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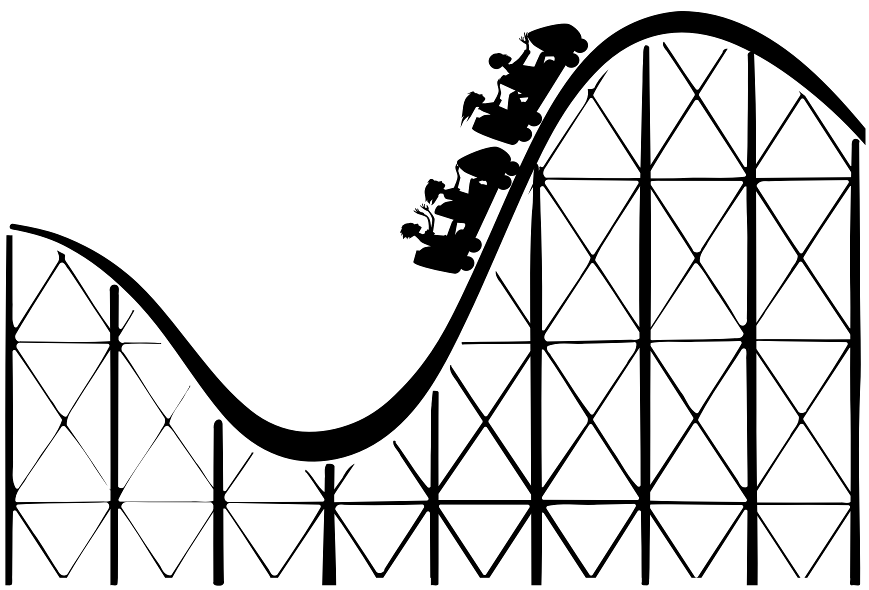
Math 110
Calculus I



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About the cover: The cover line art drawing rolls several calculus concepts elegantly together including tangent lines, area under a curve, the Riemann sum, velocity, acceleration, and the integral symbol $\int$. Also calculus is fun, just like roller coasters! The drawing is in the public domain and available from http://openclipart.org.

Contents

Chapter 1: Equations and Functions

1.1 Equations

Definition: An **equation** is two mathematical **expressions** joined with an equal sign.

Example 1-1

$$x^3 + 2x = 1 + x + \cos x$$

When an equation involves a variable (like x) equality will usually hold for only certain values of the variable. These particular values are called the **solutions** or **roots**[1] of the equation. **Solving** an equation is the act of finding the solutions of it.

The above equation is difficult to solve. The following equation is easy:

$$5x + 2 = 0\ .$$

Just isolate the variable by doing the necessary inverse operations to both sides:

$$\begin{aligned} 5x + 2 &= 0 \\ 5x &= -2 \quad \text{subtracted 2 from both sides} \\ x &= -\frac{2}{5} \quad \text{divided both sides by 5} \end{aligned}$$

The above equation is an example of a **linear equation** $\boxed{ax + b = 0}$ because it contains only powers of the variable (i.e. x^n) with the highest power being one ($x^1 = x$, $x^0 = 1$). Here a and b are **constants** – the symbols represent numbers that just have not been specified. In the above example they would be $a = 5$ and $b = 2$. The solution to the general linear equation in one variable would be $\boxed{x = -\frac{b}{a}}$.

A **quadratic equation** involves a highest power of the variable of two, for example:

$$x^2 = 5x - 6$$

The latter we rearrange by appropriate subtraction to be in the standard form $\boxed{ax^2 + bx + c = 0}$ to be:

$$x^2 - 5x + 6 = 0$$

So $a = 1$, $b = -5$, and $c = 6$. A **product** is made up of two or more **factors** multiplied together. One method of solving for x involves **factoring** the left hand side.

$$(x - 3)(x - 2) = 0$$

(Check by multiplication of the factors that the left hand side is the same as before!) This simplifies the problem since the only way a product can equal zero is if one of the factors is zero:

$$\boxed{(A)(B)(C)\cdots = 0 \iff A = 0 \text{ or } B = 0 \text{ or } C = 0 \text{ or } \ldots}$$

Factoring thus reduces the above problem to solving two simpler linear equations:

$$x - 3 = 0 \text{ or } x - 2 = 0$$

[1]Some reserve the term *root* for solutions to an equation of the form $f(x) = 0$ where f is a function of x. All equations can be rearranged into this form. The roots, in this situation, are also called the *zeros* of the function $f(x)$ meaning they are the values of x which make the function equal zero.

$$x = 3 \text{ or } x = 2$$

So the solutions of the original quadratic equation are 2 and 3. Factoring can help solve complicated equations, just make sure you rearrange things so that zero is on one side of your equation first!

A second method to solve the quadratic equation for x involves the technique called *completing the square*. It can be done once and for all in terms of the constants a, b, c to get the **quadratic formula**:

$$ax^2 + bx + c = 0 \Rightarrow \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \quad \text{Memorize me!}$$

Here, in general, using the $+$ will give one solution and the $-$ will give the other.

Other factoring problems may be solved by using the following factor **identities**:

$(Ax + Ay + \ldots) = A(x + y + \ldots)$ Pulling out a common factor of A

$x^2 + (A + B)x + AB = (x + A)(x + B)$ Factoring a quadratic with leading coefficient of 1

$ACx^2 + (AD + BC)x + BD = (Ax + B)(Cx + D)$ Factoring a general quadratic

$x^2 - y^2 = (x + y)(x - y)$ Difference of Squares

$x^2 + 2xy + y^2 = (x + y)^2$

$x^2 - 2xy + y^2 = (x - y)^2$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ Sum of Cubes

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ Difference of Cubes

Here x, y, A, and B could represent variables or constants or even a combination. A could be the factor $(x - 2)$ or y could be the number 2 or x could be $2z$ so that $x^2 = (2z)^2 = 4z^2$. Thus these factor identities have more application than at first glance.

You can verify the identities by multiplying the right side of each one out to get the left. Try it! Note that $(x + y)^2$ just means $(x + y)(x + y)$, etc.

We call these factor equations **identities** rather than just equations because they are true for not just a particular value of the variables but for all values of them. Identities can be used to manipulate expressions found in equations like we did above since they replace one expression with another which is completely equivalent.

Additional techniques beyond identities that are useful for finding solutions to polynomial equations are the **Rational Roots Test** coupled with **polynomial long division** to reduce the order of the polynomial. The latter division, which involves only linear factors, can be done efficiently through a process called **synthetic division**.

Example 1-2

Solve the following equations:

1. $x^2 + 5x - 14 = 0$ (A **quadratic equation** with leading coefficient of $a = 1$)
2. $6x^2 + 13x + 5 = 0$ (A quadratic equation with leading coefficient of $a \neq 1$)
3. $3x^2 + 2x = 2$
4. $x^3 + 2x^2 - 8x = 0$ (A **cubic equation**)

5. $x^3 - 8 = 0$
6. $x^4 - 2x^3 + x^2 - 4 = 0$ (A **quartic equation**)
7. $x^5 - 4x^3 + x^2 - 4 = 0$ (A **quintic equation**)

Solution:

1. Two numbers that multiply to get -14 and add to get 5 are 7 and -2 so
$$x^2 + 5x - 14 = 0 \implies (x+7)(x-2) = 0 \implies (x+7) = 0 \text{ or } (x-2) = 0 \implies \begin{cases} x &= -7 \\ x &= 2 \end{cases}$$
2. Two numbers that multiply to get $(6)(5) = 30$ and add to get 13 are 3, 10. Then break the $13x = 3x + 10x$ and factor by grouping the first and last two terms:
$6x^2 + 13x + 5 = 0 \implies 6x^2 + \underline{3x + 10x} + 5 = 0 \implies \underline{6x^2 + 3x} + \underline{10x + 5} = 0$
$$\implies 3x(2x+1) + 5(2x+1) = 0 \implies (3x+5)(2x+1) = 0 \implies \begin{cases} x &= -\frac{5}{3} \\ x &= -\frac{1}{2} \end{cases}$$
3. First write equation with zero on one side ($3x^2 + 2x = 2 \implies 3x^2 + 2x - 2 = 0$). Grouping as in last example will not work. Use the quadratic formula with $a = 3$, $b = 2$, $c = -2$:
$$3x^2 + 2x - 2 = 0 \implies x = \frac{-2 \pm \sqrt{4+24}}{6} = \frac{-2 \pm \sqrt{(4)(7)}}{6} = \frac{-2 \pm 2\sqrt{7}}{6} = \frac{-1 \pm \sqrt{7}}{3}$$
4. $$x^3 + 2x^2 - 8x = 0 \implies x(x^2 + 2x - 8) = 0 \implies x(x+4)(x-2) = 0 \implies \begin{cases} x &= 0 \\ x &= -4 \\ x &= 2 \end{cases}$$
5. Use cube minus cube formula: $x^3 - 8 = 0 \implies (x-2)(x^2 + 2x + 4) = 0$
$$\implies \begin{cases} x &= 2 \\ x^2 + 2x + 4 &= 0 \implies x = \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm \sqrt{-12}}{2} \implies \text{No real roots} \end{cases}$$
Thus $x = 2$ is the only real solution.
6. Try factoring quartic by grouping terms: $\underline{x^4 - 2x^3} + \underline{x^2 - 4} = 0 \implies x^3(x-2) + (x+2)(x-2) = 0$
A common factor of $(x-2)$ is found which we now pull out to get $(x-2)(x^3 + x + 2) = 0$
Using observation or the **Rational Roots Test** shows $x = -1$ makes the cubic zero so $(x - (-1)) = (x+1)$ is a factor of it. Its remaining quadratic factor can be found by inspection by starting at the coefficient of the highest power (x^2) and working downward or by using **polynomial long division** to get $(x^3 + x + 2) = (x+1)(x^2 - x + 2)$ and hence the quartic resolves to $(x-2)(x+1)(x^2 - x + 2) = 0$
$$\implies \begin{cases} x &= 2 \\ x &= -1 \\ x^2 - x + 2 &= 0 \implies x = \frac{1 \pm \sqrt{1-8}}{2} = \frac{1 \pm \sqrt{-7}}{2} \implies \text{No real roots} \end{cases} \implies \begin{cases} x &= 2 \\ x &= -1 \end{cases}$$
7. Use grouping $\underline{x^5 - 4x^3} + \underline{x^2 - 4} = 0 \implies x^3(x^2 - 4) + (x^2 - 4) = 0$
$\implies (x^2 - 4)(x^3 + 1) = 0 \implies (x+2)(x-2)(x+1)(x^2 - x + 1) = 0$
$$\implies \begin{cases} x &= -2 \\ x &= 2 \\ x &= -1 \\ x^2 - x + 1 &= 0 \implies x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} \implies \text{No real roots} \end{cases} \implies \begin{cases} x &= -2 \\ x &= 2 \\ x &= -1 \end{cases}$$

Further Questions:

Solve the following equations:

1. $x^2 - 5x + 6 = 0$
2. $x^2 - 4x + 4 = 0$
3. $2x^2 - 4x + 5 = 0$
4. $x^3 - 3x^2 - 4x = 0$
5. $x^3 - 2x^2 - 3x + 6 = 0$
6. $6x^2 - 13x - 5 = 0$
7. $x^5 - 4x^3 = 0$
8. $x^4 + 2x^3 - x - 2 = 0$
9. $x^3 + 2x^2 - 1 = 0$
10. $x^3 + 27 = 0$

Exercise 1-1

Answers: Page 237

1-10: Solve the given equations.

1. $x^2 - 6x + 9 = 0$
2. $2x^2 - 5x - 3 = 0$
3. $4x^2 + 3x + 1 = 0$
4. $x^3 - 2x - 4 = 0$
5. $x^3 - 4x^2 - 4x + 16 = 0$
6. $x^3 + 4x - 5 = 0$
7. $2x^4 - 3x^2 = 0$
8. $x^4 - 3x^2 + 2 = 0$
9. $3x^5 - 2x^3 + 3x^2 - 2 = 0$
10. $2x^5 + 5x^4 - 3x^2 = 0$

While we have focussed on factoring to find solutions, it is sometimes useful to reverse this process and use solutions to equations to factor expressions. As an example, to factor the quadratic expression $ax^2 + bx + c$, one can use the quadratic formula to find the two solutions x_+ and x_- to the quadratic equation $ax^2 + bx + c = 0$. The original quadratic polynomial then factors as

$$ax^2 + bx + c = a(x - x_+)(x - x_-)$$

where the coefficient a on the right hand side is required to make the x^2 terms equal. As a specific example, the polynomial $6x^2 - 13x - 5$ when equated to zero has solutions $x_+ = 5/2$ and $x_- = -1/3$ by the quadratic formula. It follows that $6x^2 - 13x - 5 = 6(x - 5/2)[x - (-1/3)]$ which simplifies to $(2x - 5)(3x + 1)$.[2]

[2]This factoring procedure is particularly useful if the roots generated by the quadratic equation are irrational.

1.2 Functions

1.2.1 Sets

To define a function we first define the notion of a **set**.[3]

Definition: A **set** S is a collection of distinct objects called **elements**.

In mathematics sets will often contain numbers and a finite set can be represented by listing its elements between braces, { and }. For example, the set $S = \{2, 3, 7\}$ is the set containing the three integers 2, 3, and 7. Order does not matter in a set, so $\{2, 3, 7\} = \{3, 2, 7\}$. In an infinite set we cannot list all the elements so other notation is required. For the set S of all real numbers between two values, say -2 and 5 (but not including the endpoints -2 and 5), one can use **interval notation** and write $S = (-2, 5)$. If we wish to include one or both endpoints we use square brackets [and] rather than parentheses. So $S = (-2, 5]$ includes all the numbers strictly between -2 and 5 as well as 5 itself. If we wish to indicate all the numbers that are greater than a number (say 3) we can use the infinity symbol ∞ and write the interval as $(3, \infty)$. An interval containing both endpoints, like $[-1, 1]$, is called **closed**. An interval containing neither endpoint, like $(2, 3)$ or $(-\infty, 2)$, is called **open**. For the set of all real numbers one uses the special symbol $\mathbb{R}$. Clearly $\mathbb{R} = (-\infty, \infty)$. A set containing no elements, $S = \{\,\}$, is called the **empty set** and is denoted $\varnothing$.

Given two sets S and T we can combine the elements from both sets to create a new set called the **union** of the sets, $\boldsymbol{S \cup T}$, which is the set of all the elements in S **or** T (including the elements in both). So $(1, 3) \cup (2, 6) = (1, 6)$. The **intersection** of the sets, $\boldsymbol{S \cap T}$, is the smaller set of elements that are found in both S **and** T. So $(1, 3) \cap (2, 6) = (2, 3)$. The **difference** of two sets, $\boldsymbol{S - T}$, is a new set that contains all elements in S except for those also found in T. So, for example, to describe the set of all real numbers except for the number -2 one could write either $(-\infty, -2) \cup (-2, \infty)$ or, more simply, $\mathbb{R} - \{-2\}$. Finally one can also use set notation with braces to represent infinite sets. The set $\{x \in \mathbb{R} \mid 3 < x \leq 5\}$ can be read, "The set of all x in ($\in$) the real numbers such that (|) x is between 3 and 5, including 5. In interval notation this is just the set $(3, 5]$. As a further example, the intersection of two sets could be written $S \cap T = \{x \mid x \in S \text{ and } x \in T\}$.

1.2.2 Definition of a Function

Definition: A **function** f is a rule (correspondence) that assigns to each element x in a set D **exactly one** element $y = f(x)$ in a set R.

Think of a function as a machine that takes an input value x and turns it into an output value y or $f(x)$.

Example 1-3

Let $f(x) = \dfrac{x^2 + 3}{x + 5}$. Find $f(-1)$ and $f(2)$.

Solution:

$$f(-1) = \frac{(-1)^2 + 3}{(-1) + 5} = \frac{1 + 3}{-1 + 5} = \frac{4}{4} = 1 \qquad f(2) = \frac{2^2 + 3}{2 + 5} = \frac{4 + 3}{7} = \frac{7}{7} = 1$$

[3]This naive definition of a set as a collection results in logical inconsistencies that need to be resolved using a more rigorous axiomatic approach. However, for the sets encountered in this text, both of finite and infinite size, the given definition will be sufficient.

Further Question:

Let $f(x) = 2x^2 - 3x + 5$. Find $f(3)$ and $f(-2)$.

- The variable x is called the **independent variable**.
- The variable y is called the **dependent variable**.
- The set D is called the **domain** of the function.
- The set R of all assumed y-values is called the **range** of the function.

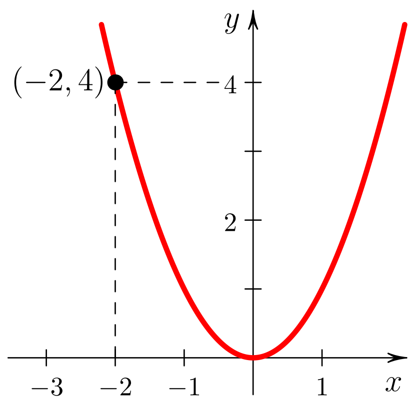
1.2.3 Representing Functions

There are different ways to represent a function:

Formulaic: The output values are given by explicit formula:

$$y = x^3 + 2x + 3 \qquad\qquad f(z) = z^3 - 5z - 10$$

Graphical (Visual): The function value for x is found by looking at the y-value of the point (x, y) on a line in the **Cartesian plane**:



Here $f(-2) = 4$.

Tabular (Numerical): The y-values are listed for each x-value. For example define a function by:

x	y
-1	2
2	0
3	5

Note for a function like $y = x^2$ it is impossible to tabulate all the possible values of y if x is allowed to be any real number. However it is often convenient to tabulate some values for graphing purposes, etc. For a function defined over only a finite domain (as in our example $D = \{-1, 2, 3\}$) a table is perfectly fine, however.

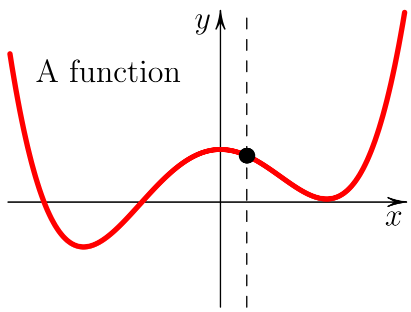
Symbolic: The function is referred to by a symbol.

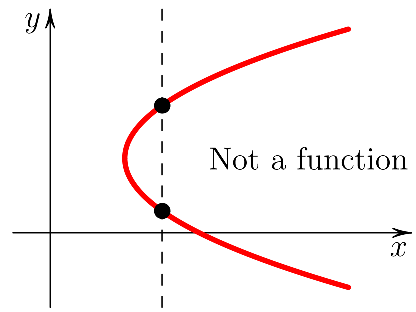
$$y = f(x) \qquad z = H(\alpha)$$

Note here that we are using functional notation. When we write $f(x)$ we do not mean multiplication; rather $f(x)$ represents the value y the function f assigns to a particular value of x.[4]

1.2.4 Vertical Line Test

Vertical line test: A curve lying in the x-y coordinate plane is the graph of a function if and only if every vertical line intersects the curve at no more than one point.





The curve at right represents the **relation** $(y-1)^2 = x - 1$. Unlike functions, relations do not require a single y-value for each x-value.

1.2.5 Determining the Domain of a Function

When a function is given as a formula with no domain specified the domain D is the set of all real numbers for which the formula makes sense and defines a real number. The impossibility of dividing by zero or taking an even root of a negative number will, for example, restrict the domain of certain functions. Deciding when an expression involving a variable is negative (< 0) or not involves working with inequalities. This is reviewed in Appendix A .

Example 1-4

Find the domain of each of the following functions:

1. $f(x) = x^3 + 3x^2 + 5$
2. $f(x) = \dfrac{x^2 - 3}{x^2 + 5x + 6}$
3. $g(t) = \sqrt{t+5}$
4. $f(u) = \sqrt{u^2 - 4}$
5. $g(t) = \sqrt{\dfrac{2t-3}{t+4}}$

[4] Also note the use of the Greek letter alpha (α) for the variable in the second example. A few useful Greek letters worth knowing are the first five of its alphabet, namely alpha: α, beta: β, gamma: γ, delta: δ, and epsilon: ϵ. Also theta: θ, phi: ϕ, and rho: ρ are also commonly used.

Solution:

1. Since $f(x) = x^3 + 3x^2 + 5$ is a sum of positive coefficients times positive integer powers of x (i.e. a polynomial) it is defined for all values of $x \implies D = \{x \in \mathbb{R} \,|\, -\infty < x < \infty\} = (-\infty, \infty)$

2. The function $f(x) = \dfrac{x^2 - 3}{x^2 + 5x + 6}$ will not be defined if we divide by zero
$\implies x^2 + 5x + 6 = 0 \implies (x+3)(x+2) = 0 \implies \begin{cases} x &= -3 \\ x &= -2 \end{cases}$
The domain is therefore $D = \{x \in \mathbb{R} \,|x \neq -3 \text{ and } x \neq -2\} = (-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$

3. The even root in $g(t) = \sqrt{t+5}$ is defined if the argument is zero or positive ("non-negative").
$t + 5 \geq 0 \implies t \geq -5 \implies D = \{t \in \mathbb{R} \,|t \geq -5\} = [-5, \infty)$

4. $f(u) = \sqrt{u^2 - 4} \implies u^2 - 4 \geq 0 \implies (u+2)(u-2) \geq 0$. The left hand side is zero if $u = -2$ or $u = 2$. It is positive if both factors are negative:
$\implies u + 2 < 0 \text{ and } u - 2 < 0 \implies u < -2 \text{ and } u < 2 \implies u < -2$
or they are both positive:
$\implies u + 2 > 0 \text{ and } u - 2 > 0 \implies u > -2 \text{ and } u > 2 \implies u > 2$
Domain is therefore $D = \{u \in \mathbb{R} | u \leq -2 \, \text{or} \, u \geq 2\} = (-\infty, -2] \cup [2, \infty)$

5. The denominator $t + 4 \neq 0$ in $g(t) = \sqrt{\frac{2t-3}{t+4}}$ implies $t \neq -4$.

   And the argument of the square root must be non-negative, $\frac{2t-3}{t+4} \geq 0$.

   The fraction vanishes if the numerator is zero ($\implies 2t - 3 = 0 \implies t = 3/2$).
To be positive its numerator and denominator must both be negative or both be positive:
$$\implies \begin{cases} 2t - 3 < 0 \text{ and } t + 4 < 0 \implies t < \frac{3}{2} \text{ and } t < -4 \implies t < -4 \\ \quad \text{or} \\ 2t - 3 > 0 \text{ and } t + 4 > 0 \implies t > \frac{3}{2} \text{ and } t > -4 \implies t > \frac{3}{2} \end{cases}$$
Domain is therefore $D = \left\{t \in \mathbb{R} | t < -4 \text{ or } t \geq \frac{3}{2}\right\} = (-\infty, -4) \cup \left[\frac{3}{2}, \infty\right)$.

Further Questions:

Find the domain of each of the following functions:

1. $f(x) = x^4 + x$
2. $f(x) = \dfrac{2}{x^2 - 9}$
3. $f(x) = \sqrt{x - 2}$
4. $h(t) = \sqrt[3]{t+4}$
5. $f(x) = \sqrt{x^2 - 3x}$
6. $g(t) = \sqrt{\dfrac{t+5}{t-2}}$

Definition: Two functions f and g are **equal** if the have the same domain D and $f(x) = g(x)$ for all x in D.

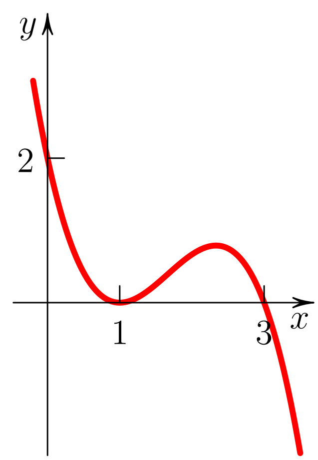
Example 1-5

1. $f(x) = \frac{1}{2}\sin 2x$ and $g(x) = \sin x \cos x$ are equal functions because they both have domain $D = \mathbb{R}$ and $\sin 2x = 2\sin x \cos x$ is a trigonometric identity so $f(x) = g(x)$ for all x.
2. $f(x) = 2x + 2$ and $g(t) = 2(t + 1)$ are equal functions because they have the same domain ($\mathbb{R}$) and $2(t + 1) = 2t + 2$ is an algebraic identity. Note that the choice of variable name used in the definition of the function (x or t) is not relevant.
3. $f(x) = (\sqrt{x})^2$ and $g(x) = x$ are not equal since the domain of $f(x)$ is $[0, \infty)$ while that of $g(x)$ is $\mathbb{R}$.
4. $f(x) = 1$ and $g(x) = \frac{x}{x}$ are not equal since the domain of $f(x)$ is $\mathbb{R}$ and that of $g(x)$ is $\mathbb{R} - \{0\}$.

1.2.6 Intercepts

Definition: A y-value at which a graph touches the y-axis is called a ***y*-intercept**. An x-value at which a graph touches the x-axis is called an ***x*-intercept**.

Note that the graph does not necessarily cross the axis at an intercept (though it typically will). The following graph has x-intercepts of $x = 1$ and $x = 3$ and a y-intercept of $y = 2$.



The function f itself can be used to find where its graph intersects an axis. A y-intercept for the graph of function $y = f(x)$ is found by evaluating $f(0)$, while x-intercepts are found by solving $f(x) = 0$ for x. For a relation one sets $x = 0$ and solves for y to find any y-intercepts and sets $y = 0$ and solves for x to find any x-intercepts.

Conversely, one can solve the equation $f(x) = 0$ approximately by graphing the function $y = f(x)$ by plotting points and then observing the approximate x-intercepts. These are the solutions to $f(x) = 0$ since at the x-intercepts one has $y = 0$. Such approximate solutions can then be further refined using numerical techniques, such as **Newton's method**, that require an initial guess for a solution as a starting point to the algorithm.

1.2.7 Symmetry of Functions

Definition: A function $f(x)$ is said to be

even if $f(-x) = f(x)$, and
odd if $f(-x) = -f(x)$.

Example 1-6

Find the intercepts of the following functions and determine if the functions are even or odd:

1. $f(x) = x^4 + 3x^2$
2. $f(x) = x^3 - 2x^2 + 3$
3. $f(x) = \dfrac{2x^2+3}{x^3-2x}$

Solution:

1. $f(x) = x^4 + 3x^2 \implies f(0) = 0 \implies y\text{–intercept: } y = 0$
$f(x) = 0 \implies x^2(x^2+3) = 0 \implies x = 0 \text{ or } x^2 = -3 \text{ (no solution)} \implies x = 0$
$\implies x\text{–intercept: } x = 0$
$f(-x) = (-x)^4 + 3(-x)^2 = (-1)^4x^4 + 3(-1)^2x^2 = x^4 + 3x^2 = f(x) \implies f \text{ is even}$

2. $f(x) = x^3 - 2x^2 + 3 \implies f(0) = 3 \implies y\text{–intercept: } y = 0$

$$\begin{aligned} f(x) = 0 &\implies x^3 - 2x^2 + 3 = 0 \\ &\implies x^3 + x^2 - 3x^2 + 3 = 0 \\ &\implies x^2(x+1) - 3(x^2-1) = 0 \\ &\implies x^2(x+1) - 3(x+1)(x-1) = 0 \\ &\implies (x+1)(x^2 - 3x + 3) = 0 \\ &\implies \begin{cases} x = -1 \\ x^2 - 3x + 3 = 0 \implies x = \frac{3\pm\sqrt{9-12}}{2} = \frac{3\pm\sqrt{-3}}{2} \implies \text{No real roots} \end{cases} \end{aligned}$$

* Therefore x–intercept: $x = -1$

$f(-x) = (-x)^3 - 2(-x)^2 + 3 = (-1)^3x^3 - 2(-1)^2x^2 + 3 = -x^3 - 2x^2 + 3 = -(x^3 + 2x^2 - 3)$
So f is neither even nor odd.

