θ) in the first quadrant we choose $\theta = 0, \pi/6 = 30°, \pi/4 = 45°$,and $\pi/3 = 60°$ at which $r = \tan\theta$ is easily evaluated to be $r = 0, 1/\sqrt{3}, 1$, and $\sqrt{3}$ respectively. Similarly points with reference angles of 30°, 45°, 60° can be found in the remaining quadrants. One notes that for angles satisfying $\pi/2 < \theta < \pi$ and $3\pi/2 < \theta < 2\pi$ the radius r will be negative. So for instance we have the points $(r, \theta) = (-1, 3\pi/4)$ and $(r, \theta) = (-1, 7\pi/4)$. By convention these points are measured along the opposite direction of the terminal ray as shown in the diagram. (All hollow points shown have radius $r = f(\theta) < 0$.) Joining these points produces a complete graph of the polar function.



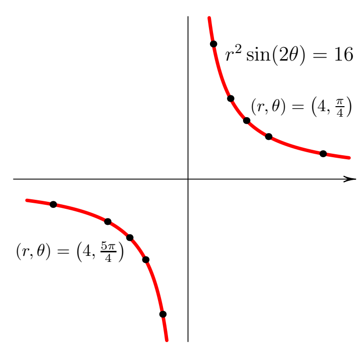
   (b) Since $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$, one has

$$r = \tan\theta \implies \sqrt{x^2+y^2} = \frac{y}{x} \implies x^2 + y^2 = \frac{y^2}{x^2} \implies x^4 + x^2y^2 = y^2$$
$$\implies y^2 - x^2y^2 = x^4 \implies y^2(1 - x^2) = x^4 \implies y^2 = \frac{x^4}{1 - x^2}$$

   Thus the Cartesian form of the given curve is $y^2 = \dfrac{x^4}{1 - x^2}$.

2. $r^2 \sin(2\theta) = 16$

   (a) Here r is defined implicitly in terms of θ. One notes that since $\sin(2\theta) = 16/r^2 > 0$ we require $0 < 2\theta < \pi$ or $2\pi < 2\theta < 3\pi$ where sine is positive. This implies $0 < \theta < \pi/2$ or $3\pi/2 < \theta < \pi$ and hence the graph only has points in Quadrants I and III. For values of θ in those quadrants we get two solutions for r, namely $r = \pm\frac{4}{\sqrt{\sin(2\theta)}}$, however the negative r points are already generated by positive r points in the opposing quadrant. The closest points to the origin are $(r, \theta) = (4, \pi/4)$ and $(4, 5\pi/4)$. Black dots show further points plotted in increments of $\Delta\theta = 0.3$ around these points.



(b) Since $r^2 = x^2 + y^2$, $\cos\theta = \frac{x}{r}$, and $\sin\theta = \frac{y}{r}$, it follows that

$$r^2\sin(2\theta) = 16 \implies (x^2+y^2)2\sin\theta\cos\theta = 16 \implies (x^2+y^2)2\left(\frac{y}{r}\right)\left(\frac{x}{r}\right) = 16$$
$$\implies (r^2)2\left(\frac{xy}{r^2}\right) = 16 \implies 2xy = 16 \implies xy = 8$$

Thus the Cartesian form of the given curve is $xy = 8$. Hence the graph is that of a hyperbola with axis along the diagonal $y = x$.

Further Questions:

1. Sketch the curve represented by the polar equation $r = 3\sin\theta$.
2. Find a Cartesian equation for this curve.

7.3.1 Symmetry in Polar Curves

Suppose a polar curve is defined either explictly by $r = f(\theta)$ or implicitly with the relation $g(r,\theta) = 0$. Then, just as we saw with Cartesian Coordinates, the curves may have certain symmetries arising from the following transformations:

$\theta \to -\theta$ **:** If $f(-\theta) = f(\theta)$ or $g(r,-\theta) = g(r,\theta)$, then the curve is symmetric about the polar axis.

$r \to -r$ **:** If $g(-r,\theta) = g(r,\theta)$ then the curve is symmetric about the origin O.

$\theta \to \pi - \theta$ **:** If $f(\pi-\theta) = f(\theta)$ or $g(r,\pi-\theta) = g(r,\theta)$, then the curve is symmetric about the vertical $\theta = \pi/2$ line.

Example 7-6

Sketch the given curve.

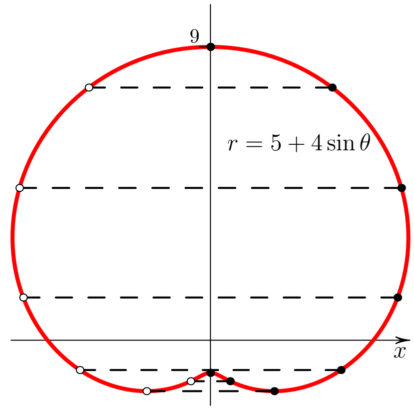
1. $r = 5 + 4\sin\theta$
2. $r^2 = 16\sin(2\theta)$

Solution:

1. $r = 5 + 4\sin\theta$
   Here $r = f(\theta)$ so r is explicitly defined with $f(\theta) = 5 + 4\sin\theta$.

$$f(-\theta) = 5 + 4\sin(-\theta) = 5 - 4\sin(\theta) \neq f(\theta)$$
$$f(\pi-\theta) = 5 + 4\sin(\pi-\theta) = 5 + 4(\sin\pi\cos\theta - \cos\pi\sin\theta) = 5 + 4\sin\theta = f(\theta)$$

   Since $f(\pi-\theta) = f(\theta)$ the graph is symmetric about the vertical line $\theta = \pi/2$ or equivalently the y-axis. We need only plot points in quadrants I and IV and reflect across that axis to get the graph of the polar function.



Note for an explicitly defined polar function $r = f(\theta)$ the transformation $r \to -r$ can never preserve the equality and consequently its graph will never be symmetric about the origin.

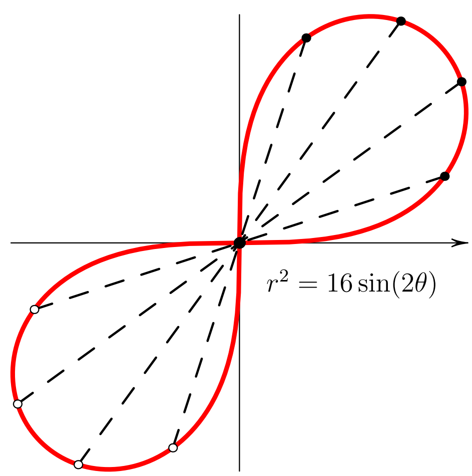
2. $r^2 = 16\sin(2\theta)$
   Here r is implicitly defined and may be written equivalently as $g(r, \theta) = 0$ where

$$g(r, \theta) = 16\sin(2\theta) - r^2 .$$

Looking for symmetries we evaluate the following:

$$\begin{aligned} g(r, -\theta) &= 16\sin(-2\theta) - r^2 = -16\sin(2\theta) - r^2 \neq g(r, \theta) \\ g(-r, \theta) &= 16\sin(2\theta) - (-r)^2 = 16\sin(2\theta) - r^2 = g(r, \theta) \\ g(r, \pi - \theta) &= 16\sin[2(\pi - \theta)] - r^2 = 16\sin(2\pi - 2\theta) - r^2 \\ &= 16\left[\sin(2\pi)\cos(2\theta) - \sin(2\theta)\cos(2\pi)\right] - r^2 = -16\sin(2\theta) - r^2 \neq g(r, \theta) \end{aligned}$$

Since $g(-r, \theta) = g(r, \theta)$ we conclude that the graph is symmetric about the origin. Since r^2 is positive we require that $\sin(2\theta) > 0$ which implies $0 \leq \theta \leq \frac{\pi}{2}$ and $\pi \leq \theta \leq \frac{3\pi}{2}$ so the graph lies only in quadrants I and III. Using the symmetry we need only plot the graph in the first quadrant and reflect the points across the origin to get the following graph:



Further Question:

Sketch the curve $r = 1 + \cos\theta$.

7.3.2 Tangents

The slope of the tangent to a polar curve described by $r = f(\theta)$ may be found by converting to Cartesian Coordinates. Since $x = r\cos\theta$ and $y = r\sin\theta$ we have

$$x = f(\theta)\cos\theta \qquad\qquad y = f(\theta)\sin\theta$$

The curve is now represented by a parametric equation with parameter θ. Applying our rules for finding the derivative dy/dx of such equations from Chapter 6 gives $dy/dx = (dy/d\theta)/(dx/d\theta)$ and so

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Vertical tangent lines occur where $\frac{dy}{d\theta} = 0$ (and $\frac{dx}{d\theta} \neq 0$) while for horizontal tangent lines $\frac{dx}{d\theta} = 0$ (and $\frac{dy}{d\theta} \neq 0$). If both derivatives vanish at some θ, consideration of the limit of $\frac{dy}{dx}$ as one approaches that value must be made to determine the nature of the tangent.

Example 7-7

For each of the following,

(a) Find the slope of the tangent line to the given curve at the given value.

(b) Find the points on the curve where the tangent line is horizontal or vertical.

1. $r = 5\cos\theta;\ \theta = \frac{\pi}{3}$

2. $r = 4\sin(2\theta);\ \theta = \frac{\pi}{6}$

Solution:

1. $r = 5\cos\theta;\ \theta = \frac{\pi}{3}$

   (a) Here $f(\theta) = 5\cos\theta,\ f'(\theta) = -5\sin\theta$ and so

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{-5\sin\theta\sin\theta + 5\cos\theta\cos\theta}{-5\sin\theta\cos\theta - 5\cos\theta\sin\theta} = \frac{\cos^2\theta - \sin^2\theta}{-2\sin\theta\cos\theta},$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3}}{-2\sin\frac{\pi}{3}\cos\frac{\pi}{3}} = \frac{\frac{1}{4} - \frac{3}{4}}{-2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

   (b) We note that due to the periodicity of cosine, $\cos(\theta + 2\pi) = \cos(\theta)$, the entire curve $r = 5\cos\theta$ is generated by θ in $[0, 2\pi)$ so we only need to find solutions in that interval. We solve $dy/d\theta = 0$ and $dx/d\theta = 0$. From (a) we have

$$\begin{aligned}
\frac{dy}{d\theta} = 0 &\implies -5\sin\theta\sin\theta + 5\cos\theta\cos\theta = 0 \\
&\implies \cos^2\theta - \sin^2\theta = 0 \implies (\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = 0 \\
&\implies \cos\theta = \pm\sin\theta \implies \begin{cases} \cos\theta = \sin\theta \implies \theta = \frac{\pi}{4}, \frac{5\pi}{4} \\ \cos\theta = -\sin\theta \implies \theta = \frac{3\pi}{4}, \frac{7\pi}{4} \end{cases} \\
\frac{dx}{d\theta} = 0 &\implies -5\sin\theta\cos\theta - 5\cos\theta\sin\theta = 0 \implies -10\sin\theta\cos\theta = 0 \\
&\implies \sin\theta\cos\theta = 0 \implies \begin{cases} \sin\theta = 0 \implies \theta = 0, \pi \\ \cos\theta = 0 \implies \theta = \frac{\pi}{2}, \frac{3\pi}{2} \end{cases}
\end{aligned}$$

The points on the curve where the tangent line is horizontal occur when $\frac{dy}{d\theta} = 0$ (and $\frac{dx}{d\theta} \neq 0$). Therefore the following points (r, θ) have horizontal tangents

$$\left(\frac{5}{\sqrt{2}}, \frac{\pi}{4}\right), \left(-\frac{5}{\sqrt{2}}, \frac{3\pi}{4}\right), \left(-\frac{5}{\sqrt{2}}, \frac{5\pi}{4}\right), \left(\frac{5}{\sqrt{2}}, \frac{7\pi}{4}\right),$$

which simplifies to $\left(\frac{5}{\sqrt{2}}, \frac{\pi}{4}\right), \left(\frac{5}{\sqrt{2}}, \frac{7\pi}{4}\right)$ when we remove identical points.