3. $f(x) = \dfrac{2x^2+3}{x^3-2x} \implies f(0) = \dfrac{3}{0} \implies f \text{ is undefined at } 0 \implies \text{No } y\text{–intercept}$
$f(x) = 0 \implies 2x^2 + 3 = 0 \implies x^2 = -3/2 \implies \text{No solution} \implies \text{No } x\text{–intercepts}$
$f(-x) = \dfrac{2(-x)^2+3}{(-x)^3-2(-x)} = \dfrac{2x^2+3}{-x^3+2x} = \dfrac{2x^2+3}{(-1)(x^3-2x)} = -\dfrac{2x^2+3}{x^3-2x} = -f(x) \implies f \text{ is odd}$

Further Questions:

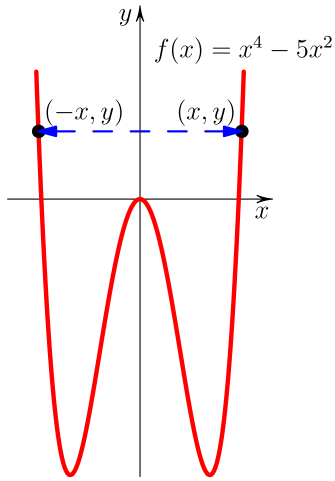
Find the intercepts of the following functions and determine if the functions are even or odd.

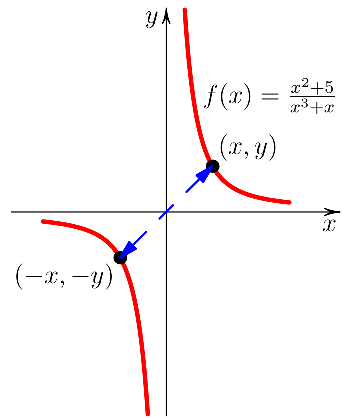
1. $f(x) = x^4 - 5x^2$
2. $f(x) = x^3 - 3x^2 + 4$
3. $f(x) = \dfrac{x^2+5}{x^3+x}$

Note:

- An even function is symmetric with respect to the y-axis. (Every point (x, y) on the graph has a corresponding point $(-x, y)$ that also sits on the graph.)
- An odd function is symmetric with respect to the origin. (Every point (x, y) on the graph has a corresponding point $(-x, -y)$ that also sits on the graph.)

The graphs of the even and odd functions from the last example illustrate these symmetries.



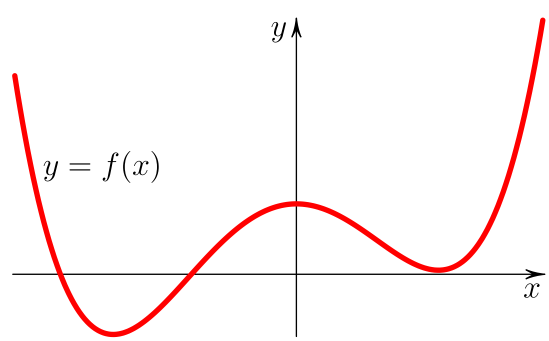


Recognizing the symmetry of a function is therefore useful when graphing, since the graph of a symmetric function to the left of the origin ($x < 0$) is completely determined by the graph to the right of it ($x > 0$). As such one only needs to calculate points on half the domain when graphing symmetric functions.

Symmetry is also helpful in solving equations. If one desires solutions to $f(x) = 0$ and one observes that $f(x)$ is symmetric (either even or odd), then if x_0 is a solution to the equation then $-x_0$ must also be a solution. This follows since, for a symmetric function $f(-x_0) = \pm f(x_0) = \pm 0 = 0$. Graphically this can be seen for the function $f(x) = x^4 - 5x^2$ above, where the function crosses the x-axis at $\sqrt{5}$ and $-\sqrt{5}$. These are both solutions to $x^4 - 5x^2 = 0$ since the graph $y = f(x)$ crosses the x-axis ($y = 0$) at those values.

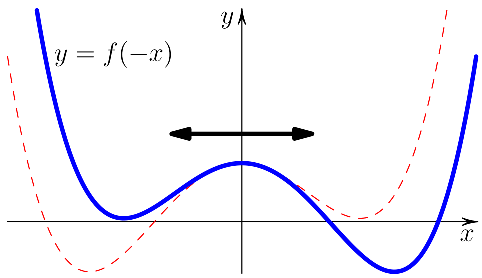
1.2.8 Reflection and Translation

It is possible to construct new functions from an original function $f(x)$ whose graphs are reflections about the y-axis or x-axis or horizontal or vertical translations of the original graph. Consider the function $f(x)$ defined by the following graph:



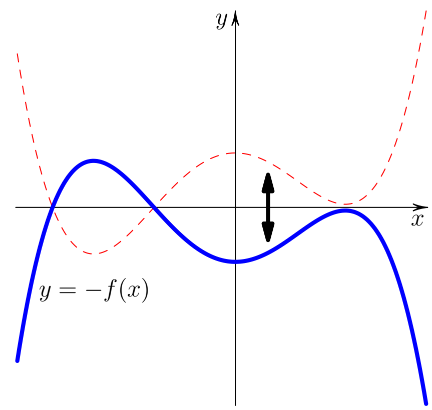
Evaluating the following transformations $g(x)$ of $f(x)$ at a few points x confirms the following results. (Alternatively pick an actual function, work out the transformation and plot the result.)

$y = g(x) = f(-x)$:



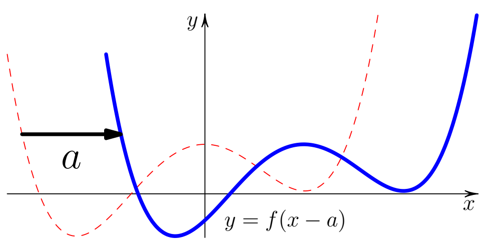
Reflection about the y-axis.

$y = g(x) = -f(x)$:



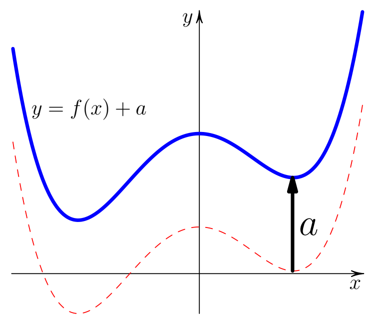
Reflection about the x-axis.

$y = g(x) = f(x - a)$:



For $a > 0$ translation is to the right, while $a < 0$ results in translation to the left.

$y = g(x) = f(x) + a$:

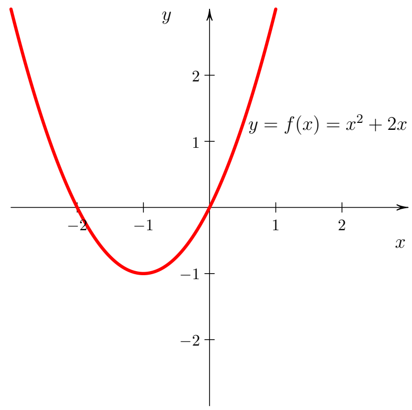


For $a > 0$ translation is upward, while $a < 0$ results in translation downward.

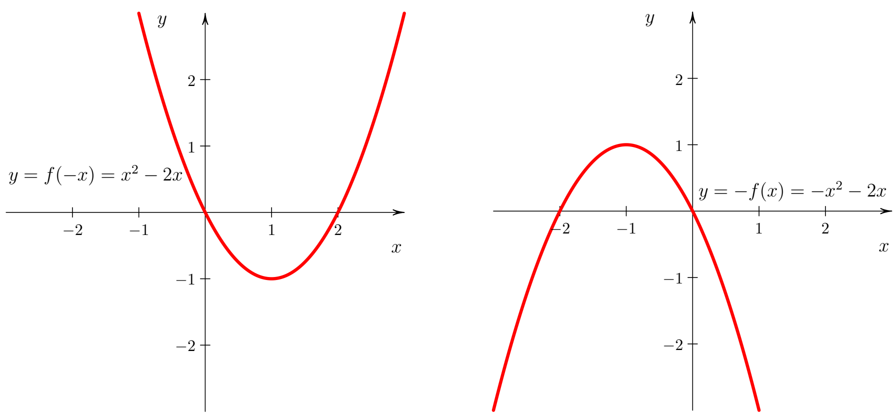
Example 1-7

Graphs of the parabola $y = x^2 - 2x$ and functions related to it by reflection and translation are shown below.

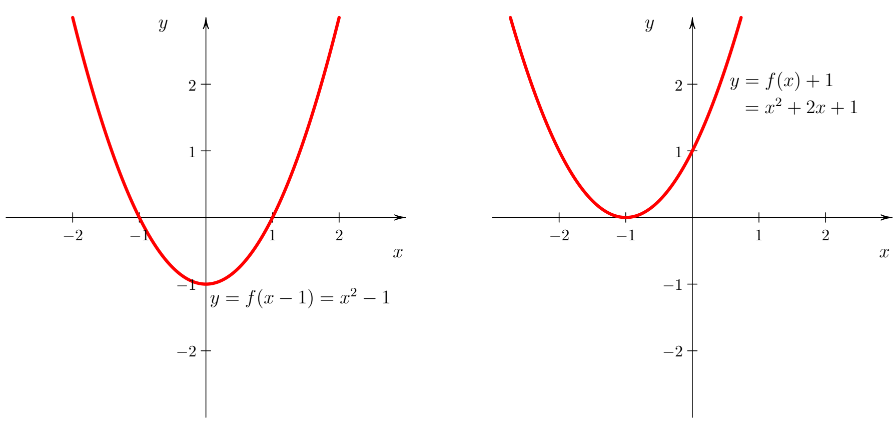
Original Function



Reflections about the *x*-axis and *y*-axis



Translations horizontally and vertically by 1



Exercise 1-2

Answers: Page 237

1-16: Find the domain and the x- and y-intercepts (if there are any) of the given functions.

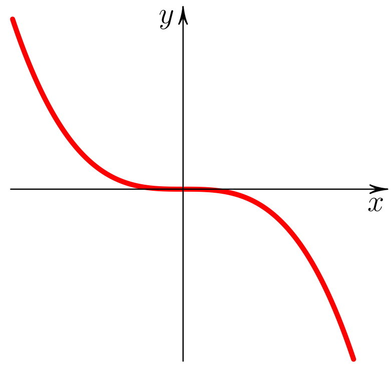
1. $f(x) = x^3$
2. $h(x) = \sqrt{x-6}$
3. $f(x) = x^2 + 4x + 4$
4. $g(x) = x^4 + 4x^3 - 5x^2$
5. $f(x) = \dfrac{1}{x-1}$
6. $h(x) = \dfrac{1}{x^2+4x+4}$
7. $p(x) = \sqrt{4-x^2}$
8. $f(x) = 3x^2 + 5x + 2$
9. $g(x) = x^3 + 3x - 4$
10. $h(x) = \dfrac{x+5}{x+7}$
11. $f(x) = \dfrac{10}{2x^2-5x-3}$
12. $g(x) = \sqrt{4-x}$
13. $p(x) = \sqrt{x^2-10}$
14. $h(x) = \sqrt{\dfrac{x+6}{2x-3}}$
15. $f(x) = \dfrac{\sqrt{x^2-10}}{x^2+10}$
16. $f(x) = \dfrac{1}{\sqrt{x+2}} - \dfrac{1}{x}$

17-32: Determine whether each of the given functions is even, odd or neither.

17. $f(x) = 3x^2$
18. $g(x) = 2x^3$
19. $h(x) = 3x^2 + 2x^3$
20. $f(t) = -6t^3$
21. $g(u) = u^3 - u^2$
22. $f(x) = x^3 + x^5$
23. $f(x) = \sqrt{4-x^2}$
24. $h(x) = \dfrac{x^2+1}{\sqrt{4-x^2}}$
25. $f(x) = 5x^2 + 3$
26. $g(t) = 4t^3 + t$
27. $h(z) = z^5 + 1$
28. $f(t) = \sqrt{t^2+5}$
29. $g(x) = \dfrac{x}{x^4+3}$
30. $h(u) = \dfrac{3+u^2}{u+1}$
31. $p(x) = \dfrac{x^5}{x^3+x}$
32. $u(s) = \sqrt{\dfrac{s^6+4}{s^{10}+7}}$

---

33. Here is a graph of a function f:



(a) Sketch the graph of $-f(x)$.
(b) Sketch the graph of $f(-x)$.
(c) Sketch the graph of $f(x+1)$.
(d) Sketch the graph of $f(x)+1$.
(e) Is f even, odd, or neither?

1.2.9 Types of Functions

Polynomials: $\boxed{f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n}$ where n is a positive integer. The a_i are constants called the **coefficients** of the polynomial while n is its **degree** (assuming $a_n \neq 0$).

Example 1-8

The following are polynomials:

$$y = 2x + 9 \quad \text{(A straight line)}$$

$$f(x) = x^4 - 5x^2 \quad \text{(See our previous plot.)}$$

$$f(x) = x^{10} + 5x^7 + 3x^4 + 10$$

Note $D = \mathbb{R}$ for a polynomial.

Rational Functions: $\boxed{f(x) = \dfrac{P(x)}{Q(x)}}$ where $P(x)$ and $Q(x)$ are polynomials.

Example 1-9

The following are rational functions.

$$f(x) = \frac{x^2+5}{x^3+x} \quad \text{(See our previous plot.)}$$

$$g(x) = \frac{x+2}{x^2-1}$$

The domain for a rational function is all the real numbers except for those where the denominator vanishes. In set notation symbols $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$. (Read $\in$ as "in" and the vertical bar (|) as "such that".)

Example 1-10

Factoring the denominator polynomials we see the domains of our rational functions are:

$$\text{Domain for } f(x) : x^3 + x = x(x^2+1) \;\Rightarrow\; D = \mathbb{R} - \{0\}$$

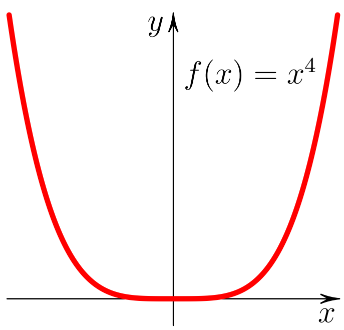
$$\text{Domain for } g(x) : x^2 - 1 = (x+1)(x-1) \;\Rightarrow\; D = \mathbb{R} - \{-1, 1\}$$

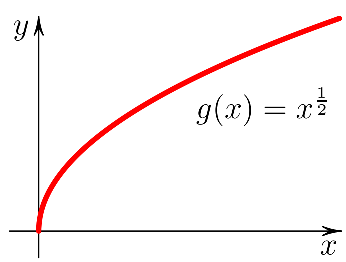
Power Functions: $\boxed{f(x) = x^r}$ where r is any real number.

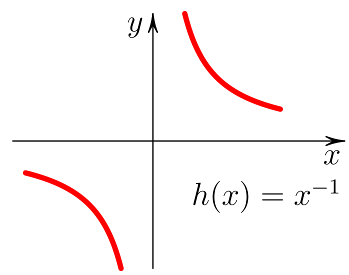
Example 1-11

The following are power functions.

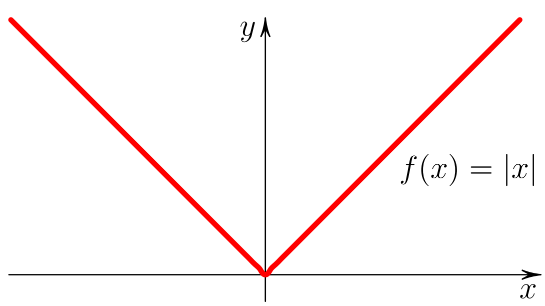
$f(x) = x^4$ $\qquad$ $g(x) = x^{\frac{1}{2}} = \sqrt{x}$ $\qquad$ $h(x) = x^{-1} = \dfrac{1}{x}$







Absolute Value Function: $\boxed{f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}}$ (A **piecewise-defined** function.)



Note that $\sqrt{x^2} \neq x$! To see this, notice that for $x = -2$ we have $\sqrt{(-2)^2} = \sqrt{4} = 2$, since square root returns a single value (as required for a function) defined to be the positive square root. So here $\sqrt{(-2)^2} = 2 = -(-2)$ and, in general, for $x < 0$ one has $\sqrt{x^2} = -x$. For $x \geq 0$ on has $\sqrt{x^2} = x$. The two possibilities can be conveniently combined using the absolute value function to get the correct result for all real values of x, namely $\boxed{\sqrt{x^2} = |x|}$. Thus the equation $x^2 = 4$ is in fact equivalent to $|x| = \sqrt{4} = 2$ which, in turn is equivalent to $x = 2$ or $-x = 2$ (so $x = -2$).[5]

[5] The underlying problem here is that the function $f(x) = x^2$ is not *invertible*. The square root function is only the inverse of $f(x) = x^2$ restricted to the domain $[0, \infty)$ (which is invertible). One cannot, therefore, take the square root of x^2 when solving an equation and get x without typically missing a solution. A related problem occurs when one (over⇒)

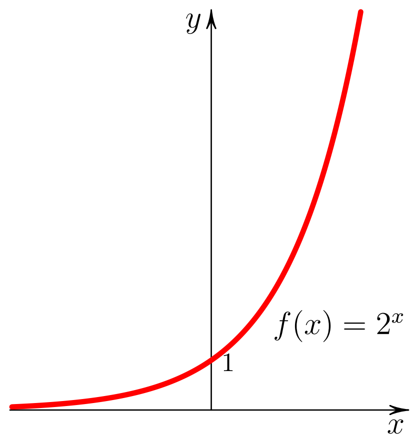
Exponential Functions: $\boxed{f(x) = b^x}$ where b is a positive real number (the **base**).

Example 1-12

The following are exponential functions.

$$f(x) = 2^x$$

$$g(x) = e^x \text{ where } e = 2.71828\ldots$$



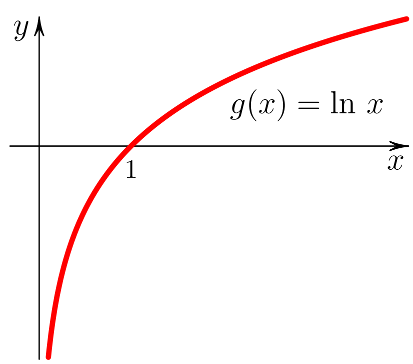
Logarithmic Functions: $\boxed{f(x) = \log_b x}$ where b is a positive real number.

Example 1-13

The following are logarithmic functions.

$$f(x) = \log_2 x$$

$$g(x) = \log_e x = \ln x$$



squares both sides of an equation to remove a radical. In this case the new equation may have solutions that are not solutions to the original equation, so-called *extraneous roots*, and one must check all solutions to the new equation back in the original equation to avoid them. As an example, the equation $\sqrt{2x+3} = x$, upon squaring both sides gives the equation $2x + 3 = x^2$ which has solutions $x = -1$ or $x = 3$. Only $x = 3$ is a solution to the original equation; $x = -1$ is extraneous. In general, care must be taken when taking even roots and powers. However, odd root and power functions (which are invertible on their natural domains) behave as expected, $\sqrt[n]{x^n} = x$ and $(\sqrt[n]{x})^n = x$ for n odd.

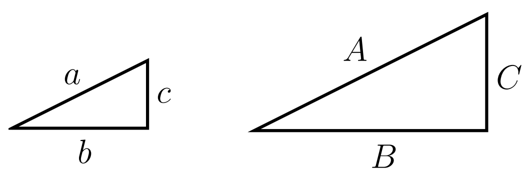
Trigonometric Functions:

Example 1-14

The following are trigonometric functions.

$$f(x) = \sin x \qquad\qquad y = \csc x$$

Trigonometric functions of angles that are less than 90° (acute angles) are defined to be functions of **similar** right triangles. Two geometrical figures are similar if they have the same shape, but potentially different sizes:



Similarity of regular polygons means that the corresponding angles in the figures are equal. The lengths of corresponding sides, however, must only form a proportion. For the similar triangles above one has

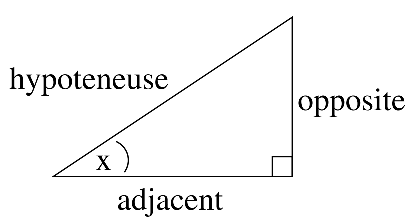
$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C}\,.$$

The first equality implies

$$\frac{a}{b} = \frac{A}{B}\,,$$

and, in general, the ratio of the lengths of two sides in a figure will be the same for the corresponding sides of all similar figures.

The basic trigonometric functions are defined as ratios of the sides of a right triangle for a given angle x. Since these ratios are the same for two similar triangles these functions are well defined.

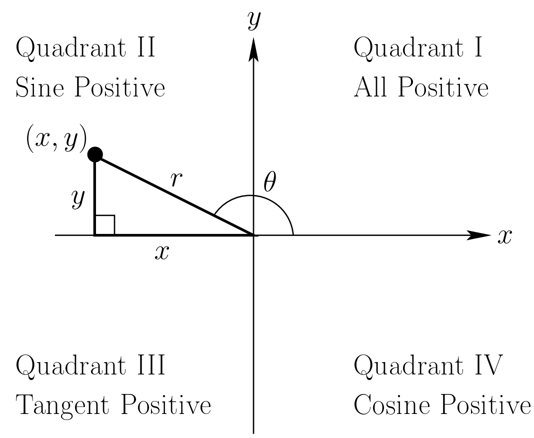


$$\sin x = \frac{opp}{hyp} \qquad \csc x = \frac{1}{\sin x} = \frac{hyp}{opp}$$

$$\cos x = \frac{adj}{hyp} \qquad \sec x = \frac{1}{\cos x} = \frac{hyp}{adj}$$

$$\tan x = \frac{opp}{adj} = \frac{\sin x}{\cos x} \qquad \cot x = \frac{1}{\tan x} = \frac{adj}{opp} = \frac{\cos x}{\sin x}$$

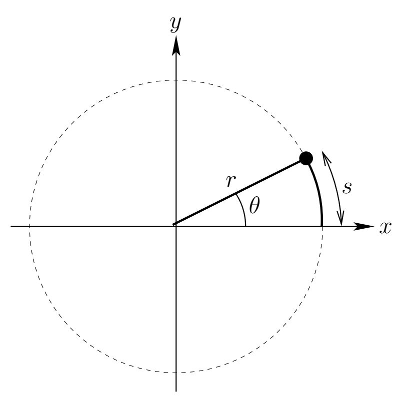
For angles larger than 90° the trigonometric functions are defined by considering Cartesian Coordinates and choosing a point (x, y) on the terminal ray of the angle and defining the functions identically as in the acute case with the replacements *adjacent* $\rightarrow x$, *opposite* $\rightarrow y$, and *hypotenuse* $\rightarrow r$ where $r = \sqrt{x^2 + y^2}$. So for instance, now using θ (pronounced theta) for the angle, $\tan\theta = \frac{y}{x}$, etc. Since x, and y can be negative the trigonometric functions themselves can also now be negative. The situation is summarized in the next diagram.



The sign of the reciprocal trigonometric functions (cosecant, secant, and cotangent) follow the same behaviour as sine, cosine, and tangent respectively.

Radian Measure

We can measure angles either in degrees (1 circle=360°) or in terms of the length of arc on a circle that an angle subtends. To do the latter the angle, in radians, is just the ratio of the length of the arc s over the radius r, so $\theta = \frac{s}{r}$.



By taking the ratio of s over r we are guaranteed that our angle calculation will be independent of the choice of the size of the circle we use since two different circles will generate two **sectors** of a circle (i.e. pieces of pie) that are similar figures.

If we draw the angle in a **unit circle** so $r = 1$ then the angle is exactly the arc length measured. Since for a unit circle the circumference is $2\pi r = 2\pi(1) = 2\pi$ the angle of an entire circle is 1 circle=2π radians. It therefore follows that

$$\pi \text{ rad} = 180°$$

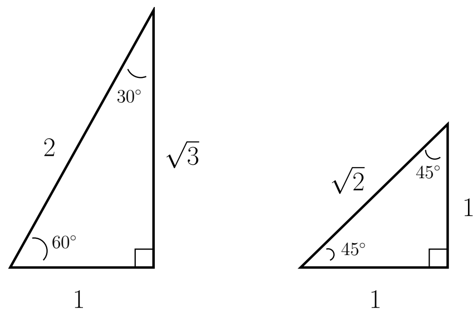
Unlike degrees, radians are a fake unit and we don't need to write it. (Both s and r have the same length units so their ratio is dimensionless.) An angle reported with no units is assumed to be in radians. When we later work with angles the formulae we derive will assume angles are given in radians.[6]

[6] Remember when doing any trigonometric evaluation formulae in calculus usually require angles to be in radians not degrees. Most calculators have a setting to put the calculator in "radian mode" so that all angles arguments (and angles returned from inverse trig functions like $\arcsin(x)$) are assumed to be in radians.

Using the previous relation some basic angle measurements in radians are:

$$30^\circ = \frac{\pi}{6} \qquad 45^\circ = \frac{\pi}{4} \qquad 60^\circ = \frac{\pi}{3} \qquad 90^\circ = \frac{\pi}{2} \qquad 180^\circ = \pi \qquad 360^\circ = 2\pi$$

Evaluating trigonometric functions of some of the common angles can be done using the $30^\circ - 60^\circ - 90^\circ$ and $45^\circ - 45^\circ - 90^\circ$ triangles:[7]

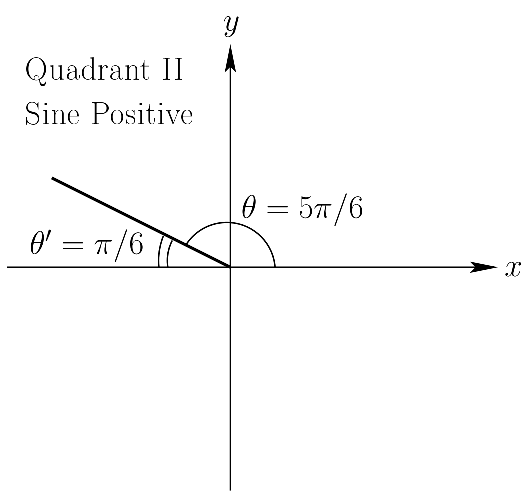


So, for instance, $\sin 60^\circ = \frac{opp}{hyp} = \frac{\sqrt{3}}{2}$. Or, in radians, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. For angles of 0 or $90^\circ = \frac{\pi}{2}$ one should know $\cos(0) = 1$, $\sin 0 = 0$, $\cos\frac{\pi}{2} = 0$, and $\sin\frac{\pi}{2} = 1$. These results may be found by considering the unit circle points $(x, y) = (1, 0)$ and $(x, y) = (0, 1)$ on the terminal ray of angles 0 and $\pi/2$ respectively on the previous quadrant diagram (so $r = 1$, $\cos\theta = x$, $\sin\theta = y$).

Trigonometric functions evaluated in quadrants other than the first may be found by evaluating the trigonometric function of the **reference angle** (the positive acute angle made between the terminal ray of the angle in question and the x-axis) multiplied by the appropriate sign of the trigonometric function for the given quadrant. So, for instance, to evaluate $\cos(5\pi/6)$ we note the angle $\theta = 5\pi/6 = 150^\circ$ is in the second quadrant and has reference angle

$$\theta' = \pi - 5\pi/6 = \pi/6 = 30^\circ\,,$$

as shown on the next diagram.



Since cosine is negative in the second quadrant we have

$$\cos(5\pi/6) = -\cos(\pi/6) = -\frac{\sqrt{3}}{2}\,.$$

[7]Note that the side lengths of the $30^\circ - 60^\circ - 90^\circ$ triangle arise from bisecting one 60° angle of an equilateral triangle of side length 2. For the $45^\circ - 45^\circ - 90^\circ$ triangle the legs of the isosceles triangle are chosen to be length 1. The remaining sides in both cases, of length $\sqrt{3}$ and $\sqrt{2}$ respectively, can then be determined by the Pythagorean Theorem.

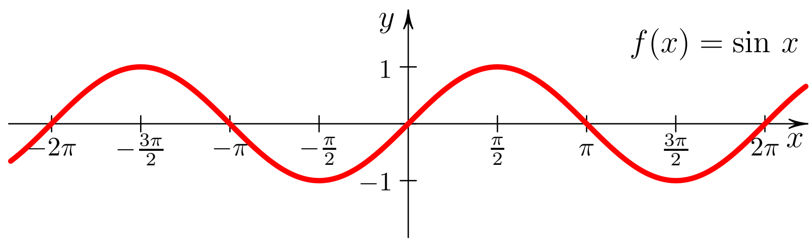
Answers:
Page 238

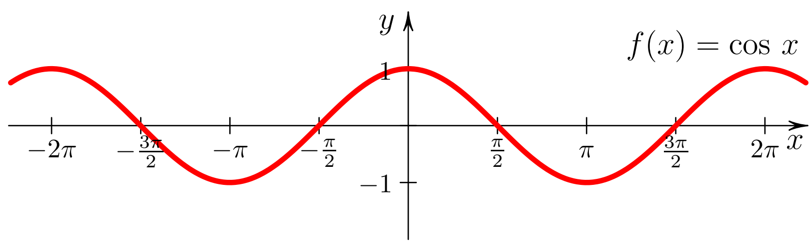
Exercise 1-3

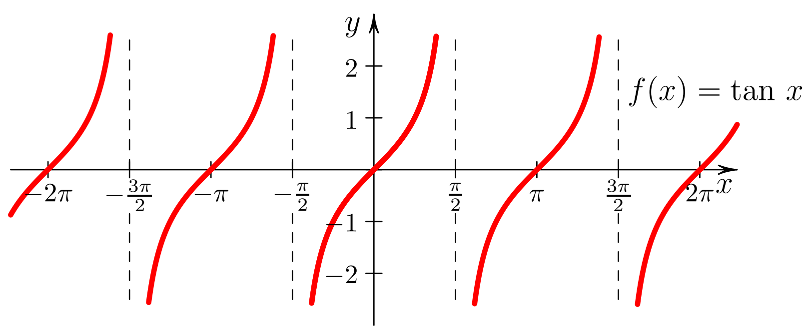
1-8: Find the exact values of the following trigonometric functions. Check your results on your calculator.

1. $\sin(\pi/3)$
2. $\cos(135°)$
3. $\sin(5\pi/6)$
4. $\cos(3\pi/2)$
5. $\csc(7\pi/4)$
6. $\cot(210°)$
7. $\sec(\pi)$
8. $\tan(13\pi/4)$

Graphs of the basic trig functions with x in radians are as follows:







The graphs of the reciprocal trig functions (csc, sec, cot) will *blow up* at any value where the corresponding trig function (sin, cos, tan) is zero.

Trigonometric Identities

Just like we saw algebraic identities, there are basic trigonometric identities which are true for all angles:[8]

Pythagorean Relations:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Symmetry Relations:

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta)$$

Periodicity:

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$$

Addition and Subtraction Formulae:

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

Double Angle Formulae:

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

From the last it follows that:

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

Note in the above identities that the notation $\sin^2 x$ means $(\sin x)^2$, etc. That is we take the sine of x first and then square that result.

From these identities other identities can be proven.

[8]Note that the various trigonometric identities on this page follow readily from the three basic identities which should be memorized:

1. $\sin^2 x + \cos^2 x = 1$
2. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
3. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

Dividing 1. by $\cos^2 x$ gives the Pythagorean identity involving tangent and secant, while dividing 1. by $\sin^2 x$ gives the identity involving cotangent and cosecant. The Symmetry Relations can be obtained by setting $x = 0$ in the subtraction versions of 2. and 3. The Periodicity results follow from setting $y = 2\pi$ in 2. and 3. and evaluating the trigonometric functions at 2π. The Double Angle Formulae follow by setting $y = x$ in the addition versions of 2. and 3. The final identities then follow from the Double Angle Formulae as mentioned above.

Example 1-15

Prove the identity.

$-\sin x = \dfrac{\cos x}{\tan x} - \csc x$

Solution:

Start by working on the more complicated side and apply identities to try to arrive at the simpler side.

$\dfrac{\cos x}{\tan x} - \csc x = \dfrac{\cos x}{\frac{\sin x}{\cos x}} - \dfrac{1}{\sin x} = \cos x \cdot \dfrac{\cos x}{\sin x} - \dfrac{1}{\sin x} = \dfrac{\cos^2 x - 1}{\sin x} = \dfrac{-\sin^2 x}{\sin x} = -\sin x$

Note that this identity is technically only true where both sides of it are actually defined since $-\sin x$ is defined for all x while the right hand side will be undefined wherever $\tan x$ is zero or $\tan x$ or $\csc x$ are undefined.

Further Questions:

Prove the following identities:

1. $1 + \sin 2\theta = (\sin\theta + \cos\theta)^2$
2. $\dfrac{\sin x}{\cot x} = \sec x - \cos x$
3. $\tan^2\theta \sin^2\theta = \tan^2\theta - \sin^2\theta$

Ultimately the usefulness of trigonometric identities, as with the factoring identities, is to simplify expressions, often to aid in the solution of trigonometric equations. See the review in Appendix B on solving trigonometric equations.

1.2.10 Composition of Functions

Given two functions f and g the **composite function**, denoted $f \circ g$, is defined by:

$$f \circ g(x) = f(g(x))$$

i.e. We replace the x in $f(x)$ by $g(x)$.

The new function $f \circ g$ is defined wherever both $g(x)$ and $f(g(x))$ are defined.

Example 1-16

For the following pairs of functions find $f \circ g$ and $g \circ f$ and the domains of the function compositions.

1. $f(x) = x^2 + 3x$, $g(x) = \sqrt{x+2}$

   Solution:

   $f \circ g(x) = f(g(x)) = f\left(\sqrt{x+2}\right) = \left(\sqrt{x+2}\right)^2 + 3(\sqrt{x+2}) = x + 2 + 3\sqrt{x+2}$

   Domain $f \circ g$: $x + 2 \geq 0 \implies x \geq -2 \implies D = [-2, \infty)$

   $g \circ f(x) = g(f(x)) = g\left(x^2 + 3x\right) = \sqrt{(x^2+3x)+2} = \sqrt{x^2+3x+2}$

   Domain $g \circ f$: $x^2 + 3x + 2 \geq 0 \implies (x+1)(x+2) \geq 0$

   $$\implies \begin{cases} x \leq -1 & \text{and } x \leq -2 \\ & \text{or} \\ x \geq -1 & \text{and } x \geq -2 \end{cases} \implies \begin{cases} x & \leq -2 \\ & \text{or} \\ x & \geq -1 \end{cases} \implies D = (-\infty, -2] \cup [-1, \infty)$$

2. $f(x) = \dfrac{x+3}{x-1}\,,\; g(x) = \dfrac{1}{x}$

   Solution:

   $$f \circ g(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \frac{\left(\frac{1}{x}\right)+3}{\left(\frac{1}{x}\right)-1} = \frac{\frac{1}{x}+3}{\frac{1}{x}-1}\cdot\frac{x}{x} = \frac{1+3x}{1-x} \quad (\text{if } x \neq 0)$$

   Domain $f \circ g$: $x \neq 0$ and $1 - x \neq 0 \implies x \neq 0$ and $x \neq 1 \implies D = \mathbb{R} - \{0, 1\}$

   $$g \circ f(x) = g(f(x)) = g\left(\frac{x+3}{x-1}\right) = \frac{1}{\left(\frac{x+3}{x-1}\right)} = \frac{x-1}{x+3} \quad (\text{if } x \neq 1)$$

   Domain: $g \circ f$: $x \neq 1$ and $x + 3 \neq 0 \implies x \neq 1$ and $x \neq -3 \implies D = \mathbb{R} - \{-3, 1\}$

Further Questions:

For the following pairs of functions find $f \circ g$ and $g \circ f$ and the domains of the function compositions.

1. $f(x) = x^2 + x - 5\,,\; g(x) = 2x + 1$
2. $f(x) = \dfrac{x^2 - 2x}{x^3 + 8}\,,\; g(x) = x + 1$

Recognizing function composition can help solve equations. For instance, to solve

$$y^{10} + y^5 - 6 = 0$$

we notice that the expression on the left hand side is a function composition as we can rewrite the equation

$$(y^5)^2 + (y^5) - 6 = 0\,.$$

If we let $x = y^5$ we can first solve $x^2 + x - 6 = 0$ to get $x = -3$ or $x = 2$ and then next solve $y^5 = -3$ or $y^5 = 2$ to get the final solutions $y = \sqrt[5]{-3} = -\sqrt[5]{3}$ or $y = \sqrt[5]{2}$. In general, solving an equation where the expression involving the variables is a composition of two functions requires solving the equation involving the outer function first followed by solving the inner function equalling those solutions found.[9]

[9] This procedure further generalizes when one has a composition of more than two functions, like $f(g(h(z))) = 0$. Simply solve $f(x) = 0$ to get solutions x_i. Next solve $g(y) = x_i$ to get solutions y_i. Then solve $h(z) = y_i$ to get solutions z_i. These are the solutions to the original equation.

Answers: Page 239

Exercise 1-4

1. Suppose that $f(x) = \dfrac{1}{x+2}$. Determine:

   (a) $f(x+2)$

   (b) $f(f(x))$

2-5: Find the composite functions $f \circ g$ and $g \circ f$ and their domains for the given functions.

2. $f(x) = 2x^3$, $g(x) = \sqrt{x^2+3}$.
3. $f(x) = 3x^2 + 6x + 4$, $g(x) = 3x - 2$
4. $f(z) = \sqrt{z^2+5}$, $g(z) = \dfrac{z}{z+1}$
5. $f(x) = \dfrac{2x+5}{x-4}$, $g(x) = x^2 + 3$

---

6. Find two functions $f(x)$ and $g(x)$ such that $f(g(x)) = \sqrt{x^2+1} - 3$.
7. Use function composition to solve the equation $\left(x^2-5\right)^2 + 7x^2 - 23 = 0$. (Hint: Note the equation can be rewritten as $\left(x^2-5\right)^2 + 7\left(x^2-5\right) + 12 = 0$.)

Chapter 1 Review Exercises

Answers: Page 239

1-5: Solve the given equations.

1. $2x^2 + 3x - 2 = 0$
2. $x^2 + x - 20 = 0$
3. $x^3 + 2x^2 - x - 2 = 0$
4. $x^4 - 5x^2 + 4 = 0$
5. $x^5 - 4x^3 - x^2 + 4 = 0$

6-9: Find the domain and the x- and y-intercepts of the given functions.

6. $h(x) = \dfrac{2x-3}{5x+4}$
7. $f(x) = \sqrt{2x^2 - 8}$
8. $g(x) = \sqrt{\dfrac{2x+1}{x+5}}$
9. $p(x) = \dfrac{\sqrt{x+8}}{x-3}$

10-13: Determine whether each of the given functions is even, odd or neither.

10. $f(x) = \dfrac{x^4+3}{x^2+1}$
11. $g(t) = t^{2/3}$
12. $h(z) = z\sqrt{z^2+1}$
13. $g(x) = \dfrac{x^3}{x^6+5} - x$

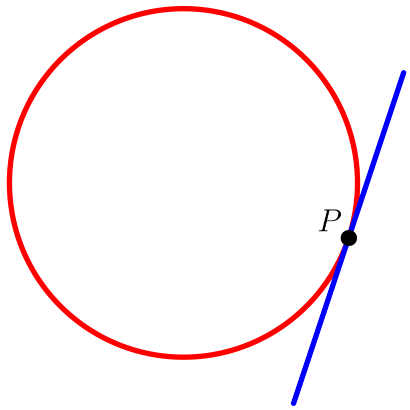
14-16: Find the composite functions $f \circ g$ and $g \circ f$ and their domains.

14. $f(x) = x^3 + 6$, $g(x) = x^{2/3}$
15. $f(t) = \dfrac{2t+5}{t-4}$, $g(t) = t^2 + 3$
16. $f(x) = \sqrt{x-1}$, $g(x) = \dfrac{x+5}{x+3}$

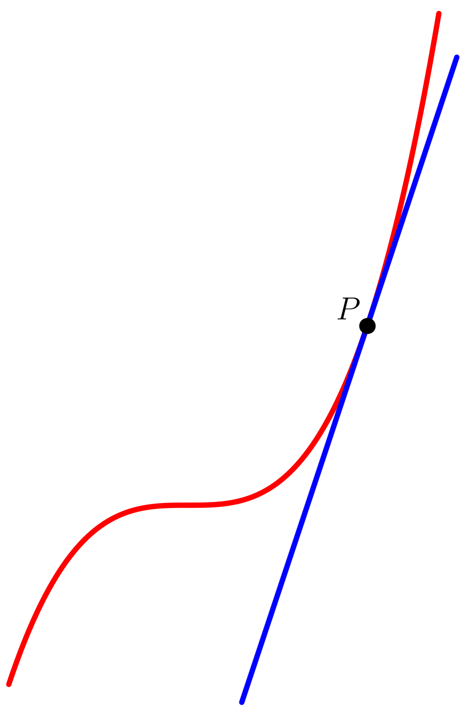
Chapter 2: Limits

2.1 Tangent to a Curve

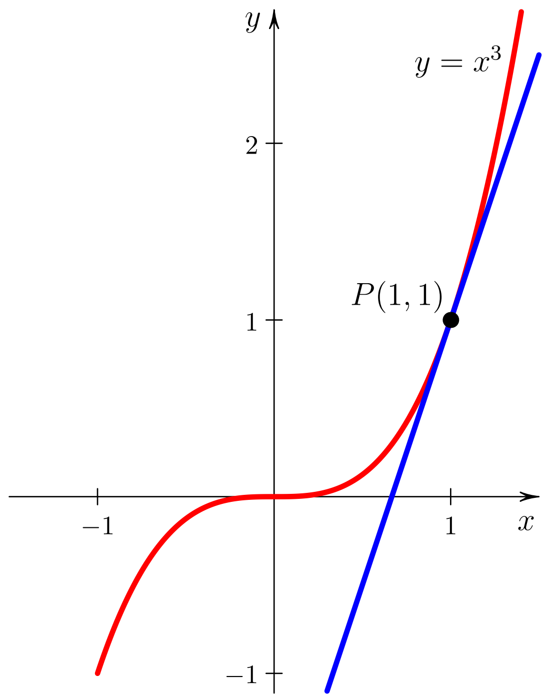
Most are familiar with a tangent line to a circle at a point P. It is the line that just clips the circle at only that point:



We can similarly imagine a tangent line to an arbitrary curve at a point P:



If we place the curve in a Cartesian Coordinate system we can represent it by the function $y = f(x) = x^3$ and the point of interest by its coordinates $P(1,1)$:



We can define our tangent line as follows:

Definition: A **tangent line** to a curve $y = f(x)$ at a point $P(x_0, y_0)$ is a (straight) line that touches the curve at P which, if extended, does not cross the curve at that point.

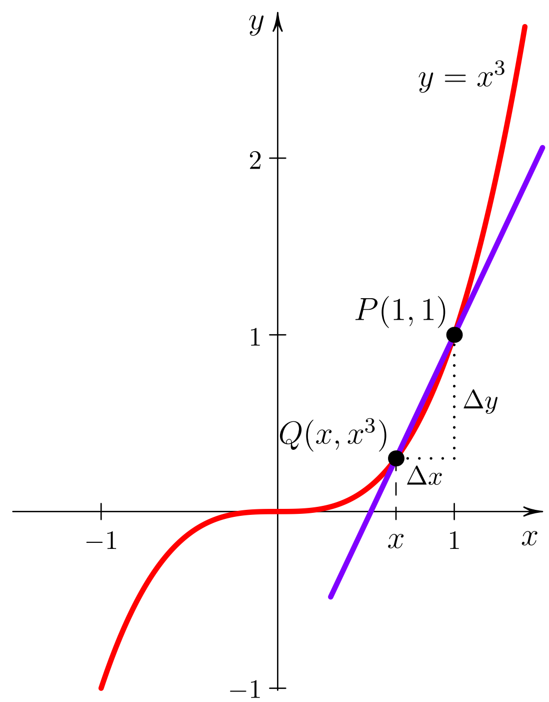
Now that we have coordinates we can ask what the tangent line at $P(1,1)$ actually is, by which we mean "What is the tangent line written as a function?". The equation of any line in two dimensions can be written:

$$\boxed{y = mx + b} \quad \Leftarrow \text{ Slope-Intercept Form}$$

where the constant $m = \frac{\Delta y}{\Delta x}$ is the **slope** which is a measure of the direction the line is going, while b is the ***y*-intercept** which indicates where to start the line on the y-axis. Another convenient recipe for determining the equation of the line is to use the slope m as before but also an arbitrary point (x_0, y_0) on the line as the starting point (no longer $(0, b)$):

$$\boxed{y = m(x - x_0) + y_0} \quad \Leftarrow \text{ Point-Slope Form}$$

In our case we know our tangent line goes through the point $(1,1)$ so this is (x_0, y_0). It remains to find the slope, call it m_t, for our tangent at P. We can approximate the slope of the tangent by taking a point Q near P and finding the slope of the line (the **secant**) between Q and P. If we let the x-coordinate of Q be x, a value close to 1, then the point Q is $Q(x, f(x)) = Q(x, x^3)$.



Then the slope of the secant line PQ is:

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{1 - x^3}{1 - x}$$

Since knowing the x-coordinate completely determines Q and consequently the slope, we see that m_{PQ} is really just a function of x:

$$m(x) = \frac{1 - x^3}{1 - x}$$

So, for instance, at the point $Q(1/2, 1/8)$ which corresponds to $x = 1/2$ the slope of the secant would be:

$$m_{PQ} = m\left(\frac{1}{2}\right) = \frac{1 - \left(\frac{1}{2}\right)^3}{1 - \frac{1}{2}} = \frac{\frac{7}{8}}{\frac{1}{2}} = \frac{7}{4}$$

Now we don't want the slope of any of these secants through P, we want the slope of the tangent m_t, so let's just plug in $x = 1$ to get it:

$$m_t = m(1) = \frac{1 - 1^3}{1 - 1} = \frac{0}{0}$$

The slope function $m(x)$ is not defined at $x = 1$! However if we take x values slightly smaller than 1 (so Q gets very close to P from the left as in the diagram) or x values slightly larger than 1 (so Q gets very close to P from the right) we have the following answers for the slope of those secants:

Approaching 1 From the Left

x	$m_{PQ} = m(x)$
0.5	1.75
0.9	2.71
0.99	2.9701
0.999	2.997001
0.9999	2.99970001

Approaching 1 From the Right

x	$m_{PQ} = m(x)$
1.5	4.75
1.1	3.31
1.01	3.0301
1.001	3.003001
1.0001	3.00030001

The results show that the closer x is to 1 (equivalently point Q is to P), the closer m_{PQ} is to 3. This suggests the slope of the tangent line at P should be $m_t = 3$. However, as shown above, this is not the value of the function $m(x)$ at $x = 1$ as that value is undefined. We need a new concept. We say the slope of the tangent line is the **limit** of the slopes of the secant lines and we write

$$m_t = \lim_{Q \to P} m_{PQ} \ .$$

Written in terms of the function $m(x)$,

$$m_t = \lim_{x \to 1} m(x) = \lim_{x \to 1} \frac{1 - x^3}{1 - x} = 3 \ .$$

We can now use the point-slope formula with $(x_0, y_0) = (1, 1)$ and $m_t = 3$ to get the equation of the tangent line at P to be

$$y = 3(x - 1) + 1 \ .$$

Multiplying this out gives the slope-intercept form of the tangent, namely

$$y = 3x - 2 \ .$$

Example 2-1

Consider the curve described by the function $y = f(x) = \dfrac{x - 4}{x^2 + 1}$.

(a) Show that the points $(0, -4)$ and $\left(2, -\frac{2}{5}\right)$ lie on the curve.

(b) Determine the slope of the secant line passing through the points $(0, -4)$ and $\left(2, -\frac{2}{5}\right)$.

(c) Let Q be the arbitrary point $(x, \frac{x-4}{x^2+1})$ on the curve. Find the slope of the secant line passing through Q and $(0, -4)$.

(d) Use your answer in (c) to determine the slope of the tangent line to the curve at the point $(0, -4)$. Evaluate the resulting limit numerically.

Solution:

(a) A point lies on a curve $y = f(x)$ if it satisfies the equation.
$(0, -4)$: $y = f(0) = \dfrac{0 - 4}{0 + 1} = -4$ $\qquad$ $\left(2, -\frac{2}{5}\right)$: $y = f(2) = \dfrac{2 - 4}{2^2 + 1} = -\dfrac{2}{5}$
So the points $(0, -4)$ and $\left(2, -\frac{2}{5}\right)$ lie on the curve.

(b) slope of secant between $(0, -4)$ and $Q = \left(2, -\frac{2}{5}\right)$: $m = \dfrac{-\frac{2}{5} - (-4)}{2 - 0} = \dfrac{-\frac{2}{5} + 4}{2} = \dfrac{\frac{18}{5}}{2} = \dfrac{9}{5}$

(c) slope of secant between $(0, -4)$ and $Q = (x, f(x))$:

$$m = \frac{\frac{x-4}{x^2+1} - (-4)}{x - 0} = \frac{\frac{x-4}{x^2+1} + 4\left(\frac{x^2+1}{x^2+1}\right)}{x} = \frac{x - 4 + 4x^2 + 4}{x(x^2 + 1)} = \frac{x(1 + 4x)}{x(x^2 + 1)} = \frac{1 + 4x}{x^2 + 1} \quad (\text{if } x \neq 0)$$

Note m itself is a function of x, $m = m(x)$, since it depends on the arbitrary x coordinate of Q.

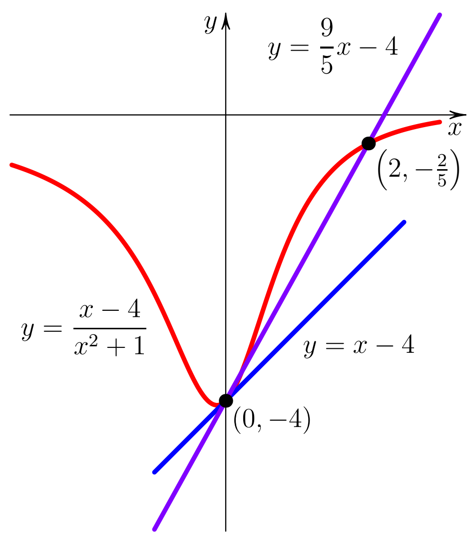
(d) The slope $m(x)$ of the arbitrary secant between $(0, -4)$ and $Q(x, f(x))$ is not defined at $x = 0$. We can evaluate points Q close to $(0, -4)$ by choosing values of x close to 0 to the left and right of that value.

Approaching 0 From the Left		Approaching 0 From the Right	
x	$m_{PQ} = m(x)$	x	$m_{PQ} = m(x)$
-0.5	-0.8	0.5	2.4
-0.1	0.594059405941	0.1	1.38613861386
-0.01	0.959904009599	0.01	1.0398960104
-0.001	0.995999004001	0.001	1.003998996
-0.0005	0.9979997505	0.0005	1.0019997495
-0.0001	0.999599990004	0.0001	1.00039999

The secant slopes approach the value 1, the slope of the tangent m_t. We write

$$m_t = \lim_{x\to 0} m(x) = \lim_{x\to 0} \frac{1+4x}{x^2+1} = 1$$

One can check the answer graphically. Since the secant and tangent lines go through $(0,-4)$ we have immediately that that the y-intercept b of both lines is $b=-4$. Using our slopes from (b) and (b) the equation of the secant between $(0,-4)$ and $\left(2,-\frac{2}{5}\right)$ is $y=\frac{9}{5}x-4$ while the equation of the tangent at $(0,-4)$ is $y=1x-4$. A graph of the original curve and these lines is:



Further Example:
Consider the curve described by the function $y = 5x^4 + 3x^2 - 10$.

(a) Show that the points $(0,-10)$ and $(1,-2)$ lie on the curve.

(b) Determine the slope of the secant line passing through the points $(0,-10)$ and $(1,-2)$.

(c) Let Q be the arbitrary point $(x, 5x^4+3x^2-10)$ on the curve. Find the slope of the secant line passing through Q and $(1,-2)$.

(d) Use your answer in (c) to determine the slope of the tangent line to the curve at the point $(1,-2)$. Evaluate the resulting limit numerically.

Answers: Page 240

Exercise 2-1

1. Consider the curve described by the function $y = x^3 + x^2 - 2x + 3$.

   (a) Show that the points $(-1, 5)$ and $(0, 3)$ lie on the curve.

   (b) Determine the slope of the secant line passing through the points $(0, 3)$ and $(-1, 5)$.

   (c) Let Q be the arbitrary point $(x, x^3 + x^2 - 2x + 3)$ on the curve. Find the slope of the secant line passing through Q and $(-1, 5)$.

   (d) Use your answer in (c) to determine the slope of the tangent line to the curve at the point $(-1, 5)$.

2.2 Limit of a Function

In the last section we came up with a function $m(x)$ and considered its limiting behaviour as $x \to 1$. In general we can explore limits numerically for an arbitrary function:

Example 2-2

Investigate the limiting behaviour of the function $f(x) = x^2 + 1$ for x near 2:

Approaching 2 From the Left		Approaching 2 From the Right	
x	$f(x)$	x	$f(x)$
1.5	3.25	2.5	7.25
1.9	4.61	2.1	5.41
1.99	4.9601	2.01	5.0401
1.999	4.996001	2.001	5.004001
1.9995	4.99800025	2.0005	5.00200025
1.9999	4.99960001	2.0001	5.00040001

From the numerical tables we can see that when x gets closer to 2 (on either side) $f(x)$ gets closer to 5. Therefore the limit of $f(x) = x^2 + 1$ as x approaches 2 is 5:

$$\lim_{x\to 2}(x^2 + 1) = 5$$

The previous discussion leads to the following non-rigorous definition of a limit.

Definition: The **limit** of $f(x)$ as x approaches a is L if we can make the values of $f(x)$ as close as we like to L by taking x-values sufficiently close but not equal to a. In symbols we write:

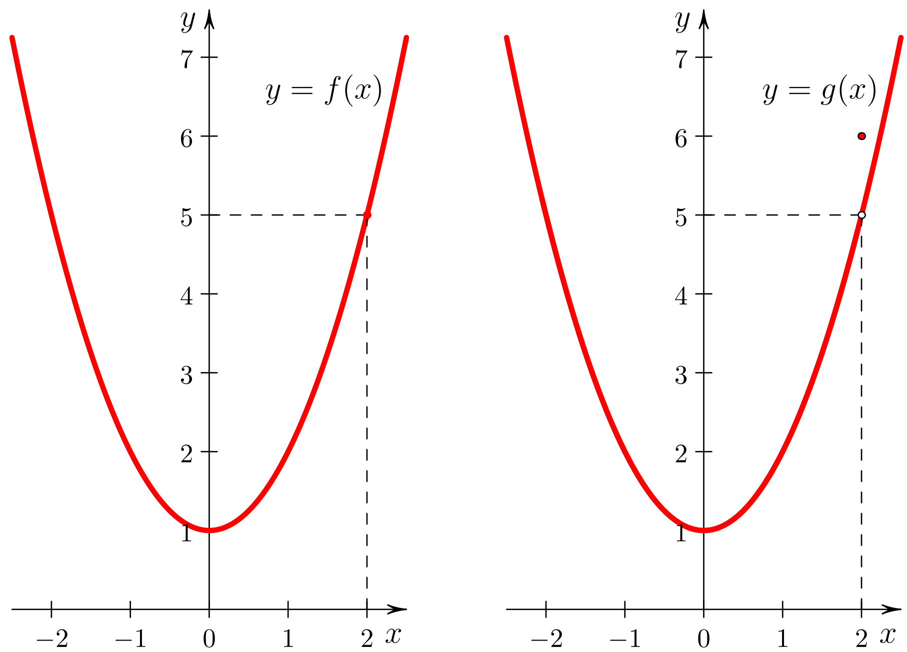
$$\lim_{x\to a} f(x) = L \ .$$

If such a limit exists then the values of $f(x)$ get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$.

Note that because the limit at $x = a$ only depends upon the values of the function $f(x)$ around a the function at a may not even be defined (i.e. a may not be in the domain of the function). This was the case of our secant slope function $m(x)$ at $x = 1$. If the function is defined it might equal the limit. In the above numerical example we found the limit was $L = 5$ at $a = 2$ and in fact $f(2) = 2^2 + 1 = 5$ also equals 5. However we could also tweak the above function to have a different value from the limit at $x = 2$ by defining it to be

$$g(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases} \ .$$

Graphically the two functions look as follows:



The function values at $x = 2$ differ since $f(2) = 5 \neq 6 = g(2)$, but the limit as $x \to 2$ for both functions is $L = 5$,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = 5 \ ,$$

since the limits only depends upon what the functions are doing near the x-value $a = 2$ which is identical for both functions.