The points on the curve where the tangent line is vertical occur when $\frac{dx}{d\theta} = 0$ (and $\frac{dy}{d\theta} \neq 0$). Therefore the following points (r, θ) have vertical tangents

$$(5, 0), \left(0, \frac{\pi}{2}\right), (-5, \pi), \left(0, \frac{3\pi}{2}\right),$$

which simplifies to $(5, 0), \left(0, \frac{\pi}{2}\right)$ once redundant points are removed.

2. $r = 4\sin(2\theta)$; $\theta = \dfrac{\pi}{6}$

(a) Here $f(\theta) = 4\sin(2\theta)$, $f'(\theta) = 8\cos(2\theta)$ and so

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{8\cos(2\theta)\sin\theta + 4\sin(2\theta)\cos\theta}{8\cos(2\theta)\cos\theta - 4\sin(2\theta)\sin\theta},$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{8\cos\frac{\pi}{3}\sin\frac{\pi}{6} + 4\sin\frac{\pi}{3}\cos\frac{\pi}{6}}{8\cos\frac{\pi}{3}\cos\frac{\pi}{6} - 4\sin\frac{\pi}{3}\sin\frac{\pi}{6}} = \frac{8\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{8\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}$$
$$= \frac{2+3}{2\sqrt{3} - \sqrt{3}} = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}.$$

As an aside, if we desire the actual tangent line at $P(x_0, y_0)$ for $\theta = \frac{\pi}{6}$ we have

$$x_0 = r\cos\theta = 4\sin(2\theta)\cos\theta = 4\sin\frac{\pi}{3}\cos\frac{\pi}{6} = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \implies x = 3$$
$$y_0 = r\sin\theta = 4\sin(2\theta)\sin\theta = 4\sin\frac{\pi}{3}\sin\frac{\pi}{6} = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) \implies y = \sqrt{3}$$

Thus the tangent line is

$$y = m(x - x_0) + y_0 = \frac{5\sqrt{3}}{3}(x - 3) + \sqrt{3} = \frac{5\sqrt{3}}{3}x - 5\sqrt{3} + \sqrt{3}$$
$$\implies y = \frac{5\sqrt{3}}{3}x - 4\sqrt{3}.$$

(b) Here again the periodicity of $\sin(2\theta)$ restricts consideration of solutions of θ to $[0, 2\pi)$. We solve $dy/d\theta = 0$ and $dx/d\theta = 0$. From (a) we have

$$\begin{aligned}
\frac{dy}{d\theta} = 0 &\implies 8\cos(2\theta)\sin\theta + 4\sin(2\theta)\cos\theta = 0\\
&\implies 8(\cos^2\theta - \sin^2\theta)\sin\theta + 8\sin\theta\cos\theta\cos\theta = 0\\
&\implies \cos^2\theta\sin\theta - \sin^3\theta + \sin\theta\cos^2\theta = 0\\
&\implies 2\cos^2\theta\sin\theta - \sin^3\theta = 0\\
&\implies \sin\theta(2\cos^2\theta - \sin^2\theta) = 0\\
&\implies \sin\theta\left(2\cos^2\theta - (1 - \cos^2\theta)\right) = 0\\
&\implies \sin\theta(3\cos^2\theta - 1) = 0
\end{aligned}$$

Then $\sin\theta = 0 \implies \theta = 0, \pi$. For $3\cos^2\theta - 1 = 0 \implies \cos\theta = \pm\sqrt{1/3}$ the reference angle is, for both $\pm$ equations, $\theta' = \cos^{-1}(1/\sqrt{3}) \approx 0.9553$ radians. The $+$ equation gives Quadrant I and IV solutions while the $-$ equation gives Quadrant II and III solutions. The four solutions of $3\cos^2\theta - 1 = 0$ in $[0, 2\pi)$ are therefore, in radians,

$$\begin{aligned} \text{I}: \theta &= \theta' = 0.9553 \\ \text{II}: \theta &= \pi - \theta' = 2.1863 \\ \text{III}: \theta &= \pi + \theta' = 4.0969 \\ \text{IV}: \theta &= 2\pi - \theta' = 5.3279 \end{aligned}$$

Thus $\frac{dy}{d\theta}$ vanishes at θ equal to

$$0, 0.9553, 2.1863, \pi, 4.0969, 5.3279\,.$$

Again from (a) we have

$$\begin{aligned} \frac{dx}{d\theta} = 0 &\implies 8\cos(2\theta)\cos\theta - 4\sin(2\theta)\sin\theta = 0 \\ &\implies 8(\cos^2\theta - \sin^2\theta)\cos\theta - 8\sin^2\theta\cos\theta = 0 \\ &\implies \cos^3\theta - \sin^2\theta\cos\theta - \sin^2\theta\cos\theta = 0 \\ &\implies \cos^3\theta - 2\sin^2\theta\cos\theta = 0 \\ &\implies \cos\theta(\cos^2\theta - 2\sin^2\theta) = 0 \\ &\implies \cos\theta(1 - \sin^2\theta - 2\sin^2\theta) = 0 \\ &\implies \cos\theta(1 - 3\sin^2\theta) = 0 \end{aligned}$$

Then $\cos\theta = 0 \implies \theta = \frac{\pi}{2}, \frac{3\pi}{2}$. For $1 - 3\sin^2\theta = 0 \implies \sin\theta = \pm\sqrt{1/3}$ the reference angle is, for both $\pm$ equations, $\theta' = \sin^{-1}(1/\sqrt{3}) \approx 0.6155$ radians. Solving as above the four solutions of $1 - 3\sin^2\theta = 0$ in $[0, 2\pi)$ are therefore $0.6155, 2.5261, 3.7571, 5.6677$ and $\frac{dx}{d\theta}$ vanishes at θ equal to

$$0.6155, \frac{\pi}{2}, 2.5261, 3.7571, \frac{3\pi}{2}, 5.6677\,.$$

The tangent line is horizontal if $\frac{dy}{d\theta} = 0$ (and $\frac{dx}{d\theta} \neq 0$). Therefore the following points in polar form (r, θ) have a horizontal tangent line

$$(0, 0), (3.7712, 0.9553), (-3.7712, 2.1863), (0, \pi), (3.7712, 4.0969), (-3.7712, 5.3279)\,.$$

Removing the identical point at the origin we are left with

$$(0, 0), (3.7712, 0.9553), (-3.7712, 2.1863), (3.7712, 4.0969), (-3.7712, 5.3279)\,.$$

The tangent line is vertical if $\frac{dx}{d\theta} = 0$ (and $\frac{dy}{d\theta} \neq 0$). Therefore the following points (r, θ) have a vertical tangent line

$$(3.7712, 0.6155), \left(0, \frac{\pi}{2}\right), (-3.7712, 2.5261), (3.7712, 3.7571), \left(0, \frac{3\pi}{2}\right), (-3.7712, 5.6677)\,.$$

Removing the identical point at the origin we are left with

$$(3.7712, 0.6155), \left(0, \frac{\pi}{2}\right), (-3.7712, 2.5261), (3.7712, 3.7571), (-3.7712, 5.6677)\,.$$

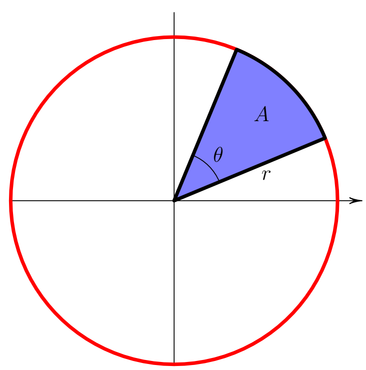
Further Questions:

Consider the curve $r = 1 + \cos\theta$.

1. Find the slope of the tangent line to the curve where $\theta = \pi/2$.
2. Find the points on the curve where the tangent line is horizontal or vertical.

7.3.3 Areas

Recall that a sector of a circle of radius r is a pie-shaped wedge between two radii subtended by an angle θ as shown:

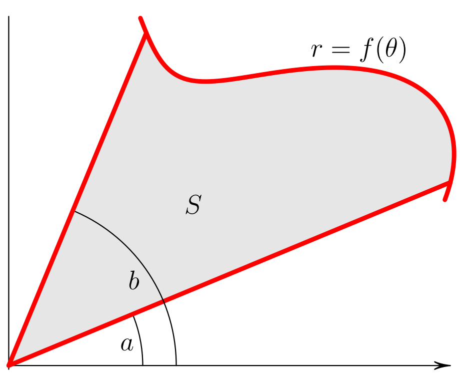


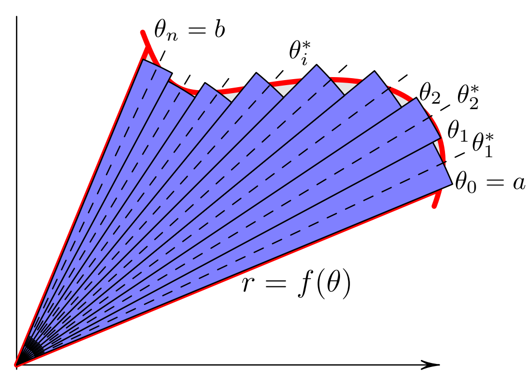
Since there are 2π radians in a circle the fraction of the area contained in the sector of angle θ must equal $\frac{\theta}{2\pi}$. As the area of the circle is πr^2 it follows that $\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$ and hence the area of the sector is

$$A = \frac{1}{2}r^2\theta\,.$$

Note θ must be in radians.

In general we may be interested in the area of a region S generated by a continuous polar function $r = f(\theta) \geq 0$ with $a < \theta < b$ and $b - a < 2\pi$ as shown below on the left.





We proceed as we did when we originally derived the area under a curve but here using the area of sectors rather than rectangles to approximate the area. First partition the θ interval $[a, b]$ into n

intervals of equal width $\Delta\theta = \frac{b-a}{n}$,

$$[\theta_0, \theta_1], [\theta_1, \theta_2], \ldots, [\theta_{i-1}, \theta_i], \ldots [\theta_{n-1}, \theta_n],$$

where $\theta_0 = a$, $\theta_n = b$, and, in general, $\theta_i = \theta_0 + i\Delta\theta$. Within the i^{th} interval $[\theta_{i-1}, \theta_i]$ select a value θ_i^* at which to evaluate the radius of the sector, $r_i = f(\theta_i^*)$. (In the diagram the midpoint of the interval has been chosen.) The i^{th} sector is subtended by angle $\Delta\theta$ and its area is therefore

$$A_i = \frac{1}{2} r_i^2 \Delta\theta\,.$$

The area A of S will be approximated by the sum of the sector areas,

$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2 \Delta\theta\,.$$

Recognizing this as a Riemann sum of the function $\frac{1}{2}[f(\theta)]^2$, let the number of intervals n approach infinity to get the exact result

$$A = \lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{1}{2}[f(\theta_i)]^2 \Delta\theta\,.$$

The area of region S is therefore the definite integral

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta\,.$$

Since by assumption f is continuous on $[a, b]$, so is $\frac{1}{2}[f(\theta)]^2$, and the latter is therefore integrable and the area is finite. The formula is easy to remember since $A = \int dA$ where the differential area dA at angle θ is, analogous to the area of the sector,

$$dA = \frac{1}{2} r^2 d\theta = \frac{1}{2}[f(\theta)]^2 d\theta\,.$$

The above results can be generalized. With regard to $r = f(\theta)$, it is sufficient for the function f to be piecewise continous on $[a, b]$. If the function is discontinuous at an endpoint of the interval or has, say, an infinite discontinuity at a point c within the interval, one can generate an improper integral of the second kind to represent the area. The latter may or may not converge.