2.3 Rigorous Definition of the Limit

For our course the previous definition of the limit which is intuitive is adequate. However if we really want to prove basic theorems of limits (such as the ones we shall state without proof in the next section) we need a more precise definition. The epsilon-delta (ϵ-δ) definition of the limit is such a definition.

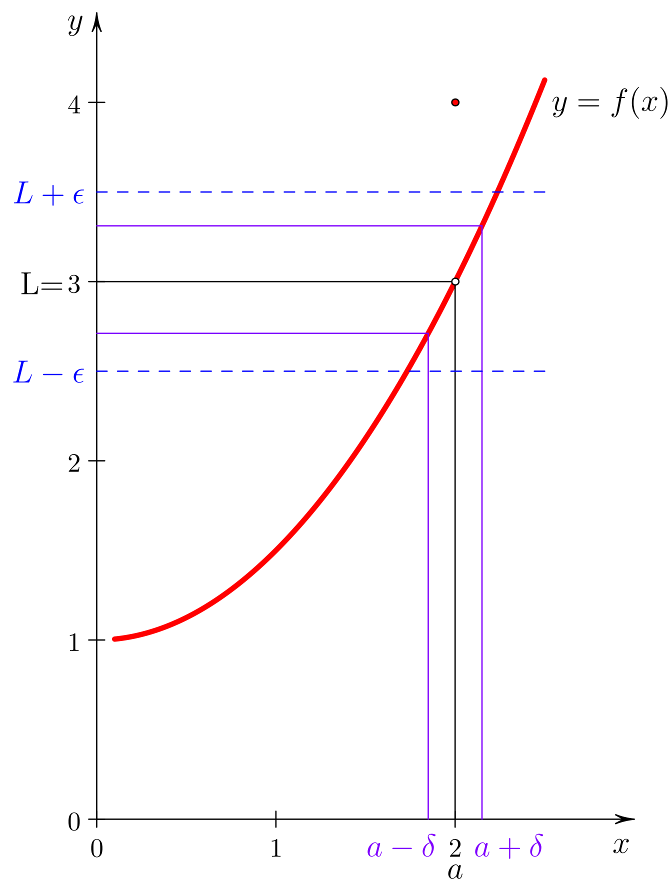
Definition: Let function f be defined on an open interval containing the number a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = L \quad \text{(the limit of } f(x) \text{ as } x \text{ approaches } a \text{ is } L\text{)}$$

if for every number $\epsilon > 0$ there exists a number $\delta > 0$ such that

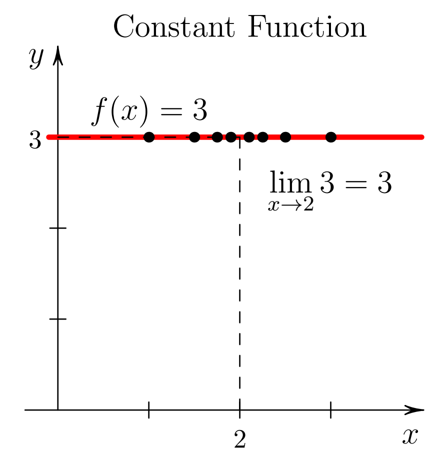
$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta \ .$$

The following diagram illustrates that for the given ϵ (here 0.5) a value of δ can be found (here 0.15) which would be sufficient to meet the criterion for the function shown. To prove that $\lim_{x\to 2} f(x) = 3$ we would need to show such a delta could be found for every epsilon.

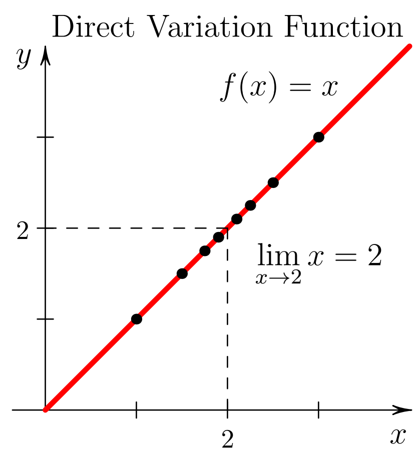


2.4 Limit Rules

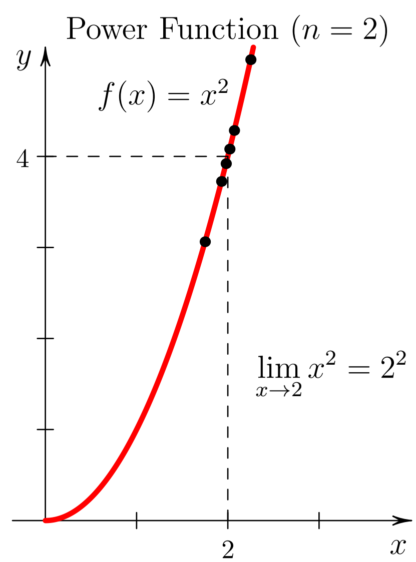
We wish to have rules for limits so we do not have to evaluate them all numerically. The following graphs of some basic functions illustrate what will be our first limit rules.



$$\Rightarrow \boxed{\lim_{x\to a} c = c}$$



$$\Rightarrow \boxed{\lim_{x\to a} x = a}$$



$$\Rightarrow \boxed{\lim_{x\to a} x^n = a^n}$$

The validity of the former limit results, whose truth is made plausible by the graphs, can be verified with proofs using the rigorous definition of the limit.[1] These and other results involving limits are summarized in the following theorem.

Theorem 2-1: Let c be a constant and suppose the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then the following limit rules hold:

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a} x^n = a^n$
4. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ (Here n is a positive integer. If n is even, then we require $a > 0$.)
5. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
6. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
7. $\lim_{x\to a} [f(x)g(x)] = \left[\lim_{x\to a} f(x)\right] \left[\lim_{x\to a} g(x)\right]$
8. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (Here we require $\lim_{x\to a} g(x) \neq 0$)
9. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
10. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ (n a positive integer. If n is even, then $\lim_{x\to a} f(x) > 0$ is required.)

[1] Note that a numerical evaluation of a limit is not rigorous as one is only checking the behaviour of a handful of x-values around a where limits require analysis of *all* the values around a.

Example 2-3

Evaluate the given limits. Consider which rules are being used at each step of the evaluation.

1. $\lim\limits_{x\to 3} 9 - 2x$
2. $\lim\limits_{x\to 2} \dfrac{1+\sqrt{2x+3x^2}}{x}$
3. $\lim\limits_{x\to -1} \dfrac{x^4-1}{x^3+1}$
4. $\lim\limits_{x\to 5} \dfrac{x-5}{\frac{1}{x}-\frac{1}{5}}$
5. $\lim\limits_{x\to 4} \dfrac{x-4}{\sqrt{x}-2}$
6. $\lim\limits_{h\to 0} \dfrac{\sqrt{9+h}-3}{h}$
7. $\lim\limits_{t\to 0} \dfrac{\frac{1}{(3+t)^2}-\frac{1}{9}}{t}$

Solution:

1. The particular rule from Theorem 2-1 being used at each step is shown below the equal sign.

$$\lim_{x\to 3} 9 - 2x \underset{⑤}{=} \lim_{x\to 3} 9 - \lim_{x\to 3} 2x \underset{⑥}{=} \lim_{x\to 3} 9 - 2\lim_{x\to 3} x \underset{①②}{=} 9 - 2(3) = 3$$

2.

$$\lim_{x\to 2} \frac{1+\sqrt{2x+3x^2}}{x} \underset{⑧}{=} \frac{\lim\limits_{x\to 2}\left(1+\sqrt{2x+3x^2}\right)}{\lim\limits_{x\to 2} x} \underset{⑤②}{=} \frac{\lim\limits_{x\to 2} 1 + \lim\limits_{x\to 2}\sqrt{2x+3x^2}}{2}$$

$$\underset{①②}{=} \frac{1+\sqrt{\lim\limits_{x\to 2}(2x+3x^2)}}{2} \underset{⑤⑥}{=} \frac{1+\sqrt{2\lim\limits_{x\to 2} x + 3\lim\limits_{x\to 2} x^2}}{2}$$

$$\underset{②③}{=} \frac{1+\sqrt{2(2)+3(2)^2}}{2} = \frac{1+\sqrt{16}}{2} = \frac{5}{2}$$

Looking at the final result of this and the previous problem suggests that the answer for many limits will be the same as just plugging the value a into the function itself. The remaining examples will show that this is often not the case. However if one can "plug in the value" and it results in a number, Theorem 2-1 can be systematically used to show that that number is the actual limit value.

3. For $\lim\limits_{x\to -1} \dfrac{x^4-1}{x^3+1}$ we can plug $x=-1$ into the function but we see we get $\frac{(-1)^4-1}{(-1)^3+1} = \frac{0}{0}$, so the function is undefined at $x=-1$. However the function not being defined at $x=-1$ does not mean the limit does not exist since the limit only depends on what the function is doing *near* $x=-1$ not *at* that value. To evaluate the limit we need a different strategy from the previous problems. A little thought suggests that the reason that the numerator and denominator are vanishing at $a=-1$ is because $(x-a)=(x-(-1))=(x+1)$ is a factor of both polynomials. Factoring confirms this and leads to the resolution of the limit:

$$\lim_{x\to -1} \frac{x^4-1}{x^3+1} = \lim_{x\to -1} \frac{(x^2)^2-1^2}{x^3+(1)^3} = \lim_{x\to -1} \frac{(x^2+1)(x^2-1)}{(x+1)(x^2-x+1)} = \lim_{x\to -1} \frac{(x^2+1)(x+1)(x-1)}{(x+1)(x^2-x+1)}$$

At this stage one can cancel the factor $(x+1)$ since near (but not at) $x=-1$ this evaluates to a non-zero number and the previous limit equals:

$$= \lim_{x\to -1} \frac{(x^2+1)(x-1)}{x^2-x+1} \underset{⑧⑦⑤①②③}{=} \frac{[(-1)^2+1](-1-1)}{(-1)^2-(-1)+1} = \frac{-4}{1+1+1} = -\frac{4}{3}$$

4. Plugging in $x = 5$ into $\frac{x-5}{\frac{1}{x}-\frac{1}{5}}$ yields $\frac{0}{0}$ showing the limit is indeterminate. Simplifying the expression by combining fractions in the denominator resolves the limit:

$$\lim_{x\to5}\frac{x-5}{\frac{1}{x}-\frac{1}{5}} = \lim_{x\to5}\frac{x-5}{\left(\frac{1}{x}\right)\left(\frac{5}{5}\right)-\left(\frac{1}{5}\right)\left(\frac{x}{x}\right)} = \lim_{x\to5}\frac{x-5}{\frac{5-x}{5x}} = \lim_{x\to5}\frac{(x-5)(5x)}{-(x-5)} = \lim_{x\to5}(-5x)$$
$$= -5(5) = -25$$

5. Also indeterminate, we can resolve this limit by factoring:

$$\lim_{x\to4}\frac{x-4}{\sqrt{x}-2} = \lim_{x\to4}\frac{(\sqrt{x})^2-2^2}{\sqrt{x}-2} = \lim_{x\to4}\frac{(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}-2} = \lim_{x\to4}(\sqrt{x}+2) = \sqrt{4}+2 = 2+2 = 4$$

6. The following indeterminate limit can be evaluated by multiplying by $1 = \frac{\text{conjugate}}{\text{conjugate}}$ where here the conjugate of $\sqrt{9+h}-3$ is $\sqrt{9+h}+3$ (i.e. change the sign between terms).

$$\lim_{h\to0}\frac{\sqrt{9+h}-3}{h} = \lim_{h\to0}\frac{\sqrt{9+h}-3}{h}\cdot\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} = \lim_{h\to0}\frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)}$$
$$= \lim_{h\to0}\frac{(\sqrt{9+h})^2+3\sqrt{9+h}-3\sqrt{9+h}-3^2}{h(\sqrt{9+h}+3)} = \lim_{h\to0}\frac{9+h-9}{h(\sqrt{9+h}+3)}$$
$$= \lim_{h\to0}\frac{h}{h(\sqrt{9+h}+3)} = \lim_{h\to0}\frac{1}{\sqrt{9+h}+3} = \frac{1}{\sqrt{9+0}+3} = \frac{1}{3+3} = \frac{1}{6}$$

7. This indeterminate form is also resolved by combining fractions:

$$\lim_{t\to0}\frac{\frac{1}{(3+t)^2}-\frac{1}{9}}{t} = \lim_{t\to0}\frac{\frac{9-(3+t)^2}{9(3+t)^2}}{t} = \lim_{t\to0}\frac{9-9-6t-t^2}{9t(3+t)^2} = \lim_{t\to0}\frac{t(-6-t)}{9t(3+t)^2} = \lim_{t\to0}\frac{-6-t}{9(3+t)^2}$$
$$= \frac{-6-0}{9(3+0)^2} = \frac{-6}{(9)(9)} = -\frac{6}{81} = -\frac{2}{27}$$

Further Questions:

Evaluate the following limits. Consider which rules are being used at each step of the evaluation.

1. $\lim_{x\to2}\left(3x^3-2x^2+3x+1\right)$
2. $\lim_{x\to4}\sqrt{x^2-4}$
3. $\lim_{x\to-1}\frac{x^2+1}{x+4}$
4. $\lim_{x\to1}\frac{x^2-1}{x^2+1}$
5. $\lim_{x\to2}\frac{x-2}{x^2-2x}$
6. $\lim_{h\to0}\frac{(1+h)^2-1}{h}$
7. $\lim_{s\to0}\frac{\sqrt{s^2+9}-3}{s^2}$
8. $\lim_{x\to-2}\frac{x^2-x-6}{x+2}$
9. $\lim_{x\to1}\frac{\frac{1}{x}-1}{x-1}$
10. $\lim_{x\to-1}\frac{x^3+x^2-x-1}{x^3+2x^2+2x+1}$
11. $\lim_{t\to-5}\frac{t^2+5t}{\sqrt{t^2-16}-3}$
12. $\lim_{x\to1}\frac{1-x^3}{1-x}$ $\Leftarrow$ Recall this was our secant slope limit: $\lim_{x\to1} m(x)$

The previous examples illustrate the general strategy for evaluating the limit $\lim_{x\to a} f(x)$:

- Evaluate $f(a)$ and see if it results in the **indeterminate form** $\mathbf{\frac{0}{0}}$. If it is not, the limit rules can often be used to show the limit really is just $f(a)$.
- If the result is an indeterminate form, try the following:
  - Simplify
  - Expand
  - Factor
  - Rationalize using the conjugate
- Of course one can also check a result numerically by evaluating the function at values close to a.

Remember: $\frac{0}{0}$ is not an answer for a limit. The indeterminate form indicates more work must be done to find the limit or else to show it does not exist.

The following example illustrates evaluating limits involving symbolic functions and constants.

Example 2-4

If $\lim_{x\to a} f(x) = 2$, $\lim_{x\to a} g(x) = -3$, and $\lim_{x\to a} h(x) = 3$ then find:

1. $\lim_{x\to a} \sqrt{f(x) + h(x)}$
2. $\lim_{x\to a} \dfrac{f(x)g(x)}{[h(x)]^2}$

Solution:

1. $\lim_{x\to a} \sqrt{f(x) + h(x)} = \sqrt{\lim_{x\to a}[f(x) + h(x)]} = \sqrt{\lim_{x\to a} f(x) + \lim_{x\to a} h(x)} = \sqrt{2+3} = \sqrt{5}$
2. $\lim_{x\to a} \dfrac{f(x)g(x)}{[h(x)]^2} = \dfrac{\lim_{x\to a}[f(x)g(x)]}{\lim_{x\to a}[h(x)]^2} = \dfrac{\left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]}{\left[\lim_{x\to a} h(x)\right]^2} = \dfrac{(2)(-3)}{(3)^2} = -\dfrac{2}{3}$

Further Questions:

Using the same limits as above find:

1. $\lim_{x\to a} \dfrac{6f(x) - 4[g(x)]^2}{g(x) - 4f(x)}$
2. $\lim_{x\to a} [f(x) + h(x)]^3$
3. $\lim_{x\to a} \dfrac{f(x)}{g(x) + h(x)}$

Answers: Page 240

Exercise 2-2

1-15: Evaluate the following limits.

1. $\lim_{x\to 3} \dfrac{x^2+4}{x+2}$
2. $\lim_{x\to -2} \dfrac{x^2+4x+4}{x+2}$
3. $\lim_{x\to 0} \dfrac{2-\sqrt{4-x}}{x}$
4. $\lim_{x\to 3} |x-3|$
5. $\lim_{x\to -2} \dfrac{x^2-3x-10}{x^2-4}$
6. $\lim_{x\to 3} \dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}$
7. $\lim_{t\to 1} \dfrac{t-\sqrt{t}}{\sqrt{t}-1}$
8. $\lim_{x\to 1} \dfrac{x^3-2x+1}{x-1}$
9. $\lim_{t\to 4} \dfrac{\sqrt{t}-2}{t-4}$
10. $\lim_{y\to 3} \dfrac{2-\sqrt{y^2-5}}{y^2-y-6}$
11. $\lim_{x\to -4} \dfrac{(x+1)^2-9}{x+4}$
12. $\lim_{u\to -5} \dfrac{\frac{1}{u}+\frac{1}{5}}{u+5}$
13. $\lim_{t\to 2} \dfrac{t^2+t-6}{\frac{2}{t}-1}$
14. $\lim_{x\to 4} \dfrac{x^3-4x^2-4x+16}{x-4}$
15. $\lim_{t\to 1} \dfrac{t^3+4t-5}{t^3-1}$

2.5 Trigonometric Limits

Theorem 2-2: The following trigonometric limits can be shown to exist:

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{\sin ax}{x}=a \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0$$

$$\lim_{x\to a}\sin x=\sin a \qquad \lim_{x\to a}\cos x=\cos a$$

Useful examples of the last two limits are:

$$\lim_{x\to 0}\sin x=0 \qquad \lim_{x\to 0}\cos x=1$$

$$\lim_{x\to \pi/2}\sin x=1 \qquad \lim_{x\to \pi/2}\cos x=0$$

$$\lim_{x\to \pi}\sin x=0 \qquad \lim_{x\to \pi}\cos x=-1$$

The trigonometric limits which involve a ratio may be verified numerically using a calculator (note the angles must be in radians), the remainder may be verified by consideration of the graphs of the functions.[2]

These basic results, our previous limit rules, and trigonometric identities may be used to find more complicated trigonometric limits.

Example 2-5

Evaluate the trigonometric limits.

1. $\displaystyle\lim_{x\to 0}\frac{3-2\sin x}{2+x}$
2. $\displaystyle\lim_{\theta\to 0}\frac{\sin(-2\theta)}{5\theta}$
3. $\displaystyle\lim_{t\to 0}\frac{t^2-t\cos t}{4t}$
4. $\displaystyle\lim_{\theta\to 0}\frac{4\theta\sin\theta-5\theta^2}{3\theta^2}$
5. $\displaystyle\lim_{x\to 0}\frac{3\sin^2 x}{1-\cos x}$

Solution:

1. Combining the basic trigonometric limits

$$\lim_{x\to a}\sin x=\sin a \qquad \lim_{x\to a}\cos x=\cos a$$

from Theorem 2-2 with the general results of Theorem 2-1 to resolve this limit:

$$\lim_{x\to 0}\frac{3-2\sin x}{2+x}=\frac{\lim_{x\to 0}(3-2\sin x)}{\lim_{x\to 0}(2+x)}=\frac{\lim_{x\to 0}3-2\lim_{x\to 0}\sin x}{\lim_{x\to 0}2+\lim_{x\to 0}x}=\frac{3-2\sin 0}{2+0}=\frac{3-0}{2}=\frac{3}{2}$$

Thus many limits involving trigonometric expressions will also equal the function evaluated at the limit point if the latter is defined. (Since limits involving tangent, cotangent, secant, and cosecant can be written as quotients involving sine and cosine this is true of those as well.)

[2]Note that the limits $\lim_{x\to a}\sin x=\sin a$ and $\lim_{x\to a}\cos x=\cos a$ reflect, as will be seen in Section 2.9, the continuity of sine and cosine.

2. The following is an indeterminate form $\left(\frac{0}{0}\right)$. Using $\lim_{x\to 0}\frac{\sin ax}{x} = a$ of Theorem 2-2 with $a = -2$ we have:

$$\lim_{\theta\to 0}\frac{\sin(-2\theta)}{5\theta} = \frac{1}{5}\lim_{\theta\to 0}\frac{\sin(-2\theta)}{\theta} = \frac{1}{5}(-2) = -\frac{2}{5}$$

An alternative approach casts uses $\lim_{x\to 0}\frac{\sin x}{x} = 1$ by making the denominator match the argument of the trigonometric function:

$$\lim_{\theta\to 0}\frac{\sin(-2\theta)}{5\theta} = \frac{1}{5}\lim_{\theta\to 0}\frac{\sin(-2\theta)}{-2\theta}(-2) \underset{(x=-2\theta)}{=} -\frac{2}{5}\lim_{x\to 0}\frac{\sin x}{x} = -\frac{2}{5}(1) = -\frac{2}{5}$$

3. This indeterminate form requires cancellation of the factor t:

$$\lim_{t\to 0}\frac{t^2 - t\cos t}{4t} = \lim_{t\to 0}\frac{t(t-\cos t)}{4t} = \lim_{t\to 0}\frac{t-\cos t}{4} = \frac{0-\cos 0}{4} = -\frac{1}{4}$$

4. $\lim_{\theta\to 0}\frac{4\theta\sin\theta - 5\theta^2}{3\theta^2} = \lim_{\theta\to 0}\frac{\theta(4\sin\theta - 5\theta)}{3\theta^2} = \lim_{\theta\to 0}\left(\frac{4\sin\theta}{3\theta} - \frac{5}{3}\right) = \frac{4}{3}(1) - \frac{5}{3} = \frac{4}{3} - \frac{5}{3} = -\frac{1}{3}$

5. This indeterminate form may be resolved by multiplying by $1 = \frac{\text{conjugate}}{\text{conjugate}}$ where conjugate is the conjugate of the denominator. A basic trigonometric identity is also required.

$$\lim_{x\to 0}\frac{3\sin^2 x}{1-\cos x} = \lim_{x\to 0}\frac{3\sin^2 x}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x} = \lim_{x\to 0}\frac{3\sin^2 x(1+\cos x)}{(1+\cos x)(1-\cos x)}$$

$$= \lim_{x\to 0}\frac{3\sin^2 x(1+\cos x)}{1-\cos^2 x} = \lim_{x\to 0}\frac{3\sin^2 x(1+\cos x)}{\sin^2 x}$$

$$= \lim_{x\to 0}3(1+\cos x) = 3(1+\cos 0) = 3(1+1) = 6$$

Further Questions:

Evaluate the following trigonometric limits:

1. $\lim_{x\to 0}\frac{\sin^2 x}{x}$
2. $\lim_{x\to 0}\frac{x}{1-\cos x}$
3. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$
4. $\lim_{x\to 0}\frac{\sec x\tan x}{x}$
5. $\lim_{x\to 0}\frac{\sin^2 x}{x^{\frac{4}{3}}}$
6. $\lim_{x\to 0}\frac{\sin 4x}{3x}$
7. $\lim_{t\to 0}\frac{2t\sin t - 5t^2}{t^2}$
8. $\lim_{x\to \frac{\pi}{2}}\frac{\cos^2 x}{1+\cos 2x}$

Answers: Page 240

Exercise 2-3

1-11: Evaluate the trigonometric limits.

1. $\lim_{\theta \to 0} \dfrac{\theta}{\sin \theta}$
2. $\lim_{\theta \to 0} \theta \cot \theta$
3. $\lim_{x \to 0} \dfrac{1 - \cos x}{\sin x}$
4. $\lim_{\theta \to 0} \dfrac{7\theta}{\sin 5\theta}$
5. $\lim_{\theta \to 0} \dfrac{7\theta}{\cos 5\theta}$
6. $\lim_{x \to 0} \dfrac{\sin x}{x \cos x}$
7. $\lim_{x \to \pi} \dfrac{\cos x}{x}$
8. $\lim_{\theta \to \frac{\pi}{4}} \dfrac{\tan \theta}{\theta}$
9. $\lim_{t \to \frac{\pi}{2}} \dfrac{\sin t - 1}{\cos t}$
10. $\lim_{x \to 0} \dfrac{\tan x}{x}$
11. $\lim_{\theta \to \pi} \dfrac{\cos \theta + 1}{\sin^2 \theta}$

2.6 The Squeeze Theorem

Theorem 2-3: If $g(x) \leq f(x) \leq h(x)$ for all x in an interval containing a (with the potential exception of $x = a$) and

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} f(x) = L$$

This is known as the **squeeze theorem**.

Example 2-6

If $3x \leq f(x) \leq x^3 + 2$ for $0 \leq x \leq 2$, evaluate $\lim_{x \to 1} f(x)$.

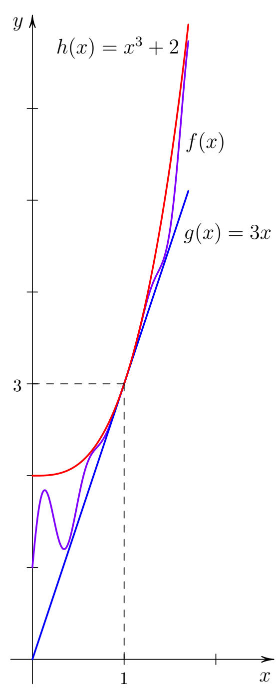
Solution:

Evaluating the limits at $a = 1$ we have

$$\lim_{x \to 1} 3x = 3(1) = 3$$
$$\lim_{x \to 1} x^3 + 2 = (1)^3 + 2 = 3$$

Since $f(x)$ is greater than or equal to $g(x) = 3x$ and less than or equal to $h(x) = x^3 + 2$ by assumption and the two limits at $a = 1$ are equal, it follows by the Squeeze Theorem that $\lim_{x \to 1} f(x) = 3$.

The following graph shows $g(x) = 3x$ and $h(x) = x^3 + 2$ and a hypothetical $f(x)$ which meets the criterion of being between the two functions.



Further Question:

If $\cos x \leq f(x) \leq \sec x$ on the interval $[-1, 1]$, evaluate $\lim_{x \to 0} f(x)$.

2.7 One-sided Limits

We can define weaker limit criteria if we restrict ourselves to approaching a from either the left or the right.[3]

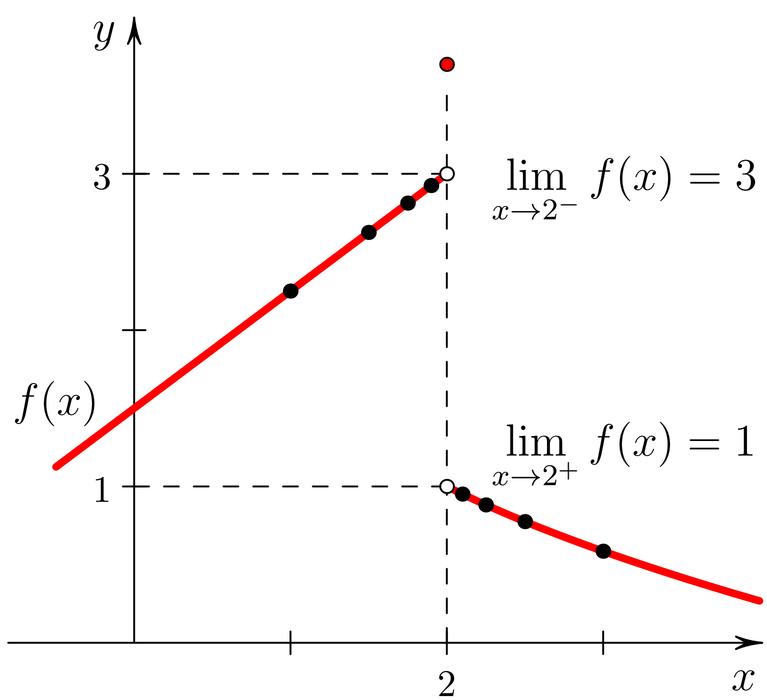
Definition: The **left-hand limit** of $f(x)$ as x approaches a is L if we can make the values of $f(x)$ as close as we like to L by taking x-values sufficiently close to a with x less than a. In symbols we write:

$$\lim_{x\to a^-} f(x) = L\ .$$

Definition: The **right-hand limit** of $f(x)$ as x approaches a is L if we can make the values of $f(x)$ as close as we like to L by taking x-values sufficiently close to a with x greater than a. In symbols we write:

$$\lim_{x\to a^+} f(x) = L\ .$$

The following graph of a hypothetical function $f(x)$ shows the behaviour of the one-sided limits at $a = 2$:



The following important theorem relates the (two-sided) limit at a to the one-sided limits at that number.

Theorem 2-4: $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$

Note that the above theorem requires that for a limit to exist both the left and right-handed limits must exist and be equal. It follows that in our above graph $\lim_{x\to 2} f(x)$ does not exist as the one-sided limits, while they exist, are not equal.

[3] More rigorous ϵ-δ definitions of the left-hand and right-hand limits can easily be formulated by replacing the condition $0 < |x - a| < \delta$ with $a - \delta < x < a$ and $a < x < a + \delta$ respectively in the (two-sided) limit definition. One now also only requires that f be defined to the left or right of a respectively.

Example 2-7

Solve the following problems involving one-sided limits.

1. Find $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$ if $f(x) = \begin{cases} 2x+3 & \text{if } x \le 3 \\ x^2-4 & \text{if } x > 3 \end{cases}$. Does $\lim_{x\to 3} f(x)$ exist?

   Solution:

   The function $f(x)$ is piecewise-defined. For $x < 3$ the function evaluates to $2x+3$, so the left-hand limit is

   $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(2x+3) = 2(3)+3 = 9$

   For $x > 3$ the function evaluates to $x^2 - 3$ so the right-hand limit is

   $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(x^2-4) = (3)^2 - 4 = 5$

   Since the left-hand and right-hand limits differ at $a = 3$ the limit $\lim_{x\to 3} f(x)$ does not exist by Theorem 2-4 .

   Note that the reader may have been concerned that we appear to be evaluating our one-sided limits using the same procedures (for which we had theorems) for evaluating our two-sided limits. How is this justified? We can actually use Theorem 2-4 in reverse. Since we know for instance that the (two-sided) limit $\lim_{x\to 3}(2x+3) = 2(3)+3 = 9$ this implies by Theorem 2-4 that $\lim_{x\to 3^-}(2x+3)$ and $\lim_{x\to 3^+}(2x+3)$ both equal 9. We used the former result above.

2. Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+}$ if $f(x) = \begin{cases} \frac{\sqrt{x+5}-2}{x+1} & \text{if } x > -1 \\ x^2 - \frac{3}{4} & \text{if } x \le -1 \end{cases}$. Does $\lim_{x\to -1} f(x)$ exist?

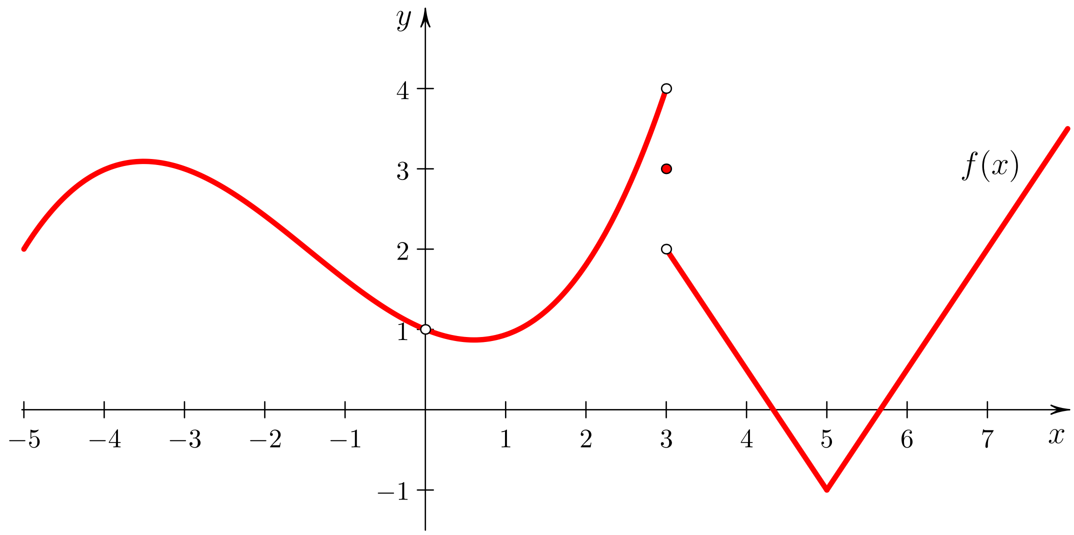
   Solution:

   $$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}\left(x^2 - \frac{3}{4}\right) = (-1)^2 - \frac{3}{4} = \frac{4}{4} - \frac{3}{4} = \frac{1}{4}$$

   $$\begin{aligned} \lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+}\frac{\sqrt{x+5}-2}{x+1} = \lim_{x\to -1^+}\frac{\sqrt{x+5}-2}{x+1}\cdot\frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} \\ &= \lim_{x\to -1^+}\frac{x+5-4}{(x+1)(\sqrt{x+5}+2)} = \lim_{x\to -1^+}\frac{x+1}{(x+1)(\sqrt{x+5}+2)} \\ &= \lim_{x\to -1^+}\frac{1}{\sqrt{x+5}+2} = \frac{1}{\sqrt{(-1)+5}+2} = \frac{1}{2+2} = \frac{1}{4} \end{aligned}$$

   Since the left-hand and right-hand limits at $a = -1$ exist and are equal, the limit exists at -1 and equals their common value, $\lim_{x\to -1} f(x) = \frac{1}{4}$.

Further Questions:



1. For the above function evaluate $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a} f(x)$, and $f(a)$ at $a = 0$, $a = 3$, and $a = 5$.

2. Find $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$ if $f(x) = \begin{cases} x^3 & \text{if } x \le 2 \\ 4 - 2x & \text{if } x > 2 \end{cases}$

   Does $\lim_{x\to 2}$ exist?

3. Suppose $f(x) = \begin{cases} \frac{9}{x^2} & \text{if } x \le -3 \\ 4 + x & \text{if } x > -3 \end{cases}$.

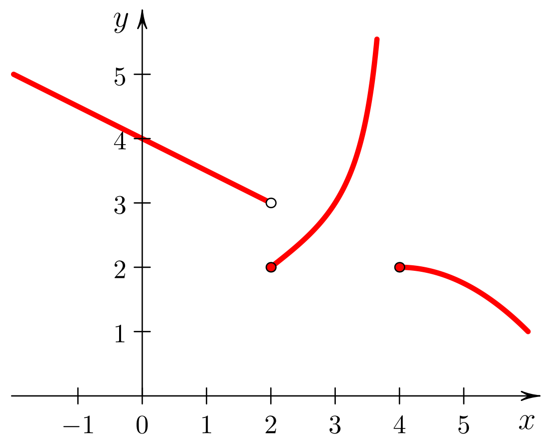
   Find $\lim_{x\to -3^-} f(x)$, $\lim_{x\to -3^+} f(x)$, and $\lim_{x\to -3} f(x)$, if they exist.

4. Show that $\lim_{x\to 7} \dfrac{|x-7|}{x-7}$ does not exist.

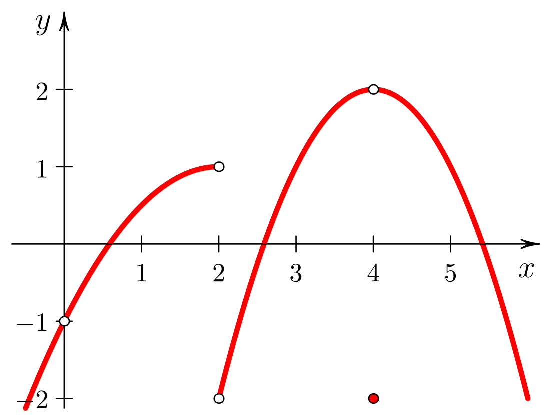
Answers: Page 240

Exercise 2-4

1. Find the one-sided limits of f at the values $x = 0$, $x = 2$ and $x = 4$.



2. Find the one-sided and two-sided limits of f at the values $x = 0$, $x = 2$ and $x = 4$.



3. Suppose $f(x) = \begin{cases} \frac{4}{x+4} & \text{if } x < 2 \\ x^2 + 1 & \text{if } x \geq 2 \end{cases}$.

   Find $\lim\limits_{x\to 2^-} f(x)$, $\lim\limits_{x\to 2^+} f(x)$, and $\lim\limits_{x\to 2} f(x)$, if they exist.

4. Suppose $f(x) = \begin{cases} x^2 + 2cx & \text{if } x \leq -1 \\ x + 5c & \text{if } x > -1 \end{cases}$.

   Find all values for the constant c that make the two-sided limit exist at $x = -1$.

2.8 Limits Involving Infinity

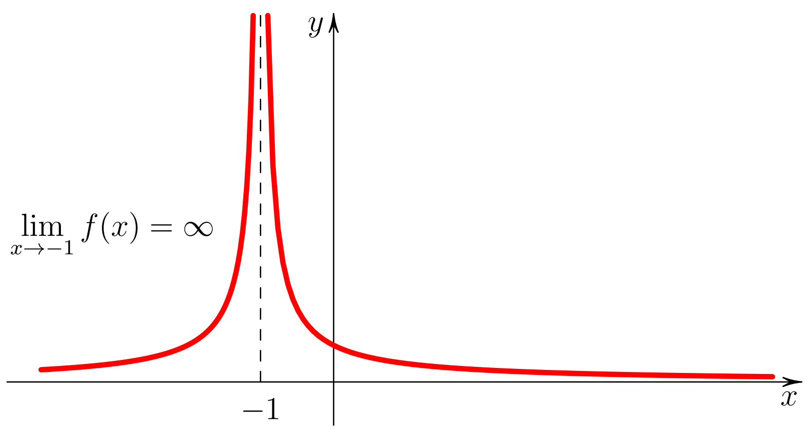
Sometimes when limits do not exist at $x = a$ they may yet show a systematic trend to larger positive or larger negative values as one approaches a. The following definitions give the notation used to indicate such trends.[4]

Definition: Suppose function $f(x)$ is defined to the left and the right of the x-value a (though perhaps not at a itself). Then

$$\lim_{x \to a} f(x) = \infty$$

if the values of $f(x)$ can be made arbitrarily large positively by taking x sufficiently close (but not equal) to a.

The following graph illustrates a function tending to positive infinity.

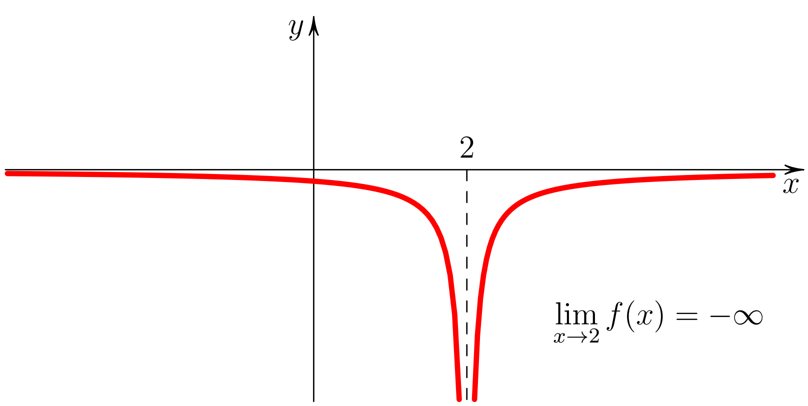


Definition: Suppose function $f(x)$ is defined to the left and the right of the x-value a (though perhaps not at a itself). Then

$$\lim_{x \to a} f(x) = -\infty$$

if the values of $f(x)$ can be made arbitrarily large negatively by taking x sufficiently close (but not equal) to a.

The following graph illustrates a function tending to negative infinity.

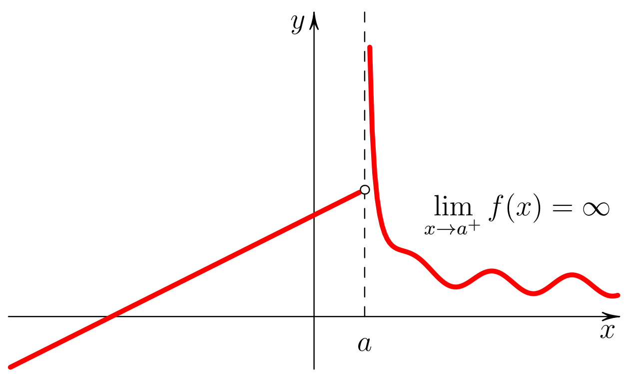


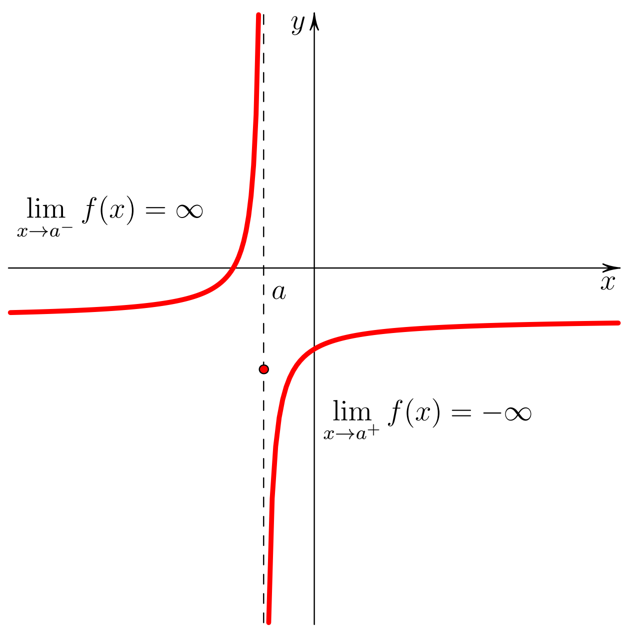
[4]The first definition (involving $+\infty$) can be made rigorous by demanding that for any $M > 0$ there exist $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - a| < \delta$. For the second definition (involving $-\infty$) we demand that for any $N < 0$ there exist $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - a| < \delta$.

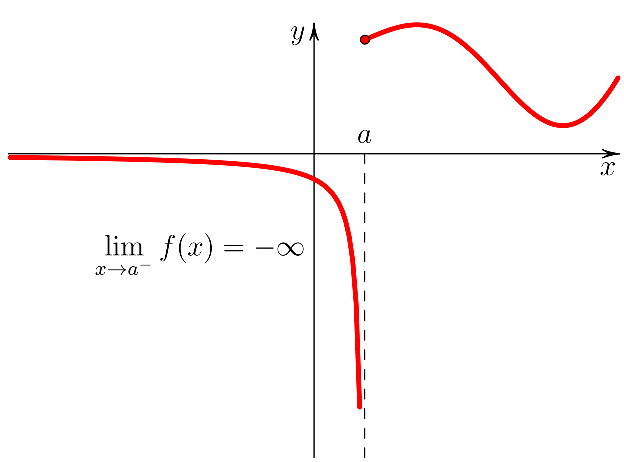
Definition: We analogously define the following **one-sided limits** involving infinity:

$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

These various cases are illustrated below.



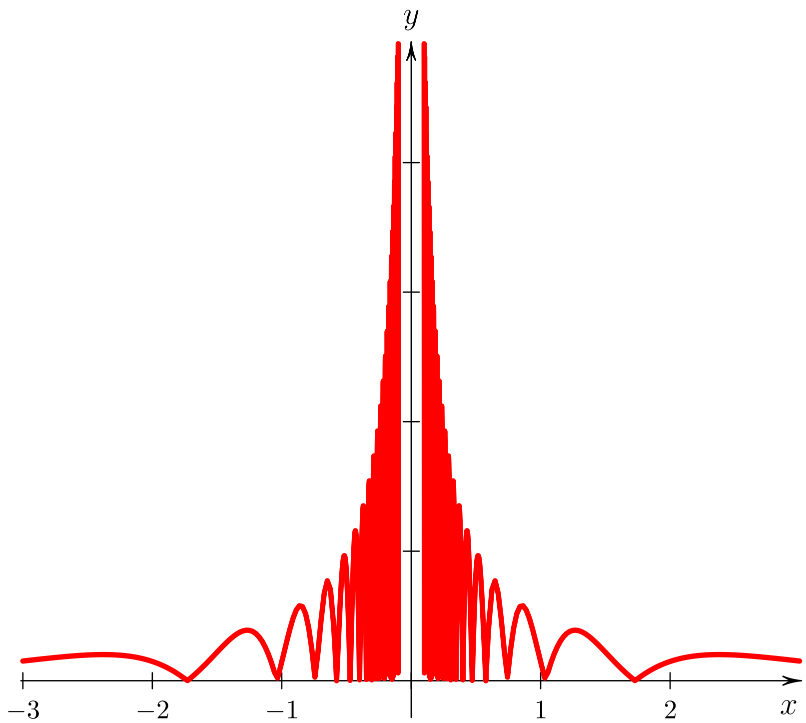




Note the following function **does not** approach infinity as x approaches zero:

$$f(x) = A\left|\frac{1}{x}\cos\left(\frac{a}{x}\right)\right|$$

where a and A are positive constants. Its graph for a choice of these constants is shown below.



If an arbitrary number is picked (say 3) there must be some interval around 0 (not including 0 itself) for which **all** the values will be above 3. The fact that the function returns to zero (despite getting very large at other points) makes it impossible to have the limit approach infinity.

Note that if $f(x)$ approaches infinity at $x = a$ the limit **does not exist** at $x = a$. A limit L must be a number. The notation ∞ indicates a trend in the function $f(x)$ toward a large magnitude at a, but it is not a number.

As we have seen, for limits $\lim_{x\to a} f(x)$ where $f(x)$ is made up of algebraic manipulations of common functions like polynomials, trigonometric functions, etc., it is of value to see what the function $f(a)$ is to help determine the behaviour of the limit at the value. If $f(a)$ is actually defined our limit laws can often be used to show that the limit is exactly that value. If the result is indeterminate, 0/0, more work must be done to evaluate the limit. If $f(a)$ is $p/0$ where p is a **non-zero** constant this is an indication the function tends to infinity at that value. Consideration of values of $f(x)$ for x a little to the left and right of a determine whether the function approaches ∞ or $-\infty$ and whether the behaviour is one-sided or not. Looking at the sign and magnitude of factors in the expression to the right and left of a can also help resolve infinite trends at a.

Example 2-8

Determine the following limits:

1. $\displaystyle\lim_{x\to 3^-} \frac{2x+3}{2x-6}$
2. $\displaystyle\lim_{x\to 4} \frac{x^2-2x-8}{x^2-3x-4}$

Solution:

1. The limit has the form $\frac{\text{non-zero}}{0}$ suggesting the limit does not exist. To determine an infinite trend one must identify the sign of the terms, in particular how the denominator is approaching 0 :

$$\lim_{x\to 3^-} \frac{2x+3}{2x-6} = \frac{6+3}{6^- - 6} = \frac{9}{0^-} = -\infty$$

Here our notation 6^- is meant to suggest we are approaching 6 from the left (5.9, 5.99, etc.) which in turn implies we will approach 0 from the left (here denoted 0^-) to get small negative values (-0.1, -.01, etc.) when we subtract 6 from this. The finite positive numerator divided by this small negative value will be large and positive. Note we can check this limit numerically by inserting values like 2.9, 2.99, etc. into the original function to confirm the trend to large negative numbers.

2. Inserting 4 in the denominator gives 0, but the numerator also vanishes so this is an indeterminate form $\left(\frac{0}{0}\right)$:

$$\lim_{x\to 4} \frac{x^2-2x-8}{x^2-3x-4} = \lim_{x\to 4} \frac{(x-4)(x+2)}{(x-4)(x+1)} = \lim_{x\to 4} \frac{x+2}{x+1} = \frac{6}{5}$$

Note that a limit with an indeterminate form *could* have an infinite trend as well. Modifying the above limit with an extra factor of $(x-4)^2$ in the denominator yields a limit in an indeterminate form that trends to positive infinity:

$$\lim_{x\to 4} \frac{x^2-2x-8}{(x^2-3x-4)(x-4)^2} = \lim_{x\to 4} \frac{(x-4)(x+2)}{(x+1)(x-4)^3} = \lim_{x\to 4} \frac{x+2}{(x+1)(x-4)^2} = \frac{6}{(5)(0^+)} = +\infty$$

Adding the square of $(x-4)$ is critical here for the two-sided limit to exist. A single $(x-4)$ factor would have made the left-hand limit approach $-\infty$ and the right-hand limit approach $+\infty$, so a (two-sided) limit in that case would have had no trend.

Further Questions:

Evaluate the following:

1. $\lim_{x\to 2} \frac{1-x^2}{x^2-4x+4}$
2. $\lim_{t\to 1} \frac{t^3+5t}{t^2+3t-4}$
3. $\lim_{x\to 0} \frac{x^2+9x}{x^3}$

Definition: If the limit as $x \to a$ of $f(x)$ tends to $\pm\infty$ from either the left or right or both, i.e.

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty\ ,$$

then the line $x = a$ is a **vertical asymptote** of the curve $y = f(x)$.

In the previous diagrams involving $f(x)$ approaching infinity on pages 53 and 54 the dashed lines are the vertical asymptotes once they are extended indefinitely upwards and downwards.

Notes:

- Because any of the above conditions in the definition can occur it is sufficient to show the limit is of the form $p/0$ with $p \neq 0$ to determine the location of a vertical asymptote, without evaluating the particular way the function is approaching infinity at a.
- When reporting a vertical asymptote, one writes $x = a$ because this is the relation representing a vertical line. (The point (a, y) clearly satisfies the equation for any real value y .)

Example 2-9

Find the vertical asymptotes of the function $f(x) = \dfrac{x^2-9}{x^2-x-6}$.

Solution:

Potential locations of a vertical asymptote occur where the denominator vanishes:

$x^2 - x - 6 = 0 \implies (x+2)(x-3) = 0 \implies x = -2$ or $x = 3$

$\boldsymbol{x = -2}$**:** $\lim\limits_{x\to -2} f(x) = \lim\limits_{x\to -2} \dfrac{x^2-9}{x^2-x-6} = \lim\limits_{x\to -2} \dfrac{(x+3)(x-3)}{(x-3)(x+2)} = \lim\limits_{x\to -2} \dfrac{x+3}{x+2} : \ \dfrac{-2+3}{-2+2} = \dfrac{1}{0}$

Therefore $x = -2$ is a vertical asymptote. Note that while we could have plugged in the value $x = -2$ into the original expression to determine this immediately, the work to get to the final expression allows one to easily see

$$\lim_{x\to -2^-} f(x) = \frac{1}{(-2)^- + 2} = \frac{1}{0^-} = -\infty \quad \text{and} \quad \lim_{x\to -2^+} f(x) = \frac{1}{(-2)^+ + 2} = \frac{1}{0^+} = +\infty\,.$$

This information is useful when graphing the function. As a further note we did not write $\lim\limits_{x\to -2} f(x) = \ldots = \dfrac{1}{0}$ as the latter is not mathematically meaningful. As the differing left-hand and right-hand limits show their is no common infinite trend to the limit and we can neither write that the limit trends to $+\infty$ or $-\infty$.

$\boldsymbol{x = 3}$**:** The limit has the indeterminate form $\frac{0}{0}$. Investigating the limit yields

$$\lim_{x\to 3} \frac{x^2-9}{x^2-x-6} = \lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)(x+2)} = \lim_{x\to 3} \frac{x+3}{x+2} = \frac{6}{5}$$

Therefore $x = 3$ is not a vertical asymptote.

Hence $x = -2$ is the only vertical asymptote of the graph $y = f(x)$. Note we write $x = -2$ and not simply -2 because an asymptote is a *line* and $x = -2$ is the equation representing this vertical line consisting of all points $(-2, y)$ which satisfy the equation.

Further Questions:

Find the vertical asymptotes of the following functions:

1. $y = \dfrac{x^2+4}{x^2-1}$
2. $y = \dfrac{x+2}{x^2+5x+6}$
3. $y = \dfrac{x^2+2x-8}{(x-2)^2}$

Answers: Page 241

Exercise 2-5

1-4: Determine the following limits. For any limit that does not exist, identify if it has an infinite trend (∞ or $-\infty$).

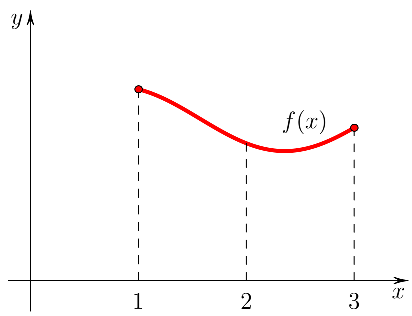
1. $\lim\limits_{x\to 2^+} \dfrac{5x+4}{2x-4}$
2. $\lim\limits_{x\to 3^-} \dfrac{x^2+2x}{x^2-5x+6}$
3. $\lim\limits_{x\to 5} \dfrac{x^2-4x-5}{x^2-3x-10}$
4. $\lim\limits_{x\to 0} \dfrac{\sec x}{x^2}$

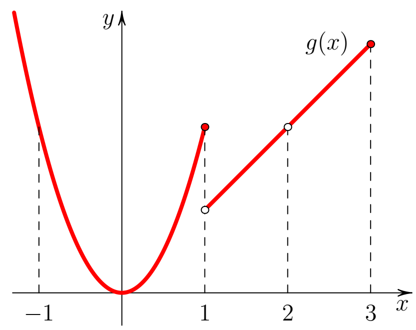
5-12: Find the vertical asymptotes of the following functions.

5. $f(x) = \dfrac{3x+3}{2x-4}$
6. $f(x) = x^3+5x+2$
7. $g(t) = \dfrac{\sqrt{t^2+3}}{t-2}$
8. $f(x) = \dfrac{x^2-2x+1}{2x^2-2x-12}$
9. $f(x) = \dfrac{\cos x}{x}$
10. $y = \dfrac{5x^2-3x+1}{x^2-16}$
11. $f(x) = \dfrac{x^3+1}{x^3+x^2}$
12. $F(x) = \dfrac{x}{\sqrt{4x^2+1}}$

2.9 Continuity

The idea of continuity of a function $f(x)$ at a value $x = a$ is intuitively seen in a graph of a function:





It is obvious from the graph of the function that the function f is continuous at $x = 2$ while g would not be (as it has a hole). Similarly g is not continuous at $x = 1$ due to the break in the function. However g would be continuous at a location like $x = -1$.

A rigorous definition of continuity that will match our intuitive one requires the concept of the limit of f at a and links it to the value of the function evaluated at a.

Definition: A function $f(x)$ is **continuous** at $x = a$ if

$$\lim_{x \to a} f(x) = f(a) \ .$$

If $f(x)$ is not continuous at a then f is **discontinuous** at a or $f(x)$ has a **discontinuity** at a.

The above definition of continuity at a requires three things:

1. $f(a)$ is defined. (I.e. a is in the domain of f .)
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

With a rigorous definition of continuity we can evaluate the continuity of a function at any point.

Example 2-10

Determine whether the function is continuous at the given value of x.

1. $f(x) = 5x + \sqrt{3x+4}$ at $x = 4$?

   Solution:

   To determine continuity we i) evaluate the function, ii) the limit, and then iii) see if they are equal.

   (i) $f(4) = 5(4) + \sqrt{3(4)+4} = 20 + 4 = 24$

   (ii) $\lim_{x \to 4} f(x) = 5(4) + \sqrt{3(4)+4} = 20 + \sqrt{16} = 20 + 4 = 24$

   (iii) $\lim_{x \to 4} f(x) = 24 = f(4)$

   $\Longrightarrow$ $f(x)$ is continuous at $x = 4$.

2. $g(t) = \dfrac{t^2 - 25}{t+5}$ at $t = -5$?