Example 7-8

Find the area of the region bounded by one loop of the graph of the given functions.

1. $r = 2\cos(3\theta)$
2. $r^2 = 16\cos(4\theta)$

Solution:

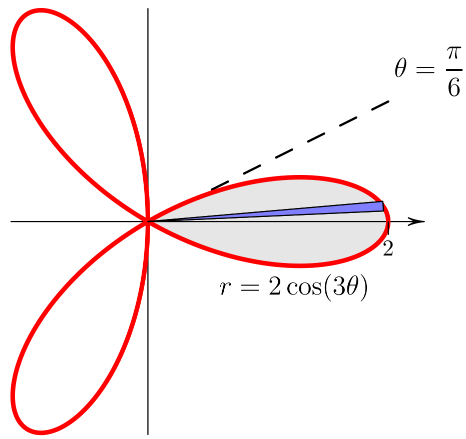
1. $r = 2\cos(3\theta)$
   The graph will start and end a loop when it goes through the origin ($r = 0$). Solving for θ gives

$$r = 0 \implies 2\cos(3\theta) = 0$$
$$\implies \cos 3\theta = 0 \implies 3\theta = \begin{cases} \frac{\pi}{2} \pm 2n\pi \\ \frac{3\pi}{2} \pm 2n\pi \end{cases} \implies \theta = \begin{cases} \frac{\pi}{6} \pm \frac{2n\pi}{3} \\ \frac{\pi}{2} \pm \frac{2n\pi}{3} \end{cases} \quad (n \text{ an integer})$$

Since $\cos(3\theta)$ is periodic solutions of interest are those in $[0, 2\pi)$:

$$\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}.$$

A graph of the polar function is



The area of the shaded loop is double the area from $\theta = 0$ to $\theta = \frac{\pi}{6}$ and therefore

$$\begin{aligned} A &= 2\int_0^{\frac{\pi}{6}} \frac{1}{2}\left[2\cos(3\theta)\right]^2 d\theta = \int_0^{\frac{\pi}{6}} 4\cos^2(3\theta)\,d\theta = 4\int_0^{\frac{\pi}{6}} \frac{1}{2}\left[1+\cos(6\theta)\right] d\theta \\ &= 2\left[\theta + \frac{1}{6}\sin(6\theta)\right]\Bigg|_0^{\frac{\pi}{6}} = 2\left\{\left[\frac{\pi}{6}+0\right] - [0+0]\right\} = \frac{\pi}{3} \text{ square units.} \end{aligned}$$

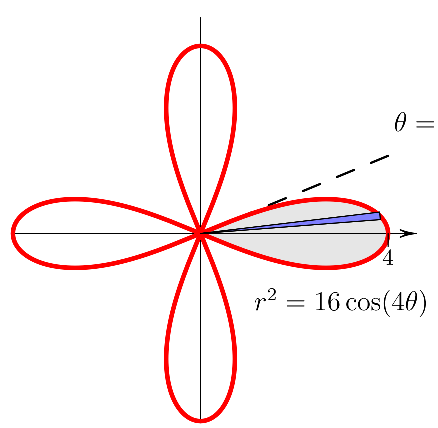
2. $r^2 = 16\cos(4\theta)$

$$r = 0 \implies 16\cos(4\theta) = 0 \implies 4\theta = \begin{cases} \frac{\pi}{2} \pm 2n\pi \\ \frac{3\pi}{2} \pm 2n\pi \end{cases} \implies \theta = \begin{cases} \frac{\pi}{8} \pm \frac{n\pi}{2} \\ \frac{3\pi}{8} \pm \frac{n\pi}{2} \end{cases} \quad (n \text{ an integer})$$

Solutions for θ in $[0, 2\pi)$ are

$$\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}.$$

A graph of the polar function is



$$A = 2\int_0^{\frac{\pi}{8}} \left(\frac{1}{2}\right) 16\cos(4\theta)\, d\theta = 16\int_0^{\frac{\pi}{8}} \cos(4\theta)\, d\theta = \ 4\sin(4\theta)|_0^{\frac{\pi}{8}} = 4\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right]$$
$$= 4 \text{ square units.}$$

Further Question:

Find the area of the curve $r = 1 + e^{-\theta}$ between $\theta = 0$ and $\theta = 2$.

Similar to our study of the area between curves $y = f(x)$ and $y = g(x)$, one may be interested in the area between an outer polar curve $r_o = f(\theta)$ and an inner one $r_i = g(\theta)$ over angular interval $[a, b]$. Then $f(\theta) \geq g(\theta) \geq 0$ on $[a, b]$ and the area between the polar curves is just the difference of the areas,

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta - \int_a^b \frac{1}{2}[g(\theta)]^2 d\theta = \int_a^b \frac{1}{2}\left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta\,.$$

One may consider this formula as $A = \int dA$ where dA is the differential area at θ between the two sectors given by

$$dA = \frac{1}{2}(r_o^2 - r_i^2) d\theta = \frac{1}{2}\left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta\,.$$

As with the Cartesian equivalent of this problem, the endpoints a and b may need to be determined by finding the values of θ where the curves intersect by solving $r_0 = f(\theta) = g(\theta) = r_i$.

Example 7-9

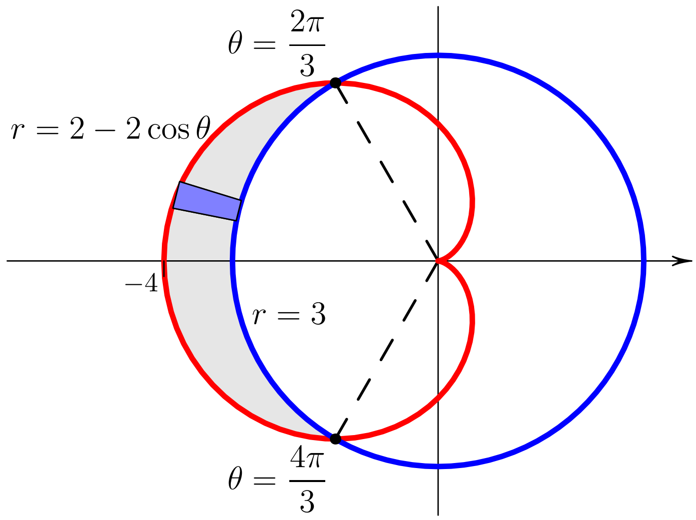
1. Find the area of the region that lies inside the cardioid $r = 2 - 2\cos\theta$ and outside the circle $r = 3$.
2. Find the area of the region that is outside of $r = 2$ and inside the graph of $r^2 = 8\sin(2\theta)$.

Solution:

1. The polar curves $r = 2 - 2\cos\theta$ and $r = 3$ intersect when they have the same radius r at the same angle θ, we therefore equate the functions and solve for θ.

$$2 - 2\cos\theta = 3 \implies -2\cos\theta = 1 \implies \cos\theta = -\frac{1}{2} \implies \text{ref angle } \theta' = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$
$$\text{cosine negative} \implies \theta = \begin{cases} \pi - \frac{\pi}{3} + 2n\pi \\ \pi + \frac{\pi}{3} + 2n\pi \end{cases} \implies \theta = \begin{cases} \frac{2\pi}{3} + 2n\pi \\ \frac{4\pi}{3} + 2n\pi \end{cases} \quad (n \text{ an integer})$$

Both functions of θ have period 2π and the curves are generated entirely by θ in $[0, 2\pi)$. Intersection points in this interval are therefore $(r, \theta) = \left(3, \frac{2\pi}{3}\right), \left(3, \frac{4\pi}{3}\right)$. Here we evaluated $r = 3$ for both angles using each equation. At $\theta = \pi$ between these values we find $r = 4$ for the first curve and $r = 3$ for the second. This confirms that the outer curve $r_o = f(\theta) = 2 - 2\cos\theta$ and the inner curve is $r_i = g(\theta) = 3$. A graph of the polar functions follows.



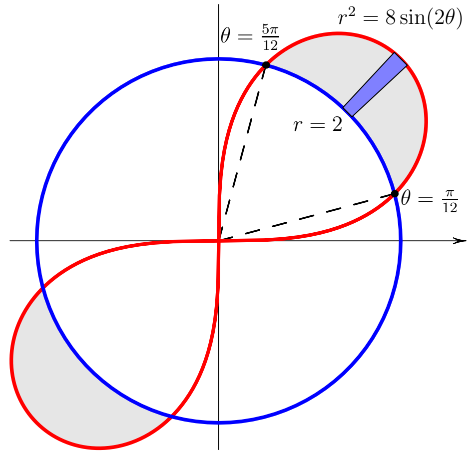
The required area is given by

$$\begin{aligned}
A &= \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \left[(2-2\cos\theta)^2 - 3^2\right]\,d\theta = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \left[4 - 8\cos\theta + 4\cos^2\theta - 9\right]\,d\theta \\
&= \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \left[-5 - 8\cos\theta + 2(1+\cos(2\theta))\right]\,d\theta = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \left[-3 - 8\cos\theta + 2\cos(2\theta)\right]\,d\theta \\
&= \frac{1}{2}\left[-3\theta - 8\sin\theta + \sin(2\theta)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\right. \\
&= \frac{1}{2}\left(\left[-4\pi - 8\sin\frac{4\pi}{3} + \sin\frac{8\pi}{3}\right] - \left[-2\pi - 8\sin\frac{2\pi}{3} + \sin\frac{4\pi}{3}\right]\right) \\
&= \frac{1}{2}\left[-2\pi - 9\sin\frac{4\pi}{3} + \sin\frac{8\pi}{3} + 8\sin\frac{2\pi}{3}\right] \\
&= \frac{1}{2}\left[-2\pi - 9\left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} + 8\frac{\sqrt{3}}{2}\right] \\
&= \frac{1}{2}\left[-2\pi + \frac{9\sqrt{3}}{2} + \frac{9\sqrt{3}}{2}\right] = \frac{9\sqrt{3}}{2} - \pi \text{ square units.}
\end{aligned}$$

2. The two curves $r = 2$ and $r^2 = 8\sin(2\theta)$ intersect when

$$8\sin 2\theta = 2^2 \implies \sin(2\theta) = \frac{1}{2} \implies 2\theta = \begin{cases} 0 + \frac{\pi}{6} + 2n\pi \\ \pi - \frac{\pi}{6} + 2n\pi \end{cases} \implies \theta = \begin{cases} \frac{\pi}{12} + n\pi \\ \frac{5\pi}{12} + n\pi \end{cases} \quad (n \text{ integer})$$

Periodicity of the curves ensures they are generated entirely by θ in $[0, 2\pi)$. Intersection points in this interval are $(r,\theta) = \left(2, \frac{\pi}{12}\right), \left(2, \frac{5\pi}{12}\right), \left(2, \frac{13\pi}{12}\right)$, and $\left(2, \frac{17\pi}{12}\right)$. At $\theta = \pi/4$ which lies within the first interval $\left[\frac{\pi}{12}, \frac{5\pi}{12}\right]$ the first curve evaluates to $r = 2$ and the second to $r = 2\sqrt{2}$. Thus the inner curve is $r_i = 2$ and the outer curve is $r_o^2 = 8\sin(2\theta)$. The same holds over the interval $\left[\frac{13\pi}{12}, \frac{17\pi}{12}\right]$. A graph of the polar functions confirms this.



The required area (shaded) is twice that in the first quadrant.

$$A = 2\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{1}{2}(r_o^2 - r_i^2)\, d\theta = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} [8\sin(2\theta) - 4]\, d\theta = \Big[-4\cos(2\theta) - 4\theta\Big|_{\frac{\pi}{12}}^{\frac{5\pi}{12}}$$
$$= \left[-4\cos\frac{5\pi}{6} - 4\left(\frac{5\pi}{12}\right)\right] - \left[-4\cos\frac{\pi}{6} - 4\left(\frac{\pi}{12}\right)\right]$$
$$= -4\left(-\frac{\sqrt{3}}{2}\right) - \frac{5\pi}{3} + 4\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{3} = 4\sqrt{3} - \frac{4\pi}{3} \text{ square units.}$$

Further Question:

Find the area in the first quadrant between the polar axis and the curves $r = \cos\theta$ and $r = \sin\theta$.

7.3.4 Arc Length

A curve represented in polar coordinates coordinates by the function $r = f(\theta)$ can be written in Cartesian coordinates, since $(x, y) = (r\cos\theta, r\sin\theta)$, as the *parametric curve*

$$x(\theta) = f(\theta)\cos\theta$$
$$y(\theta) = f(\theta)\sin\theta$$

where here θ is the parameter of the curve. If we are interested in the arc length of the curve over the interval $[a, b]$ of θ we can immediately use our result from Section 6.2.3, with $t \to \theta$ to get

$$s = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta\,.$$

Evaluating the terms in the integrand gives

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= (f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2 \\ &= [f'(\theta)]^2\cos^2\theta + [f(\theta)]^2\sin^2\theta - 2f'(\theta)f(\theta)\cos\theta\sin\theta \\ &\quad + [f'(\theta)]^2\sin^2\theta + [f(\theta)]^2\cos^2\theta + 2f'(\theta)f(\theta)\sin\theta\cos\theta \\ &= [f(\theta)]^2(\sin^2\theta + \cos^2\theta) + [f'(\theta)]^2(\cos^2\theta + \sin^2\theta) \\ &= [f(\theta)]^2 + [f'(\theta)]^2\,.\end{aligned}$$

The final result for the arc length of the polar curve $r = f(\theta)$ is therefore

$$s = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta\,.$$

One way to remember the formula is to recall $s = \int ds$, equate the integrands, and formally rearrange the differentials to get

$$ds = \sqrt{(rd\theta)^2 + (dr)^2}\,.$$

The interpretation is that as one proceeds along the curve a change of coordinates by an infinitessimal amount $d\theta$ in angle and dr in radius induces a change of $rd\theta$ in a circular direction and a (perpendicular) change of position by dr radially. The overall change in distance ds is then found by the Pythagorean formula.

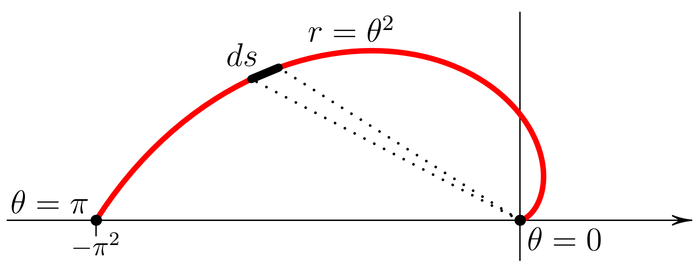
Example 7-10

Find the length of the given polar curves.

1. $r = \theta^2;\ 0 \le \theta \le \pi$
2. $r = \cos^2\left(\frac{\theta}{2}\right);\ 0 \le \theta \le \pi$

Solution:

1. A graph of the polar function $r = f(\theta) = \theta^2$ on $0 \le \theta \le \pi$ follows.



The derivative $\frac{dr}{d\theta} = f'(\theta) = 2\theta$ and so the arc length is

$$s = \int_0^\pi \sqrt{\theta^4 + 4\theta^2}\, d\theta = \int_0^\pi \sqrt{\theta^2}\sqrt{\theta^2 + 4}\, d\theta = \int_0^\pi |\theta|\sqrt{\theta^2 + 4}\, d\theta = \int_0^\pi \theta\sqrt{\theta^2 + 4}\, d\theta\,,$$

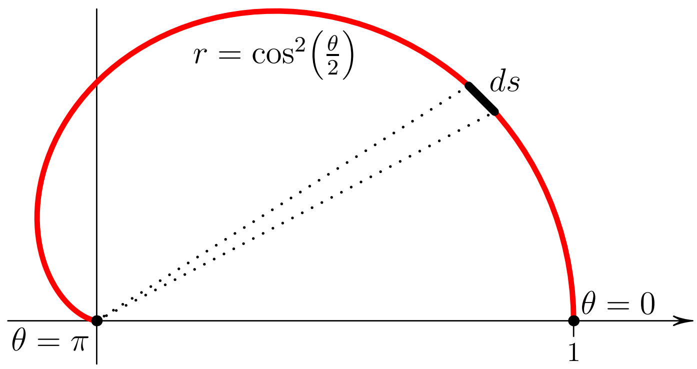
where here we used that $\theta > 0$ on $[0, \pi]$ to evaluate the absolute value. Let $u = \theta^2 + 4$, so $du = 2\theta\, d\theta \implies \frac{1}{2}du = \theta\, d\theta$. Under the change of variable the limits become

$$\theta = \pi \implies u = \pi^2 + 4$$
$$\theta = 0 \implies u = 4\,,$$

and we have

$$s = \int_4^{\pi^2+4} \sqrt{u}\left(\frac{1}{2}\,du\right) = \left(\frac{1}{2}\right)\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_4^{\pi^2+4} = \frac{1}{3}u^{\frac{3}{2}}\Big|_4^{\pi^2+4} = \frac{1}{3}\left[(\pi^2+4)^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]$$
$$= \frac{1}{3}\left[(\pi^2+4)^{\frac{3}{2}} - 8\right] \text{ units.}$$

2. A graph of the polar function $r = f(\theta) = \cos^2\left(\dfrac{\theta}{2}\right)$ on $0 \le \theta \le \pi$ is as follows.



Differentiating gives $\dfrac{dr}{d\theta} = f'(\theta) = 2\cos\left(\dfrac{\theta}{2}\right)\left[-\sin\left(\dfrac{\theta}{2}\right)\right]\left(\dfrac{1}{2}\right) = -\cos\left(\dfrac{\theta}{2}\right)\sin\left(\dfrac{\theta}{2}\right)$.

The arc length is then

$$\begin{aligned} s &= \int_0^{\pi} \sqrt{\cos^4\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\theta}{2}\right)}\, d\theta = \int_0^{\pi} \sqrt{\cos^2\left(\frac{\theta}{2}\right)}\sqrt{\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)}\, d\theta \\ &= \int_0^{\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| \sqrt{1}\, d\theta = \int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta \quad (\Leftarrow \text{since } \cos\left(\frac{\theta}{2}\right) > 0 \text{ on } [0,\pi]\,) \\ &= 2\sin\left(\frac{\theta}{2}\right)\Big|_0^{\pi} = 2\sin\frac{\pi}{2} - 2\sin 0 = 2 - 0 = 2 \text{ units.} \end{aligned}$$

Further Question:

Find the arc length of the polar curve given by $r = \tan\theta\,;\ 0 \le \theta \le \dfrac{\pi}{4}$.

Answers:
Page 274

Exercise 7-2

1-4: Find a Cartesian equation for the given curve and identify it.

1. $r = 10$
2. $r = 5\cos\theta$
3. $r^2\left(3\sin^2\theta - 7\cos^2\theta\right) = 42$
4. $r^2\cos 2\theta = 10$

5-7: Find a polar equation that has the same graph as the given Cartesian equation.

5. $x^2 + y^2 = 64$
6. $xy = 5$
7. $9x^2 + 16y^2 = 36$

8-10: Find the slope of the tangent line to the given curve at the indicated value of θ.

8. $r = 25\sin\theta; \quad \theta = \frac{\pi}{3}$
9. $r = 3\sin\theta + 5; \quad \theta = \frac{\pi}{4}$
10. $r = 1 - \sin\theta + 2\cos\theta; \quad \theta = 0$

11-12: Find the points on the curve at which the tangent line is horizontal or vertical.

11. $r = 100\cos\theta$
12. $r = 5\sin\theta - 5$

13-14: Sketch the graph of the equation and find the area of the region bounded by the graph.

13. $r = 5\cos\theta$
14. $r = 4 + 4\sin\theta$

15-16: Find the area of the region bounded by one loop of the graph of the give equation.

15. $r = 3\sin 2\theta$
16. $r^2 = 16\sin(4\theta)$

17-18: Find the area of the region that lies inside both curves.

17. $r = 4 + 4\sin\theta, \ \ r = 4 - 4\sin\theta$
18. $r = 4 + 3\sin\theta, \ \ r = 4 + 3\cos\theta$

19-20: Find the length of the given polar curve.

19. $r = e^{2\theta}; \ 0 \le \theta \le 3$
20. $r = 3 + 3\cos\theta$

---

21. We derived a formula for the arc length of a polar curve described by the function $r = f(\theta)$. Here θ is the independent variable and and r is the dependent variable.
    (a) Derive a formula for the arc length of a polar curve described by the function $\theta = g(r)$ for $c \le r \le d$. Now r is the independent variable and the angle θ is the dependent variable.
    (b) Use your formula from (a) to find the length of the curve $\theta = \ln(r); \ 1 \le r \le 4$.

7.4 Other Applications of Polar Coordinates

A point in two dimensions can be represented by Cartesian coordinates (x, y) or polar coordinates (r, θ). Physically, for example, we can locate the point P on a drum membrane with two such coordinates. Often in physical applications we are interested in the value of some quantity at such a point. At a point on a drum we could be interested in the vertical displacement z of the membrane at that point. A function $z = f(x, y)$ or $z = f(r, \theta)$ could be introduced to represent the shape of the drum surface at a particular instant in time. The motion of the drum in time would require that z further depend on time in addition to space, so $z = f(x, y, t)$ or $z = f(r, \theta, t)$. It will turn out that choosing a spatial coordinate system which reflects the symmetry of the physical system makes calculating such motion easier. In the case of a drum, most drums have a circular boundary and hence polar coordinates are preferred. If we made square drums, then Cartesian coordinates would work better. We can introduce more coordinate systems to reflect other physical situations. A drum with a boundary that was an ellipse might be analyzed easiest using an elliptic coordinate system.

In general one is able to introduce many coordinate systems that allow one to identify a location with a tuple of numbers. In three dimensions a point can be located with Cartesian coordinates (x, y, z). Polar coordinates may be generalized in three dimensions to spherical-polar coordinates (r, θ, ϕ) where a further angle is needed to identify the point. The temperature at every point inside the earth at a particular instant of time could be represented as a function of Cartesian coordinates as $T = f(x, y, z)$. However since our planet is, in rough approximation, a sphere, a function of spherical coordinates, $T = f(r, \theta, \phi)$ would likely be preferred. One might expect, for instance, that the temperature inside the earth would depend essentially only on its radial distance r from the centre. A geologist studying the evolution of the earth's temperature over geological time scales would require a time coordinate as well, $T = f(r, \theta, \phi, t)$.

The above applications involving functions at points in space and time are the domain of multivariate calculus. Spatial coordinate systems (Cartesian, polar, etc.) are fundamental to those problems. A coordinate system chosen to reflect the symmetry of the physical problem in question makes solving it much easier.

Answers: Page 275

Chapter 7 Review Exercises

1. Find Cartesian coordinates of the point $(r,\theta) = \left(7, \dfrac{5\pi}{3}\right)$.

2. Find polar coordinates of the point $(x,y) = (0,\pi)$.

3-4: Find a Cartesian equation for the given curve and identify it.

3. $r = \cos\theta - 6\sin\theta$
4. $r^2\left(\cos^2\theta - 3\sin^2\theta\right) = 16$

5-6: Find a polar equation that has the same graph as the given Cartesian equation.