Solution:

(i) $g(-5) = \dfrac{(-5)^2 - 25}{-5+5} = \dfrac{0}{0}$ so $g(-5)$ is not defined.

$\implies g(t)$ is discontinuous at $x = -5$.

Note the limit may well exist as $x \to -5$ but if the function is undefined one need not evaluate this to disprove continuity. This is why we choose to evaluate the function first as evaluating the limit is usually more work.

3. $f(x) = \dfrac{\cos(2x) + \sin(3x)}{x^2 + 1}$ at $x = 0$?

Solution:

(i) $f(0) = \dfrac{\cos(2(0)) + \sin(3(0))}{0^2 + 1} = \dfrac{1+0}{1} = 1$

(ii) $\lim\limits_{x \to 0} f(x) = \dfrac{\cos(2(0)) + \sin(3(0))}{0^2 + 1} = \dfrac{1+0}{1} = 1$

(iii) $\lim\limits_{x \to 0} f(x) = 1 = f(0)$

$\implies f(x)$ is continuous at $x = 0$.

4. $f(x) = \begin{cases} \frac{\sqrt{x-3}-1}{x-4} & \text{if } x > 4 \\ x - \frac{7}{2} & \text{if } x \le 4 \end{cases}$ at $x = 4$?

Solution:

(i) $f(4) = 4 - \dfrac{7}{2} = \dfrac{1}{2}$

(ii) $\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} \left(x - \dfrac{7}{2}\right) = 4 - \dfrac{7}{2} = \dfrac{8-7}{2} = \dfrac{1}{2}$

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \frac{\sqrt{x-3}-1}{x-4} = \lim_{x \to 4^+} \frac{\sqrt{x-3}-1}{x-4} \cdot \frac{\sqrt{x-3}+1}{\sqrt{x-3}+1}$$

$$= \lim_{x \to 4^+} \frac{x-3-1}{(x-4)(\sqrt{x-3}+1)} = \lim_{x \to 4^+} \frac{x-4}{(x-4)(\sqrt{x-3}+1)}$$

$$= \lim_{x \to 4^+} \frac{1}{\sqrt{x-3}+1} = \frac{1}{\sqrt{4-3}+1} = \frac{1}{1+1} = \frac{1}{2}$$

$$\lim_{x \to 4^-} f(x) = \frac{1}{2} = \lim_{x \to 4^+} f(x) \implies \lim_{x \to 4} f(x) = \frac{1}{2}$$

(iii) $\lim\limits_{x \to 4} f(x) = \dfrac{1}{2} = f(4)$

$\implies f(x)$ is continuous at $x = 4$.

5. $g(\theta) = \begin{cases} \frac{1-\cos\theta}{\sin^2\theta} & \text{if } \theta \ne 0 \\ 2 & \text{if } \theta = 0 \end{cases}$ at $\theta = 0$?

Solution:

(i) $g(0) = 2$

(ii) $\lim\limits_{\theta \to 0} g(\theta) = \lim\limits_{\theta \to 0} \dfrac{1-\cos\theta}{\sin^2\theta} = \lim\limits_{\theta \to 0} \dfrac{1-\cos\theta}{\sin^2\theta} \cdot \dfrac{1+\cos\theta}{1+\cos\theta} = \lim\limits_{\theta \to 0} \dfrac{1-\cos^2\theta}{\sin^2\theta(1+\cos\theta)}$

$$= \lim_{\theta \to 0} \frac{\sin^2\theta}{\sin^2\theta(1+\cos\theta)} = \lim_{\theta \to 0} \frac{1}{1+\cos\theta} = \frac{1}{1+\cos 0} = \frac{1}{1+1} = \frac{1}{2}$$

(iii) $\lim_{\theta\to 0} g(\theta) = \frac{1}{2} \neq 2 = g(0)$

$\Longrightarrow$ $g(\theta)$ is discontinuous at $x = 0$.

Further Questions:

Determine whether the following functions are **continuous** at the given values of x.

1. $f(x) = x + 2$ at $x = 1$?
2. $f(x) = \dfrac{x^2 - 4}{x - 2}$ at $x = 2$?
3. $f(x) = \begin{cases} (x-1)^3 & \text{if } x < 0 \\ (x+1)^3 & \text{if } x \geq 0 \end{cases}$ at $x = 0$?
4. $f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ x + 2 & \text{if } x > 1 \end{cases}$ at $x = 1$?
5. $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$ at $x = 0$?
6. $f(x) = \dfrac{\cos x}{x}$ at $x = 0$?

Clearly not all discontinuities have the same behaviour. A **removable discontinuity** at a is one for which, if the function f were redefined appropriately at a, would result in a continuous function at that value. These are functions whose graphs essentially have a hole in them. Questions 2, 4, and 5 in Example 2-10 are of this kind. **Jump discontinuities**, as the name suggests, occur when the function shifts from one limiting value on the left to another on the right as in Question 3. Finally an **infinite discontinuity** occurs when there is a vertical asymptote at $x = a$ such as in Question 6. The latter two discontinuities cannot be removed by redefinition of the function at a.

Definition: At $x = a$ a function $f(x)$ is **continuous from the left** if

$$\lim_{x\to a^-} f(x) = f(a) \ ,$$

and **continuous from the right** if

$$\lim_{x\to a^+} f(x) = f(a) \ .$$

Definition: A function $f(x)$ is **continuous on an interval** if it is continuous at every value in the interval. (For an endpoint of a closed interval if the domain of the function does not extend beyond the endpoint one-sided continuity is sufficient.)

Example 2-11

In our previous graphically defined functions on page 59 we see that $f(x)$ is continuous on $[1, 3]$ and any subinterval of that interval while $g(x)$ would be continuous on $(-\infty, 1]$, $[-1, 1]$, $(1, 2)$, $(2, 3]$, but not $(0, 2.5)$ nor $[1, 2]$.

Continuity of a function on its domain can also be analyzed directly from the definition.

Example 2-12

1. Is $f(x) = \begin{cases} 3x^2+5 & \text{if } x<2 \\ 2x+1 & \text{if } x \geq 2 \end{cases}$ a continuous function on $(-\infty, \infty)$?

   Solution:

   For $x<2$ and $x>2$ the function $f(x)$ evaluates to a polynomial and hence is continuous at x. It remains to evaluate the continuity at $x=2$:

   (i) $f(2) = 2(2)+1 = 4+1 = 5$

   (ii) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (3x^2+5) = 3(2)^2+5 = 12+5 = 17$

   $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (2x+1) = 2(2)+1 = 4+1 = 5$

   Since $\lim_{x\to 2^-} f(x) = 17 \neq 5 = \lim_{x\to 2^+} f(x)$ the $\lim_{x\to 2} f(x)$ does not exist.

   $\implies$ $f(x)$ is discontinuous at $x=2$. Thus $f(x)$ is not continuous on the interval $(-\infty,\infty)$.

2. Determine all values of c such that $f(x)$ is continuous on $\mathbb{R}$.

   $f(x) = \begin{cases} cx+5 & \text{if } x \leq 3 \\ cx^2+4 & \text{if } x>3 \end{cases}$

   Solution:

   For any value of c the function $f(x)$ evaluates to a (continuous) polynomial on the intervals $(-\infty,3)$ and $(3,\infty)$. Continuity at $x=3$ constrains the value of c:

   (i) $f(3) = c(3)+5 = 3c+5$

   (ii) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (cx+5) = 3c+5$

   $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (cx^2+4) = 9c+4$

   So for $\lim_{x\to 3} f(x)$ to exist requires the left-hand and right-hand limits be equal:

   $$3c+5 = 9c+4 \implies 1 = 6c \implies c = \frac{1}{6}.$$

   (iii) With $c=\frac{1}{6}$, $\lim_{x\to 3} f(x) = f(3) = \dfrac{33}{6} = \dfrac{11}{2}$ and $f(x)$ will be continuous at $x=3$ as well.

3. Determine all values of c such that f is continuous on $\mathbb{R}$.

   $f(x) = \begin{cases} c^2x+1 & \text{if } x<-1 \\ 5cx+5 & \text{if } x \geq -1 \end{cases}$

   Solution:

   (i) $f(-1) = 5c(-1)+5 = -5c+5$

   (ii) $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (c^2x+1) = c^2(-1)+1 = -c^2+1$

   $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (5cx+5) = 5c(-1)+5 = -5c+5$

   Existence of the limit at $x=-1$ requires:

   $$-c^2+1 = -5c+5 \implies 0 = c^2-5c+4 \implies 0 = (c-4)(c-1) \implies \begin{cases} c & = 4 \\ & \text{or} \\ c & = 1 \end{cases}$$

(iii) With $c = 4$ or $c = 1$ we have $\lim_{x\to -1} f(x) = f(-1)$ and $f(x)$ will be continuous at $x = -1$ (as well as the rest of $\mathbb{R}$). For $c = 4$ this value is -15 and for $c = 1$ this value is 0.

4. Find the values of m and u such that $f(x)$ is continuous on $\mathbb{R}$.

$$f(x) = \begin{cases} m & \text{if } x \leq -2 \\ 3mx + u & \text{if } -2 < x < 2 \\ ux^2 + 2 & \text{if } x \geq 2 \end{cases}$$

Solution:

For the piecewise-defined function we need to ensure continuity at $x = -2$ and $x = 2$:

(i) $f(-2) = m$, $f(2) = u(2)^2 + 2 = 4u + 2$

(ii) $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} (m) = m$
$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (3mx + u) = -6m + u$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (3mx + u) = 6m + u$
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (ux^2 + 2) = 4u + 2$

Limits of $f(x)$ existing at -2 and 2:

$$\implies \begin{cases} m & = -6m + u \\ & \text{and} \\ 6m + u & = 4u + 2 \end{cases} \implies \begin{cases} -7m + u & = 0 \;\textcircled{1} \\ \quad\text{and} & \\ 6m - 3u & = 2 \;\textcircled{2} \end{cases}$$

One can solve this linear system by substitution:
$\textcircled{1} \implies u = 7m$ into $\textcircled{2} \implies 6m - 3(7m) = 2 \implies 6m - 21m = 2$
$\implies -15m = 2 \implies m = -\dfrac{2}{15} \implies u = 7\left(-\dfrac{2}{15}\right) \implies u = -\dfrac{14}{15}$

(iii) With constant values $u = -\dfrac{14}{15}$ and $m = -\dfrac{2}{15}$ one has

$$\lim_{x\to -2} f(x) = f(-2) = -\frac{2}{15} \qquad \lim_{x\to 2} f(x) = f(2) = -\frac{26}{15}$$

and $f(x)$ is continuous on $\mathbb{R}$.

Further Questions:

1. Is $f(x) = \begin{cases} x - 1 & \text{if } x < 3 \\ 5 - x & \text{if } x \geq 3 \end{cases}$ continuous?

2. Find the value(s) of a so that the given function is continuous for all x.

$$f(x) = \begin{cases} a^2x^2 + 3x - 4 & \text{if } x \leq -1 \\ x + 2a & \text{if } x > -1 \end{cases}$$

To avoid resorting to the definition of continuity the following theorem can be used:

Theorem 2-5: The following results involving continuity are valid:

1. A constant function $f(x) = c$ is continuous for all x.
2. For n a positive integer, $f(x) = x^n$ is continuous for all x.
3. A polynomial is continuous for all x.
4. A rational function is continuous on its domain, that is it is wherever it is defined (wherever the denominator does not vanish).
5. For n a positive even integer, $\sqrt[n]{x}$ is continuous for all x with $x \geq 0$.
6. For n a positive odd integer, $\sqrt[n]{x}$ is continuous everywhere.
7. A trigonometric function is continuous on its domain (i.e. wherever it is defined.)
8. Suppose f and g are continuous at $x = a$ and let c be any constant, then the following functions are continuous at a:
$$f + g \qquad f - g \qquad cf \qquad fg \qquad \frac{f}{g} \text{ (provided } g(a) \neq 0\text{).}$$
9. If g is continuous at a and f is continuous at $g(a)$ then $f \circ g(x) = f(g(x))$ is continuous at a.

Example 2-13

Determine where the following functions are continuous.

1. $f(x) = \dfrac{2x-2}{x-1}$
2. $f(x) = \sqrt[3]{x + \sin x}$

Solution:

1. The rational function $f(x) = \frac{2x-2}{x-1}$ is undefined when $x - 1 = 0 \implies x = 1$ so its domain is $D = \mathbb{R} - \{1\} = (-\infty, 1) \cup (1, \infty)$. By Theorem 2-5 the rational function $f(x)$ is continuous on this domain.
2. The functions x and $\sin x$ are defined for all $\mathbb{R}$ and by Theorem 2-5 continuous on this domain. The theorem also implies their sum, $x + \sin x$, will also be continuous on $\mathbb{R}$. Finally the cube root function is continuous everywhere so $f(x)$, which is the composition of it and $x + \sin x$, will also be continuous on $\mathbb{R}$.

Further Questions:

Determine where the following functions are continuous:

1. $f(x) = x^2 - 2x + 1$
2. $f(x) = \dfrac{x}{x^2 - x - 6}$
3. $f(x) = \sqrt[3]{x^2 - 4}$

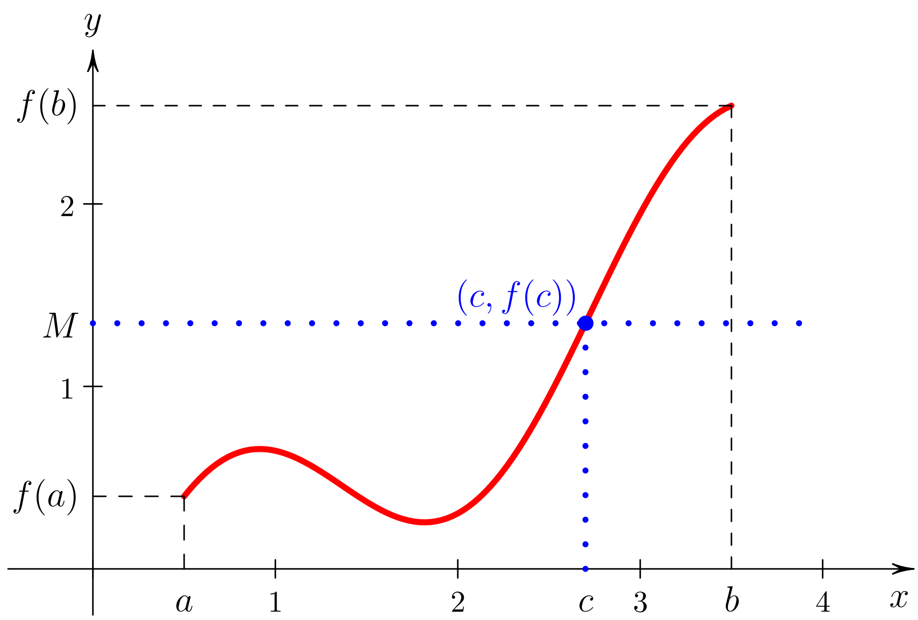
4. $f(x) = \sqrt[4]{x^2 - 4}$
5. $f(x) = \sqrt{\dfrac{x}{x-2}}$

We note that by the continuity theorems, most of the functions we would typically write down, excluding piecewise-defined functions, are continuous on their domains. Thus at any point a in the domain D of such a function, continuity there implies $\lim_{x \to a} f(x) = f(a)$. In other words, the limit can be found by evaluating the function at $x = a$, a result we previously concluded for such functions using our limit theorems.

2.10 The Intermediate Value Theorem

Theorem 2-6: Let $f(x)$ be continuous on a closed interval $[a, b]$. If M is a value strictly between $f(a)$ and $f(b)$, then there exists a number c in (a, b) such that $f(c) = M$. This is known as the **Intermediate Value Theorem**.

The theorem is depicted below for a hypothetical function:



A continuous curve on the closed interval $[a, b]$ is like a string joining the points $(a, f(a))$ and $(b, f(b))$. So if we choose the value M between $f(a)$ and $f(b)$ and extend a horizontal line across, it must, due to the continuity, cross the curve at least once. Where it does is the point $(c, f(c))$, which, since $f(c) = M$, yields the desired value c. Notice that there could be more than one value c that works. In the above graph if our value M had been 0.5 then the horizontal line would have crossed the curve 3 times, yielding three suitable values of c with $f(c) = 0.5$.

Example 2-14

For the function $f(x) = 2x^3 - 4x^2 + 5$, use the Intermediate Value Theorem to show that there is a real number c such that $f(c) = -10$.

Solution:

$f(x) = 2x^3 - 4x^2 + 5$ is a polynomial and therefore continuous on all $\mathbb{R}$ and hence on any closed interval $[a, b]$ it contains. Evaluating f at several values of x gives:

$$f(0) = 0 - 0 + 5 = 5$$
$$f(-1) = 2(-1)^3 - 4(-1)^2 + 5 = -2 - 4 + 5 = -6 + 5 = -1$$
$$f(-2) = 2(-2)^3 - 4(-2)^2 + 5 = -16 - 16 + 5 = -27$$

One observes that $M = -10$ lies between -1 and -27 so consider $[a, b] = [-1, -2]$. Then $f(x)$ is continuous on the closed interval $[-1, -2]$. Also, $f(-2) = -27 < M = -10 < f(-1) = -1$. Therefore, by the Intermediate Value Theorem there exists a value c in $(-1, -2)$ such that $f(c) = M = -10$.

Further Questions:

1. If $f(x) = x^3 + x^2 + x$, prove there is a value c in $(-1, 2)$ with $f(c) = 3$.
2. If $g(x) = x^5 - 2x^3 + x^2 + 2$ show there is a number c such that $g(c) = -1$.

A corollary of the Intermediate Value Theorem is that if $f(x)$ is a continuous function on $[a, b]$ with $f(a)$ and $f(b)$ of different sign (so $f(a) < 0$ and $f(b) > 0$ **or** $f(a) > 0$ and $f(b) < 0$) then there exists at least one c in (a, b) with $f(c) = 0$.

Now any equation (such as $2x + \sin x = 0$) can always be written in the form

$$f(x) = 0$$

by taking all the terms to the left-hand side and defining that to be $f(x)$. If we find two values a and b such that the function at those values has different sign, then, assuming the function is continuous on $[a, b]$, the intermediate value theorem corollary implies a c exists in the interval with $f(c) = 0$. But then this c is precisely a root (solution) of the equation we wished to solve.

Example 2-15

Show that the equation $x^4 - 2x^3 + 5x^2 - 6 = 0$ has a root (solution) between 1 and 2 .

Solution:

Let $f(x) = x^4 - 2x^3 + 5x^2 - 6$ and $[a, b] = [1, 2]$. The polynomial $f(x)$ is continuous on the closed interval $[1, 2]$. The value c will be a solution to the original equation if we can find $f(c) = M = 0$ since then $c^2 - 2c^3 + 5c^2 - 6 = 0$. Evaluate $f(x)$ at the endpoints:

$$f(1) = 1^4 - 2(1)^3 + 5(1)^2 - 6 = 1 - 2 + 5 - 6 = -2$$
$$f(2) = 2^4 - 2(2)^3 + 5(2)^2 - 6 = 16 - 16 + 20 - 6 = 14$$

Then $f(1) = -2 < M = 0 < f(2) = 14$. By the IVT there exists c in $(1, 2)$ with $f(c) = 0$; this c is therefore a solution to the given equation between 1 and 2 as required.

Further Question:

Show that $x^4 - x^3 - 1 = 0$ has a root between $x = -1$ and $x = 0$.

Once an interval containing a root is found one may evaluate the function at the midpoint of the interval. Its sign will differ from one of the endpoints and so one can narrow the interval which contains the root. Repeating the process is the basis of the **bisection method**, a numerical method for finding solutions to arbitrary equations.

Answers:
Page 241

Exercise 2-6

1. Define precisely what is meant by the statement "f is continuous at $x = a$".

2-7: Use the continuity definition to determine if the function is continuous at the given value. If the function is discontinuous there, decide whether it is a removable, jump, or infinite discontinuity.

2. $f(x) = x^3 + 5x + 1$, at $x = 2$

3. $g(t) = \dfrac{t+1}{t^2+4}$, at $t = -1$

4. $h(y) = \dfrac{y^2+4y+4}{y+2}$, at $y = -2$

5. $p(s) = \sqrt{s} - 4$, at $s = 2$

6. $f(x) = \begin{cases} x^2+1, & \text{if } x \le 1 \\ \frac{x+1}{x-1}, & \text{if } x > 1 \end{cases}$, at $x = 1$

7. $g(t) = \begin{cases} 2t+3, & \text{if } t \le 2 \\ \frac{t^2-5t+6}{t-2}, & \text{if } t > 2 \end{cases}$, at $t = 2$

---

8. Let c be a constant real number and f be the function

$$f(x) = \begin{cases} \sqrt{-x}+1 & \text{if } x < 0 \\ x^2 + c^2 & \text{if } x \ge 0 \end{cases}$$

   (a) Explain why, for $c = -2$, the function f is discontinuous at $x = 0$.

   (b) Determine all real numbers c for which f is continuous at $x = 0$.

9. Where is the function $f(x) = \dfrac{x^2+3x+2}{x^2-1}$ continuous?

10. Using the Intermediate Value Theorem, show there is a real number c strictly between 1 and 3 such that $c^3 + 2c^2 = 10$.

11. Show that the equation $x^2 + \cos x - 2 = 0$ has a solution in the interval $(0, 2)$.

Answers: Page 242

Chapter 2 Review Exercises

1-5: Evaluate the limits.

1. $\displaystyle\lim_{x\to 3} \frac{2x^2-5x-3}{x^2+x-12}$
2. $\displaystyle\lim_{t\to 4} \frac{\sqrt{t}-2}{t^2-2t-8}$
3. $\displaystyle\lim_{x\to -1} \frac{x^3+x^2+2x+2}{x^2-2x-3}$
4. $\displaystyle\lim_{t\to -2} \frac{\sqrt{10+3t}-2}{3t^2+4t-4}$
5. $\displaystyle\lim_{t\to 2} \frac{\frac{8}{t}-4}{3t^2-4t-4}$

6-9: Evaluate the trigonometric limits.

6. $\displaystyle\lim_{x\to 0} \frac{\sin(4x)}{\sin(5x)}$
7. $\displaystyle\lim_{\theta\to 0} \frac{\sin(3\theta)}{\tan(4\theta)}$
8. $\displaystyle\lim_{t\to \pi} \frac{2\sin^2 t}{1+\cos t}$
9. $\displaystyle\lim_{x\to 0} \frac{\cos(3x)+\cos(4x)-2}{x}$

10-12: Determine whether the functions are continuous at the given value.

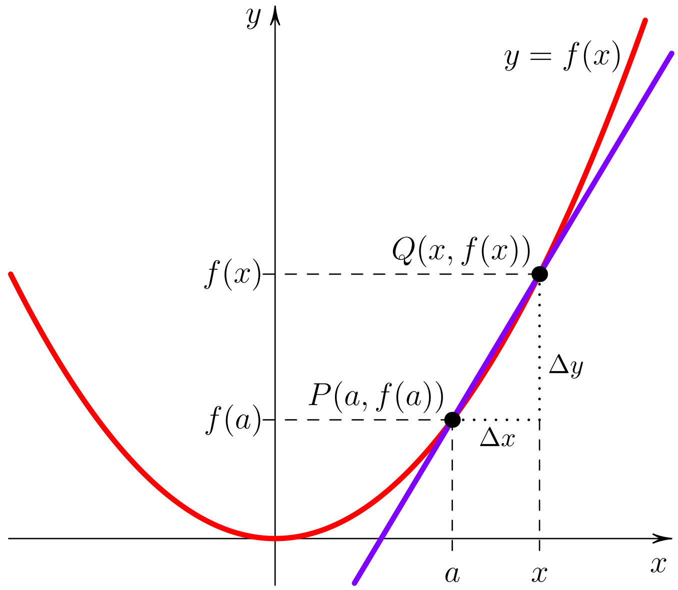
10. $f(x) = \dfrac{x+3}{\sqrt{x^2+5}}$ at $x=-1$
11. $h(t) = \dfrac{t^2+2t-1}{t-3}$ at $t=3$
12. $g(x) = \begin{cases} 3x^2-1, & \text{if } x \le 2 \\ \frac{x^2+x-6}{x-2}, & \text{if } x > 2 \end{cases}$ at $x=2$

Chapter 3: Differentiation

3.1 Motivating the Definition of the Derivative

3.1.1 Tangents

When motivating the definition of the limit we were trying to solve the equation for the tangent to a point on a curve. We now write the definition of the tangent of the curve at a point directly in terms of the limit where, from the following diagram, it follows that $\Delta y = f(x) - f(a)$ and $\Delta x = x - a$:



Definition: Let $P(a, f(a))$ be a point on the curve $y = f(x)$. The tangent line to the curve at P is the line through P having slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} ,$$

assuming the limit exists.

Example 3-1

Find the equation of the tangent line to the curve $y = x^2 - x$ at $P(2, 2)$.

Solution:

The curve is $y = f(x)$ with $f(x) = x^2 - x$. The slope of the tangent at $P(a, f(a)) = (2, 2)$ is:

$$m = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{(x^2 - x) - (2^2 - 2)}{x - 2} = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+1)}{x - 2}$$
$$= \lim_{x \to 2} (x + 1) = 2 + 1 = 3$$

Using the point-slope equation of a line, $y = m(x - x_0) + y_0$, with $(x_0, y_0) = (2, 2)$ gives the equation of the tangent line through P to be $y = 3(x - 2) + 2$ in point-slope form. Expanding gives the slope-intercept form $y = 3x - 4$.

Further Question:

Find the equation of the tangent line to the curve $y = \frac{2}{x}$ at $P(2, 1)$.

Another equivalent (and convenient) limit for the tangent slope can be found by considering the limit as Δx goes to 0. If, for brevity, we call Δx by h we have

$$h = x - a\ ,$$

so that $x = a + h$. The original limit for the tangent slope becomes $\boxed{m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}}$.

Example 3-2

Find the slope of the tangent line to the curve $y = x^2 - x$ at $P(2,2)$ of Example 3-1, but now using the limit $m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$.

Solution:

$$\begin{aligned} m &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{[(2+h)^2 - (2+h)] - (2^2 - 2)}{h} = \lim_{h\to 0} \frac{4 + 4h + h^2 - 2 - h - 2}{h} \\ &= \lim_{h\to 0} \frac{3h + h^2}{h} = \lim_{h\to 0} \frac{h(3+h)}{h} = \lim_{h\to 0}(3+h) = 3 + 0 = 3 \quad \text{(as before)} \end{aligned}$$

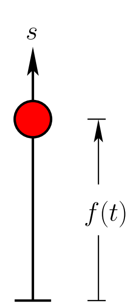
Further Questions:

1. Find the equation of the tangent line to the curve $y = x^3$ at $P(1,1)$ using the limit $m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ for the tangent slope.
2. Find the slope of the tangent line to the curve $y = x^3$ at the arbitrary point $P(a, a^3)$

3.1.2 Velocity in One Spatial Dimension

In the tangent problem the independent variable was a spatial one, x. Another useful independent variable is time t. If an object is moving along a straight line then its position in space is completely determined by its **displacement *s*** from some point along the line, the **origin**. If the object is moving in time its position at any given time t will be given by a **position function** $s = f(t)$.

An example of such motion is a ball thrown directly upwards which, in the absence of wind, will move in a vertical line where s is the height above ground:



A negative displacement would indicate the ball is positioned below ground surface.

Definition: The **average velocity** over the time interval from $t = a$ to $t = a + h$ (so $\Delta t = h$) is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h} \ .$$

Letting the time interval approach zero ($h \to 0$) we arrive at the following definition:

Definition: The (instantaneous) **velocity** $v(a)$ at time $t = a$ is:

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \ .$$

Note velocity is a signed quantity. A positive sign in our ball example indicates the ball is moving upwards and a negative sign indicates it would be falling.

Example 3-3

The displacement s (in metres) of a particle moving in a straight line is given by $s = 2t^3 + t^2 - 5$.