5. $xy + y^2 = 8$
6. $(x-1)^2 + y^2 = 4$

7-8: Find the slope of the tangent line to the given curve at the indicated value of θ.

7. $r = 3\cos^2\theta;\quad \theta = \frac{\pi}{4}$
8. $r = 3\cos\theta + \sin\theta - 2;\quad \theta = \frac{\pi}{2}$

9-10: Find the points on the curve at which the tangent line is horizontal or vertical.

9. $r = e^\theta$
10. $r = 1 + \sin\theta$

11-12: Sketch the graph of the equation and find the area of the region bounded by the graph.

11. $r = \sin\theta - 2\cos\theta$
12. $r = 3\cos\theta - 4$

13-14: Find the area of the region that lies inside both curves.

13. $r = 4\sin\theta,\ \ r = 4\cos\theta$
14. $r = 3\sin\theta + 4,\ \ r = 3\cos\theta + 4$

15-16: Find the length of the given polar curve.

15. $r = \sin^2\left(\dfrac{\theta}{2}\right)$
16. $r = \theta^4;\ \ 0 \le \theta \le 2\pi$

Answers

Chapter 1 Review Exercises (page 7)

1. $\frac{1}{4}x^4 + 2\sqrt{x} - \frac{5}{2x^2} + C$
2. $\frac{2}{7}x^{7/2} + x + 2x^{1/2} + C$
3. $2\sec\sqrt{x} + C$
4. $\frac{1}{12}$
5. $\frac{15}{4}$
6. $-\frac{1}{3}\cos\left(x^3+2\right) + C$
7. $\frac{1}{3}(2x+5)^{3/2} + C$
8. $\frac{2101}{5}$
9. $\frac{1}{3}\frac{1}{2+\cos 3x} + C$
10. $\frac{9}{2}$
11. $-\frac{1}{(1+\sqrt{x})^2} + \frac{2}{3(1+\sqrt{x})^3} + C$
12. $\frac{1}{7}(x+4)^7 - \frac{2}{3}(x+4)^6 + C$

Exercise 2-1 (page 18)

1. $f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}}$
2. $f^{-1}(x) = \frac{x}{3x-2}$
3. $f^{-1}(x) = \frac{1}{2}\left(x^2+3\right)$
4. $f^{-1}(x) = \sqrt[3]{x^{1/7}-2}$
5. (b) $f^{-1}(x) = \sqrt[5]{x-4}$, Domain=$(-\infty,\infty)$, Range=$(-\infty,\infty)$
6. (b) $f^{-1}(x) = \frac{3-x}{2x-1}$, Domain=$\mathbb{R} - \left\{\frac{1}{2}\right\}$, Range=$\mathbb{R} - \left\{-\frac{1}{2}\right\}$
7. (b) $f^{-1}(x) = \left(\frac{1-2x}{x-1}\right)^2$, Domain=$\left[\frac{1}{2},1\right)$, Range=$[0,\infty)$
   Note that for the domain when solving $x = f(y)$ for y you get $\sqrt{x} = \frac{1-2y}{y-1}$. Since $\sqrt{x} \geq 0$ this implies $\frac{1-2y}{y-1} \geq 0$. Exchanging variables means $\frac{1-2x}{x-1} \geq 0$ which restricts the domain of f^{-1}.
8. $\left(f^{-1}\right)'(7) = \frac{1}{6}$
9. $\left(f^{-1}\right)'(2) = \frac{1}{3}$
10. $\left(f^{-1}\right)'(11) = \frac{12}{13}$
11. (a) $h(x_1) = h(x_2) \implies f(g(x_1)) = f(g(x_2)) \implies f^{-1}[f(g(x_1))] = f^{-1}[f(g(x_2))] \implies g(x_1) = g(x_2) \implies x_1 = x_2$
    (b) $h^{-1}(x) = g^{-1}\left(f^{-1}(x)\right)$ so $h^{-1} = g^{-1} \circ f^{-1}$

Exercise 2-2 (page 25)

1. ∞
2. 0
3. 1
4. $f'(x) = \left(4x^3 - 2x^4 - 2\right) e^{-2x}$
5. $g'(t) = -6e^{-3t} - \dfrac{2}{t^3}$
6. $y' = \dfrac{2e^x}{\left(e^x + 3\right)^2}$
7. $f'(x) = -\left(2e^{2x} + 1\right) \sin\left(e^{2x} + x\right)$
8. $g'(x) = e^x \sec^2(e^x) + \sec^2 x\, e^{\tan x}$
9. $y' = \left(x + e^{2x}\right)\left(x^2 + e^{2x}\right)^{-1/2} \cos\sqrt{x^2 + e^{2x}}$
10. $f'(x) = \dfrac{4}{\left(e^x + 3e^{-x}\right)^2}$
11. $y = 2x - 1$
12. $y = -\dfrac{e}{e-1}x + \dfrac{2e-1}{e-1}$
13. $\frac{1}{4}e^{4x} + 2e^x - \frac{1}{2}e^{-2x} + C$
14. $\frac{1}{2}e^{x^2} + C$
15. $-\cos\left(e^x\right) + C$
16. $\dfrac{1}{3}e^{3x} + 2e^x - e^{-x} + C$
17. $\dfrac{1}{9} - \dfrac{1}{3\left(e^3 + 2\right)}$
18. $2e^2 - 2e$
19. $\tan x + e^{\tan x} + C$
20. $-\dfrac{1}{2\left(e^{2x} + 1\right)} + C$

Exercise 2-3 (page 38)

1. $4\left[\log_5(x+1) - \log_5(2x+3)\right]$
2. $2\ln(x+1) + \dfrac{1}{2}\ln(x+4) - \dfrac{1}{3}\ln(x+2)$
3. $2x + 3\ln(2x+1) - \dfrac{1}{2}\ln(e^x + 1)$
4. $\ln \dfrac{2x}{(e^x+2)^3\sqrt{x+4}}$

5. $\log_{10} \dfrac{\left(x^2+1\right)^3 \sqrt[3]{x+4}}{\sqrt[4]{3x^2+5}}$

6. $\ln \dfrac{x^3\left(x^3+2\right)^{2/\ln 3}}{\sqrt{3x+1}}$

7. $x = \pm\sqrt{e^4-3}$

8. $x = \ln 3,\ x = \ln 2$

9. $x = 1$

10. $x = \ln 4$

11. $x = \dfrac{\ln 5}{1+\ln 2}$

12. (a) Domain=$(-\infty,\infty)$, Range=$\left(\sqrt{2},\infty\right)$
    (b) $f^{-1}(x) = \ln\dfrac{x^2-2}{3}$, Domain=$\left(\sqrt{2},\infty\right)$

13. (a) Domain=$\left(-\frac{2}{3},\infty\right)$, Range=$(-\infty,\infty)$
    (b) $f^{-1}(x) = \dfrac{e^x-2}{3}$, Domain=$(-\infty,\infty)$

14. (a) Domain=$(-\infty,\infty)$, Range=$\left(-\frac{2}{3},1\right)$
    (b) $f^{-1}(x) = \ln\left(\dfrac{3x+2}{1-x}\right)$, Domain=$\left(-\frac{2}{3},1\right)$

15. (a) Domain=$\mathbb{R}-\{e^{-2}\}$, Range=$\mathbb{R}-\{1\}$
    (b) $f^{-1}(x) = e^{\frac{1-2x}{x-1}}$, Domain=$\mathbb{R}-\{1\}$

16. $f'(x) = \dfrac{2x+e^x}{x^2+e^x}$

17. $y' = \dfrac{\cos x+3}{(\ln 4)(\sin x+3x)}$

18. $g'(t) = 10^{t+2}(\ln 10)\ln(\ln t+5) + \dfrac{10^{t+2}}{t(\ln t+5)}$

19. $f'(x) = \dfrac{2}{x} + \dfrac{x}{x^2+3} + \dfrac{3e^{3x}}{e^{3x}+1}$

20. $y' = \dfrac{e^{2x}\sqrt{x^2+5}}{\sqrt[3]{x+1}}\left[2 + \dfrac{x}{x^2+5} - \dfrac{1}{3(x+1)}\right]$

21. $f'(x) = \left(x^2+e^x\right)^{\ln x}\left[\dfrac{\ln\left(x^2+e^x\right)}{x} + \dfrac{(2x+e^x)\ln x}{x^2+e^x}\right]$

22. $y' = x^{\sin x}\left[\cos x\ln x + \dfrac{\sin x}{x}\right]$

23. $-\sqrt{e^{-2x}+3}+C$

24. $\ln\left|e^x+5\right|+C$

25. $\ln \dfrac{11}{9}$

26. $-\dfrac{1}{(2+\ln x)^2} + C$

27. $\dfrac{5^{x^2+x}}{\ln 5} + C$

28. $\dfrac{180}{\ln 10}$

29. $-\dfrac{1}{2}\ln|3+\cos 2x| + C$

30. $-\dfrac{1}{(\ln x)^3} + C$

31. $\dfrac{(\ln x)^2}{2\ln 4} + C$

32. (a) $a^{\ln x} = \left[e^{\ln a}\right]^{\ln x} = e^{(\ln a)(\ln x)} = \left[e^{\ln x}\right]^{\ln a} = x^{\ln a}$

    (b) $\displaystyle\int 5^{\ln x}\,dx = \int x^{\ln 5}\,dx = \frac{1}{1+\ln 5}x^{1+\ln 5} + C$ (Integration Power Rule with $n = \ln 5$)

    (c) $u = \ln x \implies du = \frac{dx}{x}$. But solving gives $x = e^u$ so $dx = e^u\,du$.

    $$\int 5^{\ln x}\,dx = \int 5^u e^u\,du = \int [e^{\ln 5}]^u e^u\,du = \int e^{u\ln 5}e^u\,du = \int e^{(1+\ln 5)u}du$$

    $$= \int e^w \frac{dw}{1+\ln 5} = \frac{e^w}{1+\ln 5} + C = \frac{e^{(1+\ln 5)u}}{1+\ln 5} + C = \frac{e^{(1+\ln 5)\ln x}}{1+\ln 5} + C = \frac{\left(e^{\ln x}\right)^{1+\ln 5}}{1+\ln 5} + C$$

    $$= \frac{1}{1+\ln 5}x^{1+\ln 5} + C$$

    (d) Since $5^{\ln x} = x^{\ln 5}$ it is a power function because the base is the variable x and the constant $n = \ln 5$ is in the exponent.

Exercise 2-4 (page 44)

1. (a) $k = \dfrac{\ln 2}{5}$

   (b) $P(20) = 3200$ bacteria

2. $P(2) = 225$ deer

3. (a) $k = 0.02469$

   (b) $P(9) = 73465$

4. (a) $k = -\dfrac{\ln 2}{9.45} = -0.07334890$

   (b) $t = 18.9$ minutes

5. $k = -0.0743811, m(24) = 67.10$ mg

6. The half-life is 2.31 days.

Exercise 2-5 (page 53)

1. $\dfrac{\pi}{4}$
2. $\dfrac{\pi}{4}$
3. $\dfrac{\sqrt{3}}{2}$
4. $\dfrac{\sqrt{1-x^2}}{x}$
5. $f'(x) = \dfrac{e^x}{\sqrt{1-(e^x+2)^2}}$
6. $y' = -\dfrac{1}{x\sqrt{1-(\ln x+5)^2}}$
7. $g'(x) = \dfrac{\cos x}{1+\sin^2 x}$
8. $y' = \left(\sin^{-1} x\right)^{\ln x}\left[\dfrac{\ln\left(\sin^{-1} x\right)}{x} + \dfrac{\ln x}{\sin^{-1} x\ \sqrt{1-x^2}}\right]$
9. $y' = -\dfrac{y\left(1+y^2\right)}{1+(x+e^y)\left(1+y^2\right)}$
10. $y' = \dfrac{2x - \frac{1}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-y^2}} - 2y}$
11. $f'(x) = \dfrac{1}{x\sqrt{4x^2-1}}$
12. $\dfrac{1}{\sqrt{6}}\tan^{-1}\left(\sqrt{\dfrac{2}{3}}e^x\right) + C$
13. $\dfrac{2}{\sqrt{5}}\tan^{-1}\left(\dfrac{\ln x}{\sqrt{5}}\right) + C$
14. $\dfrac{3}{2}\sin^{-1}\left(x^2\right) + C$
15. $\ln\left(x^2+5\right) + \dfrac{3}{\sqrt{5}}\tan^{-1}\left(\dfrac{x}{\sqrt{5}}\right) + C$
16. $\dfrac{1}{2}\left(\sin^{-1} x\right)^2 + C$
17. $\sin^{-1}(\tan x) + C$
18. $\sqrt{2}\tan^{-1}\left(\dfrac{\sqrt{x}}{\sqrt{2}}\right) + C$
19. $\sec^{-1}(e^x) + C$
20. $\dfrac{\pi}{6}$

Exercise 2-6 (page 60)

1. $\frac{7}{5}$
2. $\frac{3}{5}$
3. 0
4. $\frac{1}{5}$
5. 0
6. 3
7. 0
8. 0
9. $-\infty$
10. -1
11. ∞
12. $\ln\frac{1}{2}$
13. e^{-10}
14. 1
15. 1
16. 1

Chapter 2 Review Exercises (page 61)

1. (a) $f(x_1)=f(x_2) \implies 3+\frac{1}{x_1^3}=3+\frac{1}{x_2^3} \implies \frac{1}{x_1^3}=\frac{1}{x_2^3} \implies x_1^3=x_2^3 \implies x_1=x_2$

   (b) $g(x)=f^{-1}(x)=\sqrt[3]{\frac{1}{x-3}}$

   (c) $g'(11)=\frac{1}{f'(g(11))}=\frac{1}{f'(1/2)}=\frac{1}{-3(1/2)^{-4}}=-\frac{1}{48}$

2. $f'(x)=\frac{2\left(e^{3x}+4\right)-3xe^{3x}(2\ln x+5)}{x\left(e^{3x}+4\right)^2}$

3. $f'(t)=\frac{4t^3+2e^{2t}}{t^4+e^{2t}+1}$

4. $g'(x)=\frac{1}{3}\left[\frac{1}{x+1}-\frac{2}{2x+4}\right]$

5. $F'(0)=2$

6. $f''(x)=2e^{-x^2}-10x^2e^{-x^2}+4x^4e^{-x^2}$

7. $y'=(2x+3)^{4x}\left[4\ln(2x+3)+\frac{8x}{2x+3}\right]$

8. $f'(t)=(\ln 10)(e^t)10^{e^t}$

9. $y'=(\ln x)^{\cos x}\left[-\sin x\ln(\ln x)+\frac{\cos x}{x\ln x}\right]$

10. $y'=\frac{xy^2e^{xy}-y}{x-x^2ye^{xy}}$

11. $f'(x)=\frac{xe^x+1}{x+x\left(e^x+\ln x\right)^2}$

12. $g'(t)=\frac{\sec^2 t}{\sqrt{1-(\tan t+3)^2}}$

13. $h'(x)=-\frac{1}{\ln 10}\frac{1}{\left(\cos^{-1}x+1\right)\sqrt{1-x^2}}$

14. $\frac{1}{2}e^{4\sqrt{x}}+C$

15. $\frac{1}{4}(2x+1)^2-(2x+1)+\frac{1}{2}\ln|2x+1|+C$

16. $\dfrac{4^x e^{2x}}{\ln(4e^2)} + C$

17. $\dfrac{1}{\ln 4}\left[4\left(4^x\right) + \sin 4^x\right] + C$

18. $\dfrac{1}{2}\left[\tan^{-1}\left(e^2\right) - \dfrac{\pi}{4}\right]$

19. $\sin^{-1}\left(\dfrac{\sin x}{\sqrt{3}}\right) + C$

20. $\ln|\sec(\ln x)| + C$

21. $-\sqrt{16 - x^2} + 2\sin^{-1}\left(\dfrac{x}{4}\right) + C$

22. $\dfrac{1}{3}\sec^{-1}\left(x^2\right) + C$

23. $\dfrac{\pi}{4}$

24. $\left(\dfrac{\sqrt{3}}{2}, \dfrac{\pi}{3}\right), \left(-\dfrac{\sqrt{3}}{2}, -\dfrac{\pi}{3}\right)$

25. ∞

26. 1

27. 1

28. ∞

29. e

30. e^2

31. $\ln\dfrac{5}{2}$

32. 64000 bacteria

33. $\lambda = -k = -\ln(3/4)\ \frac{1}{\text{hr}} \implies t = \frac{1}{k}\ln(30/400) \approx 9.0039$ hours

Exercise 3-1 (page 67)

1. $-\dfrac{1}{5}x^2 e^{-5x} - \dfrac{2}{25}xe^{-5x} - \dfrac{2}{125}e^{-5x} + C$

2. $\dfrac{1}{3}x^3\cos^{-1}x - \dfrac{1}{3}\left(1 - x^2\right)^{1/2} + \dfrac{1}{9}\left(1 - x^2\right)^{3/2} + C$

3. $te^{2\sqrt{t}} - \sqrt{t}\,e^{2\sqrt{t}} + \dfrac{1}{2}e^{2\sqrt{t}} + C$

4. $\dfrac{1}{11}x^{11}\ln x - \dfrac{1}{121}x^{11} + C$

5. $-\dfrac{1}{3}x^3\cos\left(x^3\right) + \dfrac{1}{3}\sin\left(x^3\right) + C$

6. $-\dfrac{1}{5}e^{2x}\cos 4x + \dfrac{1}{10}e^{2x}\sin 4x + C$

7. $\dfrac{\sqrt{3}\,\pi}{3} - \dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$

8. $\dfrac{1}{10}\,x\,\sin(3\ln x) - \dfrac{3}{10}\,x\,\cos(3\ln x) + C$

9. $\dfrac{1}{\ln 5}\,x^2\,5^x - \dfrac{2}{(\ln 5)^2}\,x\,5^x + \dfrac{2}{(\ln 5)^3}\,5^x + C$

10. $\dfrac{\sqrt{3}\,\pi}{3} - \ln 2$

Exercise 3-2 (page 73)

1. $-\dfrac{1}{11}\cos^{11} x + \dfrac{2}{13}\cos^{13} x - \dfrac{1}{15}\cos^{15} x + C$

2. $\dfrac{1}{5}\sin^5 x - \dfrac{1}{7}\sin^7 x + C$

3. $\dfrac{3}{8}x - \dfrac{1}{4}\sin 2x + \dfrac{1}{32}\sin 4x + C$

4. $\dfrac{1}{5}\tan^5 x + \dfrac{1}{7}\tan^7 x + C$

5. $\dfrac{1}{9}\sec^9 x - \dfrac{2}{7}\sec^7 x + \dfrac{1}{5}\sec^5 x + C$

6. $-\dfrac{1}{3}\cot^3 x - \dfrac{1}{5}\cot^5 x + C$

7. $\dfrac{1}{3}\tan^3 x - \tan x + x + C$

8. $-\dfrac{1}{4}\cos 2x - \dfrac{1}{32}\cos 16x + C$

9. $\dfrac{1}{4}\sin 2x - \dfrac{1}{16}\sin 8x + C$

10. $\dfrac{1}{2}\sin x + \dfrac{1}{18}\sin 9x + C$

Exercise 3-3 (page 77)

1. $8\sin^{-1}\left(\dfrac{x}{4}\right) + \dfrac{1}{2}\,x\sqrt{16-x^2} + C$

2. $\sqrt{9+x^2} + 3\ln|\sqrt{9+x^2} - 3| - 3\ln|x| + C$

3. $\dfrac{\sqrt{x^2-1}}{x} - \dfrac{1}{3}\left(\dfrac{\sqrt{x^2-1}}{x}\right)^3 + C$

4. $\dfrac{1}{256}\left[\dfrac{u}{\sqrt{16-u^2}} + \dfrac{1}{3}\left(\dfrac{u}{\sqrt{16-u^2}}\right)^3\right] + C$

5. $\dfrac{1}{2}\ln\left|2x + \sqrt{4x^2-9}\right| + C$

6. $\ln\left|\sqrt{4+x^2} + x\right| - \dfrac{x}{\sqrt{4+x^2}} + C$

7. $\dfrac{5}{6\sqrt{3}}\,\sin^{-1}\left(\sqrt{\dfrac{3}{5}}\,x\right) - \dfrac{1}{6}\,x\,\sqrt{5-3x^2} + C$

8. $\dfrac{1}{\sqrt{2}}\,\tan^{-1}\left(\dfrac{x-2}{\sqrt{2}}\right) + C$

9. $-\dfrac{1}{3}\left(x^2-6x+15\right)^{-3/2} + \dfrac{1}{12}\,\dfrac{x-3}{\sqrt{x^2-6x+15}} - \dfrac{1}{36}\left(\dfrac{x-3}{\sqrt{x^2-6x+15}}\right)^3 + C$

10. $\dfrac{4-x}{2\sqrt{4-4x-x^2}} + C$

Exercise 3-4 (page 88)

1. $-2\ln|x-2| + 3\ln|x-3| + C$
2. $\dfrac{1}{2}x^2 + \dfrac{13}{3}\ln|x+3| + \dfrac{14}{3}\ln|x-3| + C$
3. $\dfrac{21}{27}\ln|x-1| - \dfrac{2}{3}\dfrac{1}{x-1} - \dfrac{7}{9}\ln|x+2| + C$
4. $\ln|x+1| - \dfrac{1}{2}\ln|x^2+1| + 3\tan^{-1}x + C$
5. $-\ln|x| + \dfrac{1}{2}\ln|x^2+3| - \dfrac{3}{2}\dfrac{1}{x^2+3} + \dfrac{\sqrt{3}}{18}\tan^{-1}\dfrac{x}{\sqrt{3}} + \dfrac{1}{6}\dfrac{x}{x^2+3} + C$
6. $\dfrac{x^2}{2} + 3x + 7\ln|x-1| - \dfrac{6}{x-1} - \dfrac{3}{2}\dfrac{1}{(x-1)^2} + C$
7. $-\dfrac{2}{x} - \dfrac{2}{\sqrt{5}}\tan^{-1}\left(\dfrac{x}{\sqrt{5}}\right) + C$
8. $\dfrac{x^3}{3} - 3x + \dfrac{2}{x} + \dfrac{1}{2}\ln|x^2+3| + \dfrac{11}{\sqrt{3}}\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right) + C$
9. $-\dfrac{1}{2}\ln|x^2+3| - \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right) + \dfrac{1}{2}\ln|x^2+1| + 2\tan^{-1}x + C$
10. $\ln|x-1| + \dfrac{1}{2}\ln|x^2+1| - 3\tan^{-1}x + C$

Exercise 3-5 (page 91)

1. $2\,e^{\sqrt{x}} + C$
2. $-2\sqrt{x+5}\cos\sqrt{x+5} + 2\sin\sqrt{x+5} + C$
3. $3 + e^x - 3\ln|3+e^x| + C$
4. $\dfrac{1}{3\sqrt{5}}\,\tan^{-1}\left(\dfrac{e^{3x}}{\sqrt{5}}\right) + C$
5. $2 - 6\ln 5 + 6\ln 4$
6. $\dfrac{6}{5}\ln|\sin x - 3| + \dfrac{4}{5}\ln|\sin x + 2| + C$
7. $\dfrac{1}{2}\ln|x^2+1| - \dfrac{2}{x^2+1} + C$