1. Find the average velocity over the interval $[1, 2]$.
2. Find the (instantaneous) velocity at $t = 2$ seconds.

Solution:

1. For $s = f(t) = 2t^3 + t^2 - 5$ over time interval $[1, 2]$ one is starting at time $a = 1$ second with interval length $h = 2 - 1 = 1$ second. The average velocity over time interval $[1, 2]$ is therefore:

$$\begin{aligned}\text{average velocity} &= \frac{f(1+1) - f(1)}{h} = \frac{2(2)^3 + (2)^2 - 5 - [2(1)^3 + (1)^2 - 5]}{2 - 1} \\ &= \frac{16 + 4 - 5 - 2 - 1 + 5}{1} = 17 \, \frac{\text{m}}{\text{s}}\end{aligned}$$

2. The (instantaneous) velocity at $t = 2$ seconds is

$$\begin{aligned}v(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{2(2+h)^3 + (2+h)^2 - 5 - [2(2^3) + 2^2 - 5]}{h} \\ &= \lim_{h \to 0} \frac{2(8 + 12h + 6h^2 + h^3) + (4 + 4h + h^2) - 5 - 15}{h} \\ &= \lim_{h \to 0} \frac{16 + 24h + 12h^2 + 2h^3 + 4 + 4h + h^2 - 20}{h} \\ &= \lim_{h \to 0} \frac{2h^3 + 13h^2 + 28h}{h} = \lim_{h \to 0} (2h^2 + 13h + 28) = 28 \, \frac{\text{m}}{\text{s}}\end{aligned}$$

Further Question:

The displacement s (in metres) of a particle moving in a straight line is given by $s = t^2 - 4t + 5$ where t is measured in seconds.

1. Find the average velocity over the interval $[3, 4]$.
2. Find the (instantaneous) velocity at $t = 4$ seconds.

3.1.3 Other Rates of Change

The previous examples involving relative change in coordinate variables or in position with respect to time can be further generalized. If x changes from x_1 to x_2 then the change in x, called the **increment in $\boldsymbol{x}$**, is denoted by:

$$\Delta x = x_2 - x_1 \ .$$

If $y = f(x)$ then the corresponding **increment in $\boldsymbol{y}$** is

$$\Delta y = f(x_2) - f(x_1) \ .$$

As with the the specific example of velocity, the **average rate of change of $\boldsymbol{y}$ with respect to $\boldsymbol{x}$** is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

Taking the limit as $x_2 \to x_1$, or equivalently $\Delta x \to 0$, one defines the **rate of change of $\boldsymbol{y}$ with respect to $\boldsymbol{x}$** as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

This general definition shows there are many rates of change depending upon the choice of independent and dependent variable, velocity being only one.

Example 3-4

Recalling that the area of a circle in terms of its radius is $A = \pi r^2$,

1. Find the average rate of change of the area of a circle with respect to its radius r as r changes from 2 cm to 2.5 cm.
2. Find the instantaneous rate of change of the area with respect to the radius of the circle when $r = 2$ cm.

Solution:

1. Over the interval $[2\,\text{cm}, 2.5\,\text{cm}]$ one finds

$$\begin{array}{c}\text{Avg. Rate of Change}\\ \text{with respect to } r\end{array} = \frac{\Delta A}{\Delta r} = \frac{A(r_2) - A(r_1)}{r_2 - r_1} = \frac{\pi(2.5)^2 - \pi(2)^2}{2.5 - 2} = \frac{\pi(6.25 - 4)}{0.5} = 4.5\pi\,\frac{\text{cm}^2}{\text{cm}}$$

2. The instantaneous rate of change of area with respect to radius when $r = 2$ cm is:

$$\begin{aligned}\lim_{\Delta r \to 0} \frac{A(r + \Delta r) - A(r)}{\Delta r} &= \lim_{\Delta r \to 0} \frac{A(2 + \Delta r) - A(2)}{\Delta r} = \lim_{\Delta r \to 0} \frac{\pi(2 + \Delta r)^2 - \pi(2)^2}{\Delta r} \\ &= \lim_{\Delta r \to 0} \frac{\pi(4 + 4\Delta r + \Delta r^2) - 4\pi}{\Delta r} = \lim_{\Delta r \to 0} \frac{\pi \Delta r(4 + \Delta r)}{\Delta r} \\ &= \lim_{\Delta r \to 0} \pi(4 + \Delta r) = \pi(4 + 0) = 4\pi\,\frac{\text{cm}^2}{\text{cm}}\end{aligned}$$

Further Question:

A spherical cell has approximate volume of $V = \frac{4}{3}\pi r^3$ which increases as it grows.

1. Find the average rate of change of the volume of the cell with respect to its radius r as r changes from $5\,\mu\text{m}$ to $7\,\mu\text{m}$.

2. Find the rate of change of volume with respect to radius r when $r = 5\ \mu\text{m}$.

(Note 1 micrometre (μm) equals 1×10^{-6}m.)

Answers: Page 242

Exercise 3-1

1. Consider the function $f(x) = x^3 + x + 2$.
   (a) Find the slope of the secant line through points $(2, 12)$ and $(1, 4)$ on the graph of f.
   (b) From your graph, estimate the slope of the tangent line to the curve at the point $(2, 12)$. Next, numerically estimate the tangent slope by calculating the secant slope between the point $(2, 12)$ and a point with x value near 2.
   (c) What is the equation of the tangent line at $(2, 12)$? (Use your estimate from (b) for the slope.)

2. After t seconds, a toy car moving along a straight track has position $s(t)$ measured from a fixed point of reference given by $s(t) = t^3 + 2t^2 + 1$ cm.
   (a) How far is the car initially from the reference point?
   (b) How far from the reference point is the car after 2 seconds?
   (c) What is the average velocity of the car during its first 2 seconds of motion?
   (d) By calculator, estimate the instantaneous velocity of the car at time $t = 2$ by computing the average velocities over small time intervals near $t = 2$.

3. One mole of an ideal gas at a fixed temperature of 273 K has a volume V that is inversely proportional to the pressure P (Boyle's Law) given by

$$V = \frac{22.4}{P},$$

   where V is in litres (L) and P is in atmospheres (atm).
   (a) What is the average rate of change of V with respect to P as pressure varies from 1 atm to 3 atm?
   (b) Use a calculator to estimate the instantaneous rate of change in the volume when the pressure is 3 atm by computing the average rates of change over small intervals lying to the left and right of $P = 3$ atm.

3.2 The Derivative

The preceding examples of the tangent slope, velocity, and more generally rate of change illustrate the value in defining (and calculating) the following limit of a function $f(x)$.

Definition: The **derivative** of a function f at a value $x = a$, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

As we saw when formulating the tangent to a curve an equivalent definition is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example 3-5

Find the derivative of $f(x) = 2 + \sqrt{2x}$ at $x = a$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left(2+\sqrt{2(x+h)}\right)-\left(2+\sqrt{2x}\right)}{h} = \lim_{h\to 0}\frac{2+\sqrt{2x+2h}-2-\sqrt{2x}}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{2x+2h}-\sqrt{2x}}{h}\cdot\frac{\sqrt{2x+2h}+\sqrt{2x}}{\sqrt{2x+2h}+\sqrt{2x}} = \lim_{h\to 0}\frac{2x+2h-2x}{h\left(\sqrt{2x+2h}+\sqrt{2x}\right)} \\ &= \lim_{h\to 0}\frac{2h}{h\left(\sqrt{2x+2h}+\sqrt{2x}\right)} = \lim_{h\to 0}\frac{2}{\sqrt{2x+2h}+\sqrt{2x}} = \frac{2}{\sqrt{2x+2(0)}+\sqrt{2x}} \\ &= \frac{2}{\sqrt{2x}+\sqrt{2x}} = \frac{2}{2\sqrt{2x}} = \frac{1}{\sqrt{2x}} \end{aligned}$$

Further Question:

Find the derivative of $f(x) = \dfrac{1}{x^2+1}$ at $x = a$.

Note the following regarding $f'(a)$, the derivative of $f(x)$ at $x = a$,

1. $f'(a)$, is the **slope of the tangent** line at $P(a, f(a))$.

   It follows that if $f'(a)$ exists then the equation of the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is
   $$y = f'(a)(x - a) + f(a) \; .$$

2. If $s = f(t)$ is the position function of an object moving in one spatial dimension, then $f'(a)$ is the **velocity** of an object at time $t = a$.

3. $f'(a)$ is, in general, the **rate of change** of $y = f(x)$ with respect to x when $x = a$.

If we replace a by x in the definition of the derivative, we have the following.

Definition: For a given function $f(x)$ the **derivative function** is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

In words, associated with any function f there is a corresponding function $f'(x)$ whose evaluation gives the value of the derivative of the function f at any value x of interest.

Example 3-6

If $f(x) = \dfrac{1}{x+3} + 1$ find $f'(x)$ using the limit definition.

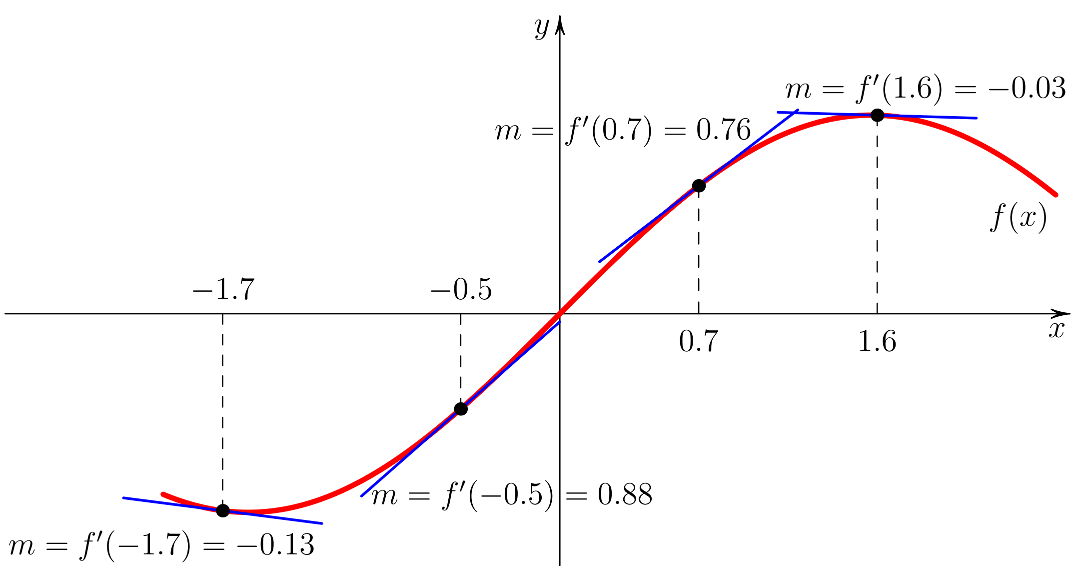
Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\left(\frac{1}{(x+h)+3} + 1\right) - \left(\frac{1}{x+3} + 1\right)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h+3} - \frac{1}{x+3}\right) \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{1(x+3) - 1(x+h+3)}{(x+h+3)(x+3)} = \lim_{h\to 0} \frac{-h}{h(x+h+3)(x+3)} = \lim_{h\to 0} \frac{-1}{(x+h+3)(x+3)} \\ &= -\frac{1}{(x+0+3)(x+3)} = -\frac{1}{(x+3)^2} \end{aligned}$$

Further Question:

If $f(x) = \sqrt{x-1}$, find $f'(x)$.

The following diagram illustrates the meaning of the derivative function $f'(x)$ graphically. Each x value has an associated point P through which there is a unique tangent with the slope given by $f'(x)$.



Derivative Notation

In addition to f' the derivative of the function $y = f(x)$ may be denoted by

$$y' \qquad \frac{dy}{dx} \qquad \frac{df}{dx} \qquad \frac{d}{dx}f \qquad Df \qquad D_x f$$

All of these are equal but have their uses in different contexts. The notation $\frac{dy}{dx}$, introduced by Gottfried Leibniz, clearly embodies the idea of the slope $\left(\frac{\Delta y}{\Delta x}\right)$. To write $f'(a)$ in this notation we often write $\left.\frac{dy}{dx}\right|_{x=a}$ where the right bar means "evaluated at". The last three notations reflect the idea of differentiation as an **operation** on a function. So $\frac{d}{dx}$ is to be understood as an operator that acts on the function following it to differentiate it. In this notation, using our last example, we could write:

$$\frac{d}{dx}\left(\sqrt{x-1}\right) = \frac{1}{2\sqrt{x-1}}$$

Notice how, in this notation, no reference must be made to a y or an f.

Definition: If $f'(a)$ exists, then function f is said to be **differentiable at *a***.

Definition: A function f is **differentiable on an interval (a, b)** if it is differentiable at every value in (a, b).

Theorem 3-1: If function $f(x)$ is differentiable at $x = a$ then f is continuous at $x = a$.

Note the converse ("If f is continuous at a then f is differentiable a") is **not** true.

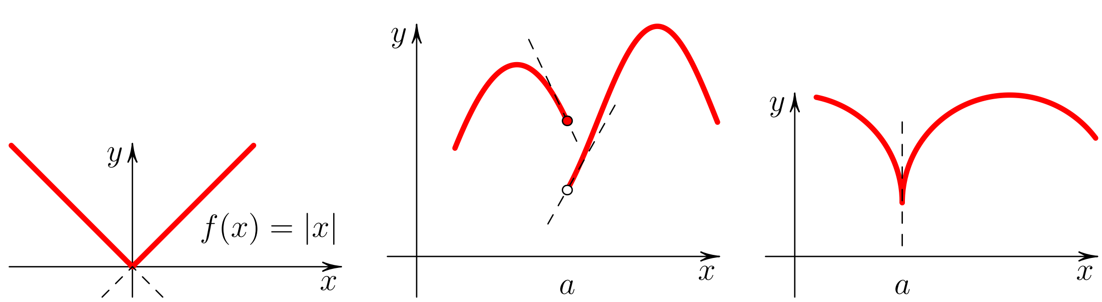
Example 3-7

The absolute value function $f(x) = |x|$ is not differentiable at $x = 0$ despite being continuous there.

Since the derivative is the slope of the tangent to $y = f(x)$ several ways f may fail to be differentiable at a value $x = a$ occur are:

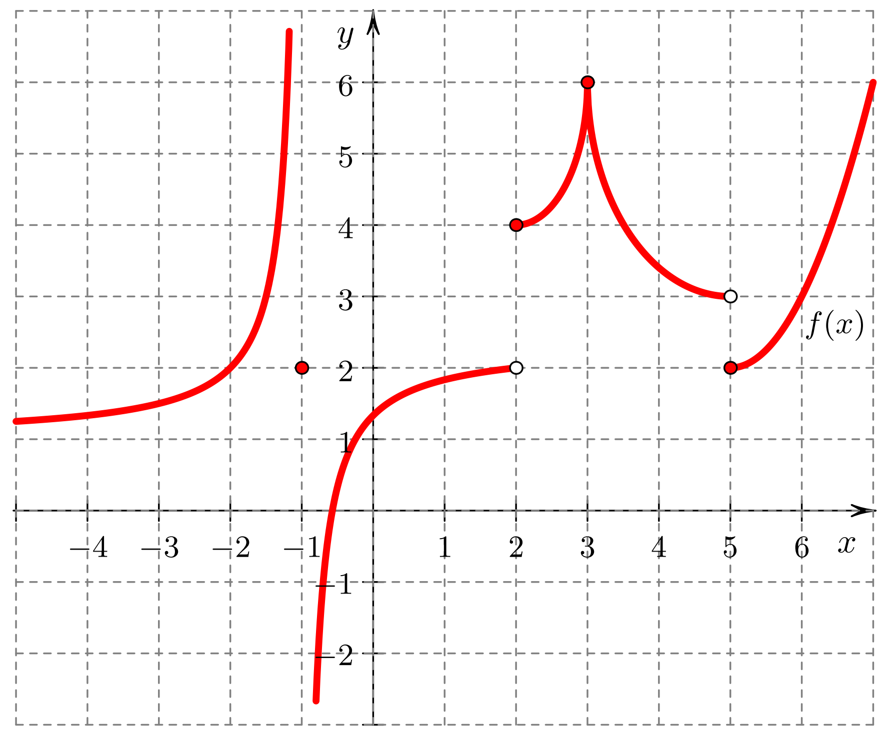
1. The graph has a corner at $(a, f(a))$.
2. The graph has a discontinuity at $x = a$
3. The graph has a vertical tangent line at $(a, f(a))$. (infinite slope)

These situations are illustrated below. Notice in the first and last example the function is continuous at a.



Example 3-8

For the graphically defined function:



1. List all values of x at which $f(x)$ fails to be continuous.
2. List all values of x at which $f(x)$ fails to be differentiable.

1. (Answer: $x = -1$, $x = 2$, $x = 5$)
2. (Answer: $x = -1$, $x = 2$, $x = 3$, $x = 5$)

Answers: Page 242

Exercise 3-2

1-2: Use the definition of the derivative at a value a to find the following.

1. $f'(2)$ if $f(x) = \dfrac{1}{x+1}$
2. $g'(4)$ if $g(x) = \sqrt{2x}$

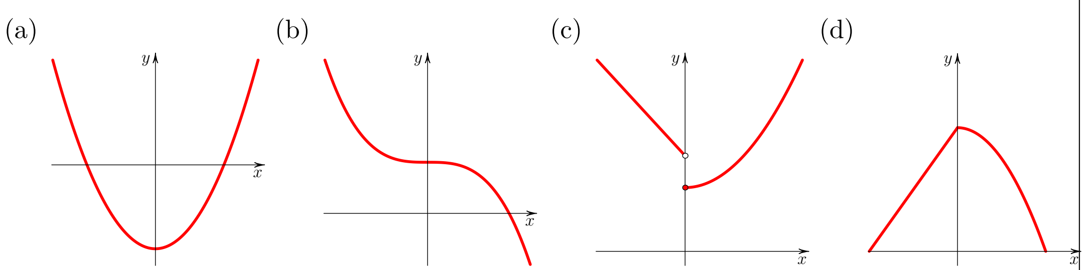
3-8: Use the definition of the derivative to calculate $f'(x)$ for each of the following functions.

3. $f(x) = x^2 + 3$
4. $f(x) = \dfrac{1}{3x}$
5. $f(x) = (x+2)^2$
6. $f(x) = \sqrt{x+2}$
7. $f(x) = \dfrac{3x+2}{x+1}$
8. $f(x) = \dfrac{1}{\sqrt{x}}$

---

9. Prove that $f(x) = \sqrt{(x-2)^2}$ is not differentiable at $x = 2$ by showing that the following left and right hand limits differ:
$$\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} \neq \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h}.$$

10. Which of the following graphs represent functions that are differentiable at $x = 0$? (Explain why or why not).



3.3 Differentiation Formulae

Several theorems for the derivatives of functions will now be introduced which will allow us to find the derivative of a function without having to resort to the limit definition.

Theorem 3-2: If f is a constant function, $f(x) = c$, then

$$f'(x) = 0 \qquad \text{or} \qquad \frac{d}{dx}(c) = 0$$

Example 3-9

$$\frac{d}{dx}(5) = 0, \qquad (\pi)' = 0, \qquad \frac{d}{dx}(-20) = 0, \qquad \left(-\frac{2}{3}\right)' = 0$$

Theorem 3-3: If f is a power function, $f(x) = x^n$, where n is any real number, then

$$f'(x) = nx^{n-1} \qquad \text{or} \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$

This is known as the **Power Rule**.

Note as a particular example of the last theorem, when $n = 1$ we have $f(x) = x$ and

$$f'(x) = 1x^{1-1} = x^0 = 1 \ .$$

In Leibniz notation this is easy to remember $\frac{d}{dx}x = 1$ because $\frac{dx}{dx}$ should be 1!

The last theorem can be demonstrated from first principles as demonstrated in the following examples.

Example 3-10

Use the definition of the derivative to prove the power function derivative $\dfrac{d}{dx}\left(x^2\right) = 2x$.

Solution:

If $f(x) = x^2$ then

$$\begin{aligned} f'(x) &= \lim_{h\to0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to0}\frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to0}\frac{h(2x+h)}{h} \\ &= \lim_{h\to0}(2x+h) = 2x + 0 = 2x \end{aligned}$$

Further Question:

Use the definition of the derivative to prove the power function derivative $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$.

(Recall $\dfrac{1}{x} = x^{-1}$ so this is a power function.)

Differentiation involving the Power Rule often requires rewriting an expression as a power. Here is a summary of some relevant identities involving powers:

$$\begin{aligned} x^0 &= 1 \\ x^m \cdot x^n &= x^{m+n} \\ \frac{x^m}{x^n} &= x^{m-n} \\ \sqrt[n]{x} &= x^{\frac{1}{n}} \\ \frac{1}{x^n} &= x^{-n} \\ (x^m)^n &= x^{mn} \\ (xy)^n &= x^n y^n \\ \left(\frac{x}{y}\right)^n &= \frac{x^n}{y^n} \end{aligned}$$

Example 3-11

Use the Power Rule directly to find the following derivatives.

1. $\left(x^4\right)'$
2. $\dfrac{d}{dt}\left(\sqrt[3]{t^2}\right)$
3. $\dfrac{d}{dx}\left(\dfrac{1}{x^c}\right)$ Here c is a constant.

Solution:

1. $\left(x^4\right)' = 4x^{4-1} = 4x^3$
2. $\dfrac{d}{dt}\left(\sqrt[3]{t^2}\right) = \dfrac{d}{dt}\left(t^2\right)^{\frac{1}{3}} = \dfrac{d}{dt}\left(t^{\frac{2}{3}}\right) = \dfrac{2}{3}t^{\frac{2}{3}-1} = \dfrac{2}{3}t^{\frac{2}{3}-\frac{3}{3}} = \dfrac{2}{3}t^{-\frac{1}{3}} = \dfrac{2}{3t^{\frac{1}{3}}} = \dfrac{2}{3\sqrt[3]{t}}$
3. $\dfrac{d}{dx}\left(\dfrac{1}{x^c}\right) = \dfrac{d}{dx}\left(x^{-c}\right) = -cx^{-c-1} = -cx^{-(c+1)} = -\dfrac{c}{x^{c+1}}$

Further Questions:

Use the Power Rule directly to find the following derivatives.

1. $\dfrac{d}{dx}\left(x^5\right)$
2. $\left(\dfrac{1}{x^{10}}\right)'$
3. $\left(\sqrt[5]{x}\right)'$
4. $\left(\dfrac{1}{\sqrt{x}}\right)'$
5. $\dfrac{d}{dx}\left(x^{23}\right)$
6. $\dfrac{d}{dt}\left(t^{\pi}\right)$
7. $\dfrac{d}{dy}\left(y^{\sqrt{3}}\right)$

Theorem 3-4: Suppose c is a constant and $f'(x)$, $g'(x)$ exist at a value x.

1. If $h(x) = cf(x)$, then $h'(x)$ exists and $h'(x) = cf'(x)$. Equivalently:
$$(cf)' = cf' \qquad \text{or} \qquad \frac{d}{dx}(cf) = c\frac{df}{dx}$$
2. If $h(x) = f(x) + g(x)$, then $h'(x)$ exists and $h'(x) = f'(x) + g'(x)$. Equivalently:
$$(f+g)' = f' + g' \qquad \text{or} \qquad \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$
3. If $h(x) = f(x) - g(x)$, then $h'(x)$ exists and $h'(x) = f'(x) - g'(x)$. Equivalently:
$$(f-g)' = f' - g' \qquad \text{or} \qquad \frac{d}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}$$

We note in the previous theorem that the functions f and g need not be differentiable everywhere – the theorem applies wherever their derivatives exist. Also it follows from the theorem that the result may be generalized to sums or differences of a finite number of terms. (i.e. $(f+g\ldots+h)' = f'+g'+\ldots+h'$, etc.) Mathematically, the previous theorem shows that differentiation is a **linear operation**.

Example 3-12

Use the previous theorems to evaluate the following derivatives directly.

1. $\frac{d}{dx}\left(4\sqrt{x}\right)$
2. $g'(t)$ if $g(t) = 4t + \frac{1}{4t}$
3. y' if $y = m(x - x_0) + y_0$ Here m, x_0, and y_0 are the straight line constants.

Solution:

1. $\frac{d}{dx}\left(4\sqrt{x}\right) = 4\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \frac{2}{\sqrt{x}}$
2. $g'(t) = \left(4t^1 + \frac{1}{4}\cdot\frac{1}{t}\right)' = 4\left(t^1\right)' + \frac{1}{4}\left(t^{-1}\right)' = 4(1)t^0 + \frac{1}{4}(-1)t^{-2} = 4 - \frac{1}{4t^2}$
3. $y' = [m(x-x_0)+y_0]' = \left[mx^1 - mx_0 + y_0\right]' = m(1)x^0 + 0 + 0 = m(1) = m$

   In other words we found that the slope of the tangent line to any point on a line is just the slope m of the line, which of course it is since the tangent line to any point on a straight line will just be the straight line itself.

Further Questions:

Use the previous theorems to evaluate the following derivatives directly:

1. $\frac{d}{dx}\left(3x^4\right)$
2. $\frac{d}{dx}\left(-5x^2\right)$
3. $\frac{d}{dx}\left(2x^2 - 3x + 1\right)$
4. y' if $y = \frac{3x^3 - x^2 + 5}{x}$
5. $f'(x)$ if $f(x) = x^4 - \frac{3}{x^2} + \sqrt[5]{x} + \frac{2\sqrt{x}}{x}$
6. $\frac{d}{dx}\left(\frac{x^{a^2+2}}{\pi + \ln 5} + \frac{2e}{x^b}\right)$

Answers: Page 243

Exercise 3-3

1-8: Differentiate the following functions involving powers, sums and constant multiplication. (Any value that is not the function variable should be considered a constant.)

1. $f(x) = \sqrt{x} - x^{12}$
2. $g(x) = \dfrac{1}{\sqrt{x^5}}$
3. $y = \dfrac{4x^3}{5}$
4. $f(u) = u^{-4} + u^4$; Also find $f'(1)$.
5. $f(x) = (x^3 + x)^2$ (Hint: Expand first.)
6. $f(x) = \sqrt{3x} + \sqrt[5]{\dfrac{x}{3}}$

   (Hint: $\sqrt[n]{xy} = (\sqrt[n]{x})(\sqrt[n]{y})$ and $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$.)
7. $f(x) = \sin(\pi/15)x^{2a}$
8. $s(t) = -\dfrac{g}{2}t^2 + v_0 t + s_0$

---

9. Calculate the instantaneous rates of change given in problems 1(b), 2(d), and 3(b) of Exercise 3-1 directly using the derivative.
10. Find the equation of the tangent line to the curve $y = x + \sqrt{x}$ at the point $P(1, 2)$.
11. Find the value(s) of x for which the curve $y = 2x^3 - 4x^2 + 5$ has a horizontal tangent line.

12-13: The following problems consider the meaning of the derivative as a rate of change.

12. The concentration of carbon dioxide in the Earth's atmosphere has been observed at Mauna Loa Observatory in Hawaii to be steadily increasing (neglecting seasonal oscillations) since 1958. A best fit curve to the data measurements taken from 1982 to 2009 yields the following function for the concentration in parts per million (ppm) as a function of the year t:

$$C(t) = 0.0143(t - 1982)^2 + 1.28(t - 1982) + 341$$

   (a) What was the level of CO_2 in the air in the year 2000?

   (b) At what rate was the CO_2 level changing with respect to time in the year 2000?

   (c) By what percentage did the CO_2 level change between 2000 and 2005?

13. A conical tank has a height of 5 metres and radius at the top of 2 metres.

   (a) Show that the volume of liquid in the tank when it is filled to a depth y is given by

$$V = \frac{4\pi}{75}y^3$$

   (b) What is the rate of change of volume with respect to depth when the tank is filled to 4 metres?