8. $\ln\left|\dfrac{\sqrt{e^{2x}+4e^x+6}}{\sqrt{2}}+\dfrac{e^x+2}{\sqrt{2}}\right|+C$

9. $\dfrac{\ln 6}{2}+\dfrac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}-\dfrac{1}{2}\ln 3-\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$

10. $-\dfrac{2}{(x+1)^2+8}-\dfrac{\sqrt{8}}{32}\tan^{-1}\left(\dfrac{x+1}{\sqrt{8}}\right)-\dfrac{1}{4}\dfrac{x+1}{(x+1)^2+8}+C$

Exercise 3-6 (page 99)

1. convergent
2. divergent
3. convergent
4. convergent
5. convergent
6. convergent
7. convergent
8. divergent
9. divergent
10. divergent

Chapter 3 Review Exercises (page 100)

1. $\dfrac{1}{2}x^2\tan^{-1}x-\dfrac{1}{2}x+\dfrac{1}{2}\tan^{-1}x+C$
2. $\dfrac{1}{3}\sec^3 x+C$
3. $6\ln 2-2$
4. $\dfrac{1}{1296}\left[\dfrac{x}{\sqrt{x^2+36}}-\dfrac{1}{3}\left(\dfrac{x}{\sqrt{x^2+36}}\right)^3\right]+C$
5. $-\dfrac{1}{3}\dfrac{1}{(x+1)^3}+\dfrac{3}{4}\dfrac{1}{(x+1)^4}+C$
6. $2\ln|x|-\ln|x^2+2|+C$
7. $\ln\left|\dfrac{\sqrt{x^2+6x+12}+x+3}{\sqrt{3}}\right|+C$
8. $\dfrac{1}{2}+\ln\dfrac{3}{2}$
9. $-5\ln|x-1|+6\ln|x-2|+3\ln|x-3|+C$
10. $-\ln|x^2+1|+3\tan^{-1}x+4\ln|x^2+4|-\dfrac{7}{2}\tan^{-1}\left(\dfrac{x}{2}\right)+C$
11. $\dfrac{2}{7}x^{7/2}\ln x-\dfrac{4}{49}x^{7/2}+C$
12. $\ln|\sqrt{1+\sin^2 x}+\sin x|+C$
13. convergent
14. convergent
15. convergent

Exercise 4-1 (page 110)

1. $a_1 = \frac{4}{7}, a_2 = \frac{20}{21}, a_3 = \frac{54}{59}, a_4 = \frac{112}{133}, \lim_{n\to\infty} a_n = \frac{1}{2}$
2. $a_1 = \frac{2e}{3e+1}, a_2 = \frac{2e^2}{3e^2+1}, a_3 = \frac{2e^3}{3e^3+1}, a_4 = \frac{2e^4}{3e^4+1}, \lim_{n\to\infty} a_n = \frac{2}{3}$
3. $a_1 = -\frac{4}{7}, a_2 = \frac{7}{16}, a_3 = -\frac{12}{37}, a_4 = \frac{19}{76}, \lim_{n\to\infty} a_n = 0$
4. $\lim_{n\to\infty} a_n = \frac{1}{2}$, convergent
5. $\lim_{n\to\infty} a_n = 0$, convergent
6. $\lim_{n\to\infty} a_n = 0$, convergent
7. $\lim_{n\to\infty} a_n = 0$, convergent
8. decreasing
9. increasing
10. increasing

Exercise 4-2 (page 117)

1. convergent, Sum=6
2. divergent
3. divergent
4. convergent, Sum=20
5. convergent, Sum=$\frac{3}{4}$
6. divergent
7. convergent, Sum=6
8. divergent
9. divergent
10. divergent
11. convergent, Sum=sin(1)

Exercise 4-3 (page 128)

1. divergent
2. convergent
3. divergent
4. divergent
5. convergent
6. convergent
7. convergent
8. divergent
9. convergent
10. divergent

Exercise 4-4 (page 131)

1. convergent
2. convergent
3. divergent
4. divergent
5. convergent
6. divergent

7. convergent
8. divergent
9. convergent
10. convergent

Exercise 4-5 (page 137)

1. divergent
2. convergent
3. convergent
4. divergent
5. convergent
6. convergent
7. absolutely convergent
8. absolutely convergent
9. absolutely convergent
10. divergent
11. absolutely convergent
12. absolutely convergent

Exercise 4-6 (page 142)

1. divergent
2. conditionally convergent
3. absolutely convergent
4. divergent
5. divergent
6. divergent
7. convergent
8. divergent
9. absolutely convergent
10. absolutely convergent
11. divergent
12. absolutely convergent
13. conditionally convergent
14. absolutely convergent

Exercise 4-7 (page 150)

1. $I = [-1, 1),\ R = 1$
2. $I = [0, 2],\ R = 1$
3. $I = (-\infty, \infty),\ R = \infty$
4. $I = \left(-\dfrac{1}{2}, 1\right),\ R = \dfrac{3}{4}$
5. $I = \{1\},\ R = 0$

Exercise 4-8 (page 156)

1. $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n},\ I = (-1, 1)$
2. $\displaystyle\sum_{n=0}^{\infty}\frac{4^n}{5^{n+1}}x^{n+1},\ I = \left(-\frac{5}{4}, \frac{5}{4}\right)$
3. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{n+1}}x^{2n+1},\ I = \left(-\sqrt{2}, \sqrt{2}\right)$

4. $-\frac{1}{6}\sum_{n=0}^{\infty}(-1)^n \frac{n3^n}{2^n}x^{n-1},\ I=\left(-\frac{2}{3},\frac{2}{3}\right)$

5. $\sum_{n=0}^{\infty}(-1)^n \frac{2^{n+1}}{(n+1)3^{n+1}}x^{n+1}+\ln 3,\ I=\left(-\frac{3}{2},\frac{3}{2}\right)$

Exercise 4-9 (page 160)

1. $\sum_{n=0}^{\infty}(-1)^n \frac{2^n}{n!}x^n,\ R=\infty$

2. $\sum_{n=0}^{\infty}\frac{2^n}{n!}x^{n+2},\ R=\infty$

3. $\frac{1}{2}+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}}{(2n)!}x^{2n},\ R=\infty$

4. $\sum_{n=0}^{\infty}(-1)^n \frac{4^{2n+1}}{(2n+1)!}x^{2n+3},\ R=\infty$

5. $\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}x^{6n},\ R=\infty$

Exercise 4-10 (page 165)

1. $\frac{1}{3}-\frac{1}{3}(x-\ln 3)+\frac{1}{6}(x-\ln 3)^2-\frac{1}{18}(x-\ln 3)^3+\dots$

2. $1-\frac{\pi^2}{2!}\left(x-\frac{1}{2}\right)^2+\frac{\pi^4}{4!}\left(x-\frac{1}{2}\right)^4+\dots$

3. $1+\frac{1}{e}(x-e)-\frac{1}{2e^2}(x-e)^2+\frac{1}{3e^3}(x-e)^3+\dots$

4. $2+2\ln 2\,(x-1)+(\ln 2)^2(x-1)^2+\frac{1}{3}(\ln 2)^3(x-1)^3+\dots$

5. $\ln 4+\frac{1}{4}(x-1)-\frac{1}{32}(x-1)^2+\frac{1}{192}(x-1)^3+\dots$

Chapter 4 Review Exercises (page 166)

1. convergent

2. divergent

3. convergent

4. decreasing

5. decreasing

6. (a) $\lim_{k\to\infty} a_k=\lim_{k\to\infty}\ln\left(\frac{k+1}{k}\right)=\ln\left(\lim_{k\to\infty}\frac{k+1}{k}\right)=\ln(1)=0$. Since the terms converge to zero this tells us nothing about the convergence of the series $\sum a_k$.

(b) $S_n = \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{n} [\ln(k+1) - \ln(k)]$

$$= [\cancel{\ln(2)} - \underbrace{\ln(1)}_{=0}] + [\cancel{\ln(3)} - \cancel{\ln(2)}] + [\cancel{\ln(4)} - \cancel{\ln(3)}] + \ldots + [\ln(n+1) - \cancel{\ln(n)}] = \ln(n+1)$$

(c) $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = \infty$. Series diverges.

7. convergent
8. divergent
9. divergent
10. convergent
11. divergent
12. convergent
13. convergent
14. convergent
15. divergent
16. convergent
17. divergent
18. convergent
19. convergent
20. convergent
21. convergent
22. $I = (-\infty, \infty)\,,\ R = \infty$
23. $I = [6, 8]\,,\ R = 1$
24. $I = [9, 11)\,,\ R = 1$
25. $I = [-5, -4]\,,\ R = \frac{1}{2}$
26. $\frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}}{(2n)!}x^{2n},\ R = \infty$
27. $\sum_{n=0}^{\infty} (-1)^n\, \frac{5^n}{n!}x^{n+1},\ R = \infty$
28. $\sum_{n=0}^{\infty} (-1)^n\, \frac{5^n\, e^{-10}}{n!}(x-2)^n$
29. $\ln 2 + \frac{1}{2}\, x - \frac{1}{8}\, x^2 + \frac{1}{24}\, x^3 + \ldots$

Exercise 5-1 (page 174)

1. $\frac{32}{3}$
2. $\frac{31}{6}$
3. $\frac{125}{6}$
4. $\frac{50}{3}$
5. 6

Exercise 5-2 (page 189)

1. 40π
2. $\frac{32\pi}{5}$
3. $\frac{896\pi}{15}$
4. 8π
5. $\frac{96\pi}{5}$
6. $\frac{166\pi}{15}$
7. $\frac{1088\pi}{15}$
8. $\frac{64\pi}{15}$
9. $\frac{14\pi}{5}$
10. $\frac{17\pi}{21}$

Exercise 5-3 (page 196)

1. $\frac{27\pi}{2}$
2. $\frac{256\pi}{15}$
3. $\frac{5\pi}{14}$
4. $\frac{3\pi}{10}$
5. $\frac{40\pi}{3}$
6. $\frac{24\pi}{5}$
7. $\frac{9\pi}{2}$
8. $\frac{53\pi}{5}$
9. 64π
10. 16π

Exercise 5-4 (page 199)

1. $\frac{117\pi}{5}$
2. 128π
3. $\frac{640\pi}{3}$
4. $\frac{1408\pi}{15}$
5. $\frac{875\pi}{6}$
6. $\frac{1080\pi}{3}$
7. $\frac{1664\pi}{15}$
8. $\frac{25\pi}{21}$
9. $\frac{128\pi}{3}$
10. $\frac{61\pi}{2}$

Exercise 5-5 (page 206)

1. $\frac{1}{27}[40^{3/2} - 13^{3/2}]$
2. $\frac{14}{3}$
3. $\ln|\sqrt{2}-1| - \ln|2-\sqrt{3}|$
4. $\frac{61}{18}$
5. $\frac{67}{10}$
6. $\frac{e-1}{\sqrt{e}}$
7. $s(x) = \frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln|\sqrt{1+4x^2} + 2x|\right)$

8. $s(x) = \sqrt{x}\,\sqrt{1+x} + \ln|\sqrt{1+x} + \sqrt{x}| - \sqrt{2} - \ln|\sqrt{2}+1|$

9. $\frac{1}{3}(x-2)^{3/2} + (x-2)^{1/2} - \frac{4}{3}$

10. $\frac{1}{6}(4x+17)^{3/2} - \frac{1}{6}(17)^{3/2}$

11. Let $s(x) = \ln|\sec x + \tan x|$. Then

$$\begin{aligned} s(-x) &= \ln|\sec(-x)+\tan(-x)| = \ln|\sec x - \tan x| = \ln\left|\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right| = \ln\left|\frac{1-\sin x}{\cos x}\right| \\ &= \ln\left|\left(\frac{\cos x}{1-\sin x}\right)^{-1}\right| = \ln\left(\left|\frac{\cos x}{1-\sin x}\right|^{-1}\right) = -\ln\left|\frac{\cos x}{1-\sin x}\right| \\ &= -\ln\left|\left(\frac{\cos x}{1-\sin x}\right)\left(\frac{1+\sin x}{1+\sin x}\right)\right| = -\ln\left|\left(\frac{\cos x}{1-\sin^2 x}\right)(1+\sin x)\right| \\ &= -\ln\left|\left(\frac{\cos x}{\cos^2 x}\right)(1+\sin x)\right| = -\ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| = -\ln|\sec x + \tan x| = -s(x) \end{aligned}$$

Therefore $s(x)$ is an odd function of x.

Exercise 5-6 (page 213)

1. $28\sqrt{5}\pi$
2. $\frac{18\pi}{5}[8^{5/2} - 5^{5/2}]$
3. $\frac{4907\pi}{1600}$
4. $\frac{\pi}{6}[21^{3/2} - 17^{3/2}]$
5. $\frac{2}{9}\pi[5^{3/2} - 1]$
6. 2π
7. $\frac{3\pi}{2}$
8. $\frac{\pi}{2}\left[\sqrt{82} + \ln\left|9 + \frac{\sqrt{82}}{9}\right| - \sqrt{2} - \ln\left|1+\sqrt{2}\right|\right]$

Chapter 5 Review Exercises (page 214)

1. $\frac{8}{3}$
2. $\frac{15}{4}$
3. $\frac{32}{3}$
4. $2\sqrt{2} - 2$
5. 256π
6. $\frac{64\pi}{3}$
7. $\frac{88\pi}{3}$
8. $\frac{8\pi}{3}$
9. $\frac{92\pi}{3}$
10. 4π
11. $\frac{1}{27}[31^{3/2} - 8]$
12. $\frac{17}{12}$
13. $s(x) = \ln\left(\frac{\sqrt{1+x^2}-1}{x}\right) + \sqrt{1+x^2} - \ln\left(\sqrt{2}-1\right) - \sqrt{2}$
14. $8\sqrt{5}\pi$
15. $\frac{\pi}{2\sqrt{3}}\left[28\sqrt{3} + \ln|4\sqrt{3}+7|\right]$

Exercise 6-1 (page 220)

1. (b) line segment: $y = x - 3, \quad 9 \le x \le 18$
2. (b) parabola: $y = (x-2)^2 + 4$
3. (b) circle: $x^2 + y^2 = 9$
4. (b) ellipse: $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$
5. (b) hyperbola: $\dfrac{(x-1)^2}{4} + \dfrac{(y-2)^2}{9} = 1$

Exercise 6-2 (page 231)

1. $y = \dfrac{1}{3}x + \dfrac{16}{3}$
2. $y = -\dfrac{1}{2}x + 4$
3. $y = 4x - 2\sqrt{3}$
4. $y = -\sqrt{3}x + 4\sqrt{3}$
5. Horizontal if $t = \pm 2$, Vertical if $t = 0$
6. Horizontal if $t = \ln 2$, Vertical if $t = 0$, $t = \ln 3$
7. Concave up for all $t \ge 0$
8. Concave up for $t > \ln 2$ and concave down for $0 \le t < \ln 2$
9. $A = \dfrac{32}{3}$
10. $A = \pi$
11. $s = \dfrac{99}{10}$
12. $s = \sqrt{2} + \ln(1 + \sqrt{2})$

Chapter 6 Review Exercises (page 232)

1. $y = 3x - 2$
2. $y = -x + \dfrac{\pi}{2} + 1$
3. Horizontal if $t = -\ln 2$, Vertical if $t = \ln 3$
4. Horizontal if $t = \dfrac{\pi}{2}$, Vertical if $t = \dfrac{\pi}{3}$, $t = \dfrac{5\pi}{3}$
5. $A = 10 \ln 10 - 9$
6. $A = 61 - 5e^{-4}$
7. $s = 12 + \ln 5$
8. $s = \sqrt{2}\,(e^{\pi} - 1)$

Exercise 7-1 (page 238)

1. $(x, y) = \left(1, \sqrt{3}\right)$
2. $(x, y) = (0, -3)$
3. $(x, y) = \left(-2\sqrt{2}, -2\sqrt{2}\right)$
4. $(x, y) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
5. $(r, \theta) = \left(\sqrt{2}, \frac{7\pi}{4}\right)$
6. $(r, \theta) = \left(4, \frac{\pi}{6}\right)$
7. $(r, \theta) = (5, \pi)$
8. $(r, \theta) = \left(8, \frac{4\pi}{3}\right)$

Exercise 7-2 (page 252)

1. circle: $x^2 + y^2 = 100$
2. circle: $\left(x - \dfrac{5}{2}\right)^2 + y^2 = \dfrac{25}{4}$
3. hyperbola: $\dfrac{y^2}{14} - \dfrac{x^2}{6} = 1$
4. hyperbola: $x^2 - y^2 = 10$
5. $r = 8$
6. $r^2 \sin(2\theta) = 10$
7. $r^2 \left(9 + 7\sin^2\theta\right) = 36$
8. $-\sqrt{3}$
9. $-\dfrac{6 + 5\sqrt{2}}{5\sqrt{2}}$
10. -3
11. Horizontal if $\theta = \dfrac{\pi}{4} + n\pi, \;\; \theta = \dfrac{3\pi}{4} + n\pi$
    Vertical if $\theta = n\pi, \;\; \theta = \dfrac{\pi}{2} + 2n\pi, \;\; \theta = \dfrac{3\pi}{2} + 2n\pi$
12. Horizontal if $\theta = \dfrac{3\pi}{2} + 2n\pi, \;\; \theta = \dfrac{\pi}{6} + 2n\pi, \;\; \theta = \dfrac{5\pi}{6} + 2n\pi$
    Vertical if $\theta = \dfrac{\pi}{2} + 2n\pi, \;\; \theta = \dfrac{7\pi}{6} + 2n\pi, \;\; \theta = \dfrac{11\pi}{6} + 2n\pi$
13. $A = \dfrac{25\pi}{4}$
14. $A = 24\pi$
15. $A = \dfrac{9\pi}{8}$
16. $A = 4$
17. $A = 24\pi - 64$
18. $A = \dfrac{41\pi}{2} - 24\sqrt{2}$
19. $s = \dfrac{\sqrt{5}}{2}\left(e^6 - 1\right)$

20. $s = 24$

21. (a) $s = \int_c^d \sqrt{1 + r^2[g'(r)]^2}\, dr = \int_c^d \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr$
    Note that formally the integrand is what is expected if we "pulled out" the dr from $ds = \sqrt{(r d\theta)^2 + (dr)^2}$.
    (b) $s = 3\sqrt{2}$

Chapter 7 Review Exercises (page 255)

1. $(x, y) = \left(\frac{7}{2}, -\frac{7\sqrt{3}}{2}\right)$
2. $(r, \theta) = \left(\pi, \frac{\pi}{2}\right)$
3. circle: $\left(x - \frac{1}{2}\right)^2 + (y - 3)^2 = \frac{37}{4}$
4. hyperbola: $x^2 - 3y^2 = 16$
5. $r^2 \left[\sin(2\theta) + 2\sin^2\theta\right] = 16$
6. $r^2 - 2r\cos\theta = 3$
7. $-\frac{1}{3}$
8. -3
9. Horizontal if $\theta = \frac{3\pi}{4} + 2n\pi,\ \ \theta = \frac{7\pi}{4} + 2n\pi$
   Vertical if $\theta = \frac{\pi}{4} + 2n\pi,\ \ \theta = \frac{5\pi}{4} + 2n\pi$
10. Horizontal if $\theta = \frac{\pi}{2} + 2n\pi,\ \ \theta = \frac{7\pi}{6} + 2n\pi,\ \ \theta = \frac{11\pi}{6} + 2n\pi$
    Vertical if $\theta = \frac{3\pi}{2} + 2n\pi,\ \ \theta = \frac{\pi}{6} + 2n\pi,\ \ \theta = \frac{5\pi}{6} + 2n\pi$
11. $A = \frac{5\pi}{4}$
12. $A = \frac{41\pi}{2}$
13. $A = 2\pi - 4$
14. $A = \frac{41\pi}{2} - 24\sqrt{2}$
15. $L = 4$
16. $L = \frac{32}{5}\left(\pi^2 + 4\right)^{5/2} - \frac{128}{3}\left(\pi^2 + 4\right)^{3/2} + \frac{2048}{15}$

Table of Derivatives

1. $\dfrac{d}{dx}(c) = 0$

2. $\dfrac{d}{dx}(x^n) = nx^{n-1}$

---

3. $\dfrac{d}{dx}\left(e^x\right) = e^x$

4. $\dfrac{d}{dx}\left(a^x\right) = a^x \ln a$

5. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

6. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$

---

7. $\dfrac{d}{dx}(\sin x) = \cos x$

8. $\dfrac{d}{dx}(\cos x) = -\sin x$

9. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

10. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

11. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

12. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

---

13. $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$

14. $\dfrac{d}{dx}\left(\cos^{-1} x\right) = -\dfrac{1}{\sqrt{1-x^2}}$

15. $\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}$

16. $\dfrac{d}{dx}\left(\sec^{-1} x\right) = \dfrac{1}{x\sqrt{x^2-1}}$

17. $\dfrac{d}{dx}\left(\csc^{-1} x\right) = -\dfrac{1}{x\sqrt{x^2-1}}$

18. $\dfrac{d}{dx}\left(\cot^{-1} x\right) = -\dfrac{1}{1+x^2}$

---

19. $\dfrac{d}{dx}(cf) = c\dfrac{df}{dx}$

20. $\dfrac{d}{dx}(f \pm g) = \dfrac{df}{dx} \pm \dfrac{dg}{dx}$

21. $\dfrac{d}{dx}(fg) = \dfrac{df}{dx}g + f\dfrac{dg}{dx}$ (**Product Rule**)

22. $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$ (**Quotient Rule**)

23. $\dfrac{d}{dx}[f(x)]^n = n\,[f(x)]^{n-1}\, f'(x)$ (**Generalized Power Rule**)

24. $\dfrac{d}{dx}f(g(x)) = f'\,(g(x)) \cdot g'(x)$ (**Chain Rule**)

Here c, n, and $a > 0$ are constants, f and g are functions of x, and primes (f', g') denote differentiation.

Table of Indefinite Integrals

1. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$

2. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

---

3. $\displaystyle\int e^x\,dx = e^x + C$

4. $\displaystyle\int a^x\,dx = \frac{1}{\ln a}a^x + C$

---

5. $\displaystyle\int \sin x\,dx = -\cos x + C$

6. $\displaystyle\int \cos x\,dx = \sin x + C$

7. $\displaystyle\int \tan x\,dx = \ln|\sec x| + C$

8. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$

9. $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x| + C$

10. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

---

11. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

12. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$

13. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

14. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

---

15. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

16. $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

17. $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$

18. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$

19. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

20. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

---

21. $\displaystyle\int cf(x)\,dx = c\int f(x)\,dx$

22. $\displaystyle\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

23. $\displaystyle\int f(g(x))g'(x)\,dx = \int f(u)\,du$ where $u = g(x)$ (**Substitution Rule**)

24. $\displaystyle\int u\,dv = uv - \int v\,du$ (**Integration by Parts**)

Here c, n, and $a > 0$ are constants, f and g are functions of x, and primes (g') denote differentiation.

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