3.4 Product and Quotient Rules

Theorem 3-5: If $h(x) = f(x)g(x)$ and $f'(x)$ and $g'(x)$ exist at a value x then $h'(x) = f'(x)g(x) + f(x)g'(x)$. Equivalently:

$$(fg)' = f'g + fg' \qquad \text{or} \qquad \frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

This is known as the **Product Rule**.[1]

Note:

- The derivative of the product is **not** the product of the derivatives, $(fg)' \neq f'g'$!
- Memorize the product rule in words, namely

  "*The derivative of a product is the derivative of the first function times the second plus the first times the derivative of the second.*"

Example 3-13

Use the Product Rule in evaluating the following:

1. If $f(t) = \left(\frac{1}{t} + t\right)\left(\frac{3}{t^2} + 2t^3\right)$ find $\frac{df}{dt}$.
2. If $y = x^3\left(\frac{x^4+1}{\sqrt{x}}\right)$ find y'

Solution:

1. $$\begin{aligned}\frac{df}{dt} &= \frac{d}{dt}\left[\left(t^{-1}+t^1\right)\left(3t^{-2}+2t^3\right)\right] = \frac{d}{dt}\left(t^{-1}+t^1\right)\cdot\left(3t^{-2}+2t^3\right) + \left(t^{-1}+t^1\right)\cdot\frac{d}{dt}\left(3t^{-2}+2t^3\right)\\ &= \left[(-1)t^{-2}+1t^0\right]\left(3t^{-2}+2t^3\right) + \left(t^{-1}+t^1\right)\left[3(-2)t^{-3}+2(3)t^2\right]\\ &= \left(-t^{-2}+1\right)\left(3t^{-2}+2t^3\right) + \left(t^{-1}+t^1\right)\left(-6t^{-3}+6t^2\right)\end{aligned}$$
2. $$\begin{aligned}y' &= \left[x^3\left(\frac{x^4+1}{\sqrt{x}}\right)\right]' = \left[x^3\left(\frac{x^4}{\sqrt{x}}+\frac{1}{\sqrt{x}}\right)\right]' = \left[x^3\left(\frac{x^4}{x^{\frac{1}{2}}}+\frac{1}{x^{\frac{1}{2}}}\right)\right]' = \left[x^3\left(x^{\frac{7}{2}}+x^{-\frac{1}{2}}\right)\right]'\\ &= (3x^2)\left(x^{\frac{7}{2}}+x^{-\frac{1}{2}}\right) + (x^3)\left(\frac{7}{2}x^{\frac{5}{2}} - \frac{1}{2}x^{-\frac{3}{2}}\right)\end{aligned}$$

[1] To prove the Product Rule we use our limit theorems as follows. For clarity call $H(x) = f(x)g(x)$.

$$\begin{aligned}H'(x) &= \lim_{h\to 0}\frac{H(x+h)-H(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}g(x+h)+f(x)\frac{g(x+h)-g(x)}{h}\right]\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot\lim_{h\to 0}g(x+h)+\lim_{h\to 0}f(x)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)g(x)+f(x)g'(x)\end{aligned}$$

Note that since g is differentiable at x it is continuous at x and so $\lim_{h\to 0} g(x+h) = g(x)$ follows.

Further Questions:

Use the Product Rule to evaluate the following:

1. If $f(x) = x^2$, $g(x) = x^3$, and $h(x) = f(x)g(x) = x^5$, find $h'(x)$ directly and using the Product Rule. Confirm that $h'(x)$ is not the product $f'(x)g'(x)$.
2. If $y = (x^2 + 3x + 1)(x^3 - 2)$, find y'.
3. If $f(t) = (t^6 - 2t^3 + t^2 - 3)(\sqrt{t} - 2)$, find $f'(t)$.
4. Evaluate $\dfrac{d}{dz}\left[\left(\dfrac{1}{z^4} - 2z^3 + \sqrt{z}\right)\left(z^2 + 5\right)\right]$
5. If $g(z) = \left(\dfrac{1}{z} + \dfrac{2}{z^5}\right)\left(z^3 - 4z^2 + 10\right)$, find $\frac{dg}{dz}$.

Theorem 3-6: If $h(x) = \dfrac{f(x)}{g(x)}$ with $g(x) \neq 0$ and $f'(x)$ and $g'(x)$ exist at a value x then

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \ .$$

Equivalently:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$$

This is known as the **Quotient Rule**.

Note:

- The derivative of a quotient is **not** equal to the quotient of the derivatives, $\left(\dfrac{f}{g}\right)' \neq \dfrac{f'}{g'}$.
- The quotient rule is not symmetric under interchange of $f \leftrightarrow g$ like the product rule is because $f/g \neq g/f$. The numerator includes a minus sign (so order matters) and only the original denominator appears in the denominator of the result.
- Memorize in words:

  "*The derivative of a quotient equals the the derivative of the top times the bottom minus the top times the derivative of the bottom all over the bottom squared.*"

Example 3-14

Evaluate the following derivatives using the Quotient Rule.

1. If $h(y) = \dfrac{y^2+5}{y+2}$ find $h'(y)$
2. If $y = \dfrac{x^2+2x}{\sqrt[4]{x}+9}$ find $\dfrac{dy}{dx}$

Solution:

1. $$h'(y) = \frac{(y^2+5)'(y+2)-(y^2+5)(y+2)'}{(y+2)^2} = \frac{2y(y+2)-(y^2+5)(1)}{(y+2)^2} = \frac{2y^2+4y-y^2-5}{(y+2)^2}$$
$$= \frac{y^2+4y-5}{(y+2)^2}$$
2. $$\frac{dy}{dx} = \frac{\frac{d}{dx}\left(x^2+2x\right)\cdot\left(x^{\frac{1}{4}}+9\right)+\left(x^2+9\right)\cdot\frac{d}{dx}\left(x^{\frac{1}{4}}+9\right)}{\left(x^{\frac{1}{4}}+9\right)^2}$$
$$= \frac{(2x+2)\left(x^{\frac{1}{4}}+9\right)+\left(x^2+9\right)\left(\frac{1}{4}x^{-\frac{3}{4}}\right)}{\left(x^{\frac{1}{4}}+9\right)^2}$$

Further Questions:

Evaluate the following derivatives using the Quotient Rule.

1. If $f(x) = \dfrac{x^2-3x+5}{x-1}$ find $f'(x)$.
2. If $g(t) = \dfrac{t^4+2t}{t^3-5t^2+5}$ find $\dfrac{dg}{dt}$.
3. Evaluate $\dfrac{d}{dz}\left(\dfrac{\sqrt{z}+1}{z^2+5}\right)$.

When differentiating products and quotients where one of the functions is a constant function it is much easier to pull the constant out front directly using our constant rule $(cf)' = cf'$.

Example 3-15

You do not need to use the quotient rule to evaluate the following derivative with a constant denominator:

$$\frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{d}{dx}\left(\frac{1}{2}x^2\right) = \frac{1}{2}\frac{d}{dx}\left(x^2\right) = \frac{1}{2}(2x^1) = x\,.$$

If one evaluates this using the quotient rule a zero arises due to the differentiation of the constant.

As with the other rules, the product and quotient rules may need to both be applied or applied repeatedly when evaluating a given derivative.

Example 3-16

Differentiate $f(x) = (x^2)\left(\sqrt{x} - \frac{1}{x}\right)(3x + 7)$

Solution:

$$\begin{aligned}
f'(x) &= \left[\underbrace{(x^2)\left(x^{\frac{1}{2}} - x^{-1}\right)}\underbrace{(3x+7)}\right]' \quad \Leftarrow \text{ Apply Product Rule on braced factors.} \\
&= \left[(x^2)\left(x^{\frac{1}{2}} - x^{-1}\right)\right]'(3x+7) + \left[(x^2)\left(x^{\frac{1}{2}} - x^{-1}\right)\right](3x+7)' \\
&= \left[(x^2)'\left(x^{\frac{1}{2}} - x^{-1}\right) + (x^2)\left(x^{\frac{1}{2}} - x^{-1}\right)'\right](3x+7) + \left[(x^2)\left(x^{\frac{1}{2}} - x^{-1}\right)\right](3x+7)' \\
&= (x^2)'\left(x^{\frac{1}{2}} - x^{-1}\right)(3x+7) + (x^2)\left(x^{\frac{1}{2}} - x^{-1}\right)'(3x+7) + (x^2)\left(x^{\frac{1}{2}} - x^{-1}\right)(3x+7)' \\
&= (2x)\left(x^{\frac{1}{2}} - x^{-1}\right)(3x+7) + (x^2)\left(\frac{1}{2}x^{-\frac{1}{2}} + x^{-2}\right)(3x+7) + (x^2)\left(x^{\frac{1}{2}} - x^{-1}\right)(3) \\
&= 2x\left(x^{\frac{1}{2}} - x^{-1}\right)(3x+7) + x^2\left(\frac{1}{2}x^{-\frac{1}{2}} + x^{-2}\right)(3x+7) + 3x^2\left(x^{\frac{1}{2}} - x^{-1}\right)
\end{aligned}$$

Further Questions:

Differentiate the following functions:

1. $f(x) = \dfrac{(\sqrt{x}+1)(x^2 - 3x + 1)}{x^3 + 4}$
2. $g(x) = (\sqrt[3]{x}+1)(x^3 + 4x^2 - 4)(x^2 + 1)$

As the last example shows one can generalize the Product Rule to three (or more) terms as follows $(fgh)' = f'gh + fg'h + fgh'$, etc.

Exercise 3-4

Answers: Page 243

1-8: Differentiate the following functions involving products and quotients. (Any value that is not the function variable should be considered a constant.)

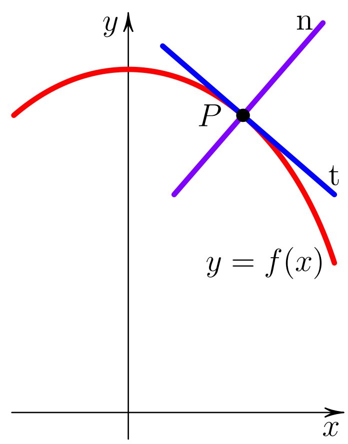
1. $f(x) = (x^4 - 3x^2 + 2)\left(x^{\frac{1}{3}} - x\right)$
2. $h(x) = (x^3 + \pi x + 2)\left(2 + \frac{1}{x^3}\right)$
3. $y = (x^2 - 1)(x^3 + 2)(2x^2 + \sqrt{x})$
4. $f(x) = \dfrac{x-4}{x-6}$
5. $f(\theta) = \dfrac{\theta^2 + 3\theta - 4}{\theta^2 - 7}$
6. $g(x) = \dfrac{1 + x^2}{\sqrt{x}}$; Also find $g'(4)$.
7. $f(v) = \dfrac{(2v+3)(v + 4/v)}{v^2 + v}$
8. $h(x) = cx^2 + (3\sqrt{x} + 2)(2x^2 + x)$

3.5 Further Geometrical Applications of the Derivative

Note the following properties of the slope of a line:

- The slope of **horizontal** line is **zero** ($\boldsymbol{m = 0}$).
- If two lines are **parallel** then their slopes are **equal** ($\boldsymbol{m_1 = m_2}$)
- If two lines are **perpendicular** then their slopes satisfy $\boxed{\boldsymbol{m_1 \cdot m_2 = -1}}$, that is they are the **negative reciprocals of each other**, $\left(\boldsymbol{m_2 = -\frac{1}{m_1}}\right)$.

Definition: The **normal line** to a curve at point P is the line through P perpendicular to the tangent line.



With these definitions in mind we can solve various problems arising from geometry, keeping in mind that the derivative at x gives the slope of the tangent at $P(x, f(x))$.

Example 3-17

1. Find the equation of the normal line to the curve $y = 3x^2 - 8x + 2$ at $P(1, -3)$.
2. At what point does the previous curve have a horizontal tangent?

Solution:

1. $$y' = 6x - 8 \implies m_t = y'(1) = 6(1) - 8 = -2 \implies m_n = -\frac{1}{m_t} = -\frac{1}{(-2)} = \frac{1}{2}$$

   $$y = m(x - x_0) + y_0 \implies y = \frac{1}{2}(x-1) - 3 \quad \text{(Point-Slope Form)}$$

   $$\implies y = \frac{1}{2}x - \frac{7}{2} \quad \text{(Slope-Intercept Form)}$$

2. Horizontal tangent implies slope $m_t = 0$. Since the slope of the tangent is the derivative, $0 = y' = 6x - 8 \implies 6x = 8 \implies x = \frac{4}{3}$. The y-coordinate is

   $$y = 3\left(\frac{4}{3}\right)^2 - 8\left(\frac{4}{3}\right) + 2 = \frac{16 - 32 + 6}{3} = -\frac{10}{3},$$

   so at the point $P(4/3, -10/3)$ the tangent line is horizontal.

Further Questions:

1. Find the equation of the tangent and normal lines to the curve $y = 2x^3 + 3x^2 - 2$ when $x = 1$.
2. For what values of x does the graph of $f(x) = 2x^3 - 3x^2 - 6x + 87$ have a horizontal tangent line?
3. Find a tangent line to the graph of $y = 3x^2 + 4x - 6$ that is parallel to the line $5x - 2y - 1 = 0$.

Exercise 3-5

Answers: Page 244

1. Find the line through the point $P(2,1)$ that is parallel to the tangent to the curve $y = 3x^2 + 2x + 1$ at the point $Q(1,6)$.
2. Find the normal line to the curve $y = \sqrt{x} + x^2$ at the point $P(1,2)$.
3. Find any points on the curve $y = x^3 - 4$ with normal line having slope $-\dfrac{1}{12}$.

3.6 General Power Rule

Theorem 3-7: If n is any real number and $f(x)$ is differentiable at x then

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x) .$$

This is the **General Power Rule**.[2]

Example 3-18

Differentiate the following:

1. $F(t) = \sqrt{t^2+3t+2}$
2. $f(x) = (2x+3)^2(x+5)$
3. $y = \dfrac{3}{(\sqrt{x}+7x)^3}$

Solution:

1. $F'(t) = \left[\left(t^2+3t+2\right)^{\frac{1}{2}}\right]' = \frac{1}{2}(t^2+3t+2)^{-\frac{1}{2}}(2t+3) = \dfrac{2t+3}{2\sqrt{t^2+3t+2}}$
2. $f'(x) = \left[(2x+3)^2\right]'(x+5) + (2x+3)^2(x+5)'$ $\Leftarrow$ Apply Product Rule First
$= 2(2x+3)^1(2)(x+5) + (2x+3)^2(1+0)$
$= 4(2x+3)(x+5) + (2x+3)^2 = (2x+3)(4x+20+2x+3) = (2x+3)(6x+23)$
3. $y' = \left[3\left(x^{\frac{1}{2}}+7x\right)^{-3}\right]' = 3(-3)\left(x^{\frac{1}{2}}+7x\right)^{-4}\cdot\left(\frac{1}{2}x^{-\frac{1}{2}}+7\right) = -\dfrac{9}{(\sqrt{x}+7x)^4}\left(\dfrac{1}{2\sqrt{x}}+7\right)$

Further Questions:

Differentiate the following:

1. $y = \left(x^2-2x+5\right)^8$
2. $y = \dfrac{2}{\sqrt{x^3-2x^2+3}}$
3. $f(t) = \left[\dfrac{1}{t}+(2-t)^5\right]^{-4}$
4. $g(x) = \dfrac{\left(x^2+3\right)^5}{\sqrt[3]{x+1}}$

[2]For positive integer n the General Power Rule follows from the generalized Product Rule since

$$([f(x)]^n)' = [\underbrace{f(x)f(x)\cdots f(x)}_{n\text{ factors}}]' = \underbrace{f'(x)f(x)\cdots f(x) + f(x)f'(x)\cdots f(x) + \cdots + f(x)f(x)\cdots f'(x)}_{n\text{ equal terms}} = n[f(x)]^{n-1}f'(x) .$$

The rule is also just a special case of the Chain Rule (Section 3.8) with the outer function taken to be $f(u) = u^n$ so that $\frac{d}{dx}u^n = nu^{n-1}\cdot\frac{du}{dx}$.

Answers: Page 244

Exercise 3-6

1-6: Differentiate the following functions requiring use of the General Power Rule. (Any value that is not the function variable should be considered a constant.)

1. $f(x) = \left(x^2 + 3\right)^9$
2. $g(x) = \dfrac{1}{x + \sqrt{x}}$
3. $f(t) = \dfrac{7}{\sqrt{2t^2 + 3t + 4}}$
4. $y = \left(\dfrac{4x + 3}{x^2 + x}\right)^{-\frac{1}{7}}$
5. $h(x) = \sqrt[3]{5x^n + 4c}$
6. $f(x) = \left[\left(2x + \sqrt{x}\right)^4 + 3x\right]^5$

3.7 Derivatives of Trigonometric Functions

Using our derivative laws, the trigonometric identities, and recalling the following trigonometric limits,

$$\lim_{\theta\to 0}\sin\theta = 0 \qquad \lim_{\theta\to 0}\cos\theta = 1 \qquad \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0\ ,$$

the next result may be shown.

Theorem 3-8: The basic trigonometric functions have the following derivatives:

1. $\dfrac{d}{dx}(\sin x) = \cos x$
2. $\dfrac{d}{dx}(\cos x) = -\sin x$
3. $\dfrac{d}{dx}(\tan x) = \sec^2 x$
4. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$
5. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
6. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

Proof:

1. Let $f(x) = \sin x$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\ &= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0}\left[\sin x\frac{(\cos h - 1)}{h} + \cos x\frac{\sin h}{h}\right] \\ &= (\sin x)(0) + (\cos x)(1) \\ &= \cos x \end{aligned}$$

3. $\dfrac{d}{dx}(\tan x) = \dfrac{d}{dx}\left(\dfrac{\sin x}{\cos x}\right) = \dfrac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{[\cos x]^2} = \dfrac{1}{\cos^2 x} = \left(\dfrac{1}{\cos x}\right)^2 = \sec^2 x$

5. $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{d}{dx}(\cos x)^{-1} = -(\cos x)^{-2}(-\sin x) = \dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x} = \sec x \tan x$

Once the basic trigonometric derivatives are known one may use them to differentiate any function involving a trigonometric function.

Example 3-19

Evaluate the following:

1. y' if $y = x^3 \sec x$
2. $g'(\theta)$ if $g(\theta) = (3\csc\theta + \sin\theta)^4$
3. $f'(x)$ if $f(x) = \dfrac{\cos x}{1+\tan x}$
4. $\dfrac{dH}{dz}$ if $H(z) = \left(z^2 + \dfrac{1}{z^2} + \cot z\right)^{-4}$

Solution:

1. $y' = 3x^2 \sec x + x^3 \sec x \tan x$
2. $g'(\theta) = 4(3\csc\theta + \sin\theta)^3(-3\csc\theta\cot\theta + \cos\theta)$
3. $f'(x) = \dfrac{-\sin x(1+\tan x) - \cos x\left(0 + \sec^2 x\right)}{(1+\tan x)^2} = \dfrac{-\sin x - \sin x\tan x - \cos x\sec^2 x}{(1+\tan x)^2}$
4. $\dfrac{dH}{dz} = -4\left(z^2 + \dfrac{1}{z^2} + \cot z\right)^{-5}\left(2z - \dfrac{2}{z^3} - \csc^2 z\right)$

Further Questions:

Evaluate the following:

1. $\dfrac{d}{dx}(\sin x \cos x)$
2. $\left(\dfrac{x^2+3}{\sin x}\right)'$
3. z' if $z = \cos^2\theta + \cos\theta$
4. $\dfrac{dz}{d\theta}$ if $z = \tan^3\theta$

Example 3-20

Suppose $f(x) = \sin^3[g(x)]$ and $g(2) = \pi$ and $g'(2) = 4$. Find $f'(2)$.

Solution:

By the General Power Rule we have, since $\sin^3[g(x)] = (\sin[g(x)])^3$,

$$\begin{aligned} & f'(x) = 3\sin^2[g(x)] \cdot g'(x) \\ \implies & f'(2) = 3\sin^2[g(2)] \cdot g'(2) \\ & \quad = 3\sin^2(\pi)(4) = 3(-1)^2(4) = 12 \end{aligned}$$

Further Question:

Suppose $f\left(\dfrac{\pi}{3}\right) = 4$ and $f'\left(\dfrac{\pi}{3}\right) = -2$ and let $g(x) = f(x)\sin x$. Find $g'\left(\dfrac{\pi}{3}\right)$.

Example 3-21

Find the tangent line to the curve $y = \sin x + 2\cos x$ at the point $P(\pi/2, 1)$.

Solution:

$$\begin{aligned} y' = \cos x - 2\sin x &\implies m = y'(\pi/2) = \cos(\pi/2) - 2\sin(\pi/2) = 0 - 2(1) = -2 \\ y = m(x - x_0) + y_0 &\implies y = -2(x - \pi/2) + 1 \quad \text{(Point-Slope Form)} \\ &\implies y = -2x + (\pi + 1) \quad \text{(Slope-Intercept Form)} \end{aligned}$$

Further Question:

Find the (x-coordinates of the) points on the curve $y = \dfrac{\cos x}{2 + \sin x}$ at which the tangent is horizontal.

Answers: Page 244

Exercise 3-7

1. Find the derivative $f'(x)$ of $f(x) = \sin 4x$ using the definition of the derivative.

2-5: Differentiate the following functions involving trigonmetric functions.

2. $f(x) = x^2 \cos x$
3. $f(t) = \dfrac{t^3}{\sin t + \tan t}$
4. $H(\theta) = \csc\theta \cot\theta$; Also find $\left.\dfrac{dH}{d\theta}\right|_{\theta=\pi/3}$.
5. $f(x) = (\sin x + \cos x)(\sec x - \cot x)$

---

6. Calculate $\dfrac{df}{d\theta}$ for the function $f(\theta) = \sin^2\theta + \cos^2\theta$
   (a) Directly by using the rules of differentiation.
   (b) By first simplifying f with a trigonometric identity and then differentiating.

7. For the curve $y = x \tan x$ and point $P\left(\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$,
   (a) Confirm the point P lies on the curve. (b) Find the equation of the tangent line at P.

3.8 The Chain Rule

Suppose

$$h(x) = \sqrt{3x^2 + 2x + 1}$$

Since $h(x) = \left(3x^2 + 2x + 1\right)^{\frac{1}{2}}$ the derivative could be found using the General Power Rule. Notice $h(x)$ can be written as a composition of two functions. Let

$$y = f(u) = \sqrt{u}$$

and

$$u = g(x) = 3x^2 + 2x + 1$$

then

$$y = h(x) = f\left(g(x)\right)$$

The following rule generalizes the General Power Rule to the case where the external function in the composition (written f here) is an arbitrary differentiable function of u, not just $f(u) = u^n$.

Theorem 3-9: If the derivatives $g'(x)$ and $f'\left(g(x)\right)$ both exist, and $h(x) = f \circ g(x) = f\left(g(x)\right)$ is the composite function then

$$h'(x) = f'\left(g(x)\right) \cdot g'(x)\ .$$

Equivalently, if $y = f(u)$ and $u = g(x)$ are both differentiable then this result may also be stated

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}\,,$$

where $\dfrac{dy}{du}$ is evaluated at $g(x)$. This is known as the **Chain Rule**.

Example 3-22

Solve the following using the Chain Rule where possible to find the derivatives:

1. If $y = \cot\left(x^3 + 5x\right)$ find y'.
2. If $y = u^2 + 2u$ and $u = \sqrt{x} + 1$:
   (a) Find $y(x)$.
   (b) Find $\dfrac{dy}{dx}$.
3. If $M(x) = [\sin x + x^2 + 5]^{\frac{3}{4}}$ find $M'(x)$ and $M'(0)$.
4. If $g(\theta) = \sqrt{\sin\left(\theta^2 + 4\right)}$ find $\dfrac{dg}{d\theta}$.
5. If $f(x) = \tan^8\left(\sqrt{x^3 + 1}\right)$ find $f'(x)$.

Solution:

1. $y' = \left[\cot\left(x^3 + 5x\right)\right]' = \sec^2\left(x^3 + 5x\right) \cdot \left(x^3 + 5x\right)' = \sec^2\left(x^3 + 5x\right)\left(3x^2 + 5\right)$
2. Since $y = u^2 + 3u$ and $u = 2\sqrt{x} + 1$,
   (a) $y(x) = \left(2\sqrt{x} + 1\right)^2 + 3\left(\sqrt{x} + 1\right)$
   (b) $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = (2u + 3)\left[(2)\left(\dfrac{1}{2}\right)x^{-\frac{1}{2}} + 0\right] = \left[2\left(2\sqrt{x} + 1\right) + 3\right]x^{-\frac{1}{2}}$
3. $M'(x) = \dfrac{3}{4}[\sin x + x^2 + 5]^{-\frac{1}{4}}(\cos x + 2x) \qquad M'(0) = \dfrac{3}{4}[0 + 0 + 5]^{-\frac{1}{4}}(1 + 0) = \dfrac{3}{4\sqrt[4]{5}}$

4. In this example we have a composition of three functions and must apply the Chain Rule twice. We show the differentiation at each step but once the pattern is observed one can arrive at the final line in a single step.

$$\begin{aligned}\frac{dg}{d\theta} &= \frac{d}{d\theta}\left\{\left[\sin\left(\theta^2+4\right)\right]^{\frac{1}{2}}\right\}\\ &= \frac{1}{2}[\sin\left(\theta^2+4\right)]^{-\frac{1}{2}}\frac{d}{d\theta}\left[\sin\left(\theta^2+4\right)\right]\\ &= \frac{1}{2}[\sin\left(\theta^2+4\right)]^{-\frac{1}{2}}\cos\left(\theta^2+4\right)\frac{d}{d\theta}\left(\theta^2+4\right)\\ &= \frac{1}{2}[\sin(\theta^2+4)]^{-\frac{1}{2}}\cos(\theta^2+4)(2\theta) = \frac{\theta\cos(\theta^2+4)}{\sqrt{\sin(\theta^2+4)}}\end{aligned}$$

5. $$\begin{aligned}f'(x) &= \left[\tan^8\left(\sqrt{x^3+1}\right)\right]' = 8\tan^7\left(\sqrt{x^3+1}\right)\sec^2\left(\sqrt{x^3+1}\right)\frac{1}{2}\left(x^3+1\right)^{-\frac{1}{2}}(3x^2)\\ &= \frac{12x^2\tan^7\left(\sqrt{x^3+1}\right)\sec^2\left(\sqrt{x^3+1}\right)}{\sqrt{x^3+1}}\end{aligned}$$

Further Questions:

Solve the following using the **Chain Rule** where possible to find the derivatives:

1. If $h(x)=\sqrt{3x^2+2x+1}$ find $h'(x)$.
2. If $y=u^3+u^2+1$ and $u=(2x^2-1)$,
   (a) Find $y(x)$.
   (b) Find the derivative $\dfrac{dy}{dx}$.
   (c) Find $\left.\dfrac{dy}{dx}\right|_{x=2}$.
3. If $y=u^4+3u^2-3$ and $u=\sqrt{x}-1$, find $\dfrac{dy}{dx}$.
4. If $h(x)=\cot\left(3x^2+5\right)$ find $h'(x)$.
5. If $y=5\sqrt[3]{1+\sqrt{t}}$ find y' .
6. If $z=\tan\sqrt{1+x^3}$ find $\dfrac{dz}{dx}$.
7. Evaluate $\dfrac{d}{d\theta}\left(\cos^3\theta+\cos\theta^3\right)$.
8. Evaluate $\left(\sqrt[3]{\dfrac{x^2+1}{x^3+5x}}\right)'$.
9. If $y=\dfrac{1}{\sin(x-\sin x)}$, find y' .
10. If $h(z)=\left(z^2-4z+5\right)^4\cdot\sec(3z)$, find $h'(z)$.

Answers: Page 245

Exercise 3-8

1-13: Differentiate the following functions requiring use of the Chain Rule. (Any value that is not the function variable should be considered a constant.)

1. $f(x) = \left(x^8 - 3x^4 + 2\right)^{12}$
2. $g(x) = \sqrt{3x^2 + 2}$; Also find $g'(2)$.
3. $f(\theta) = \sin(\theta^2)$
4. $h(\theta) = \cot^2 \theta$
5. $f(x) = \sec\left[\left(x^3 + 3\right)\left(\sqrt{x} + x\right)\right]$
6. $y = 4\cos\sqrt[3]{x}$
7. $f(x) = \dfrac{1}{3 + \sin^2 x}$
8. $y = (\csc x + 2)^5 + x^2 + x$
9. $y = \pi \tan\theta + \tan(\pi\theta)$
10. $g(x) = \left(\sqrt{x + \sqrt{x}}\right)\left(x^4 - 1\right)^7$
11. $f(x) = \left(\dfrac{x - 3}{x + 1}\right)^3$
12. $A(t) = \cos(\omega t + \phi)$; Also find $\left.\dfrac{dA}{dt}\right|_{t=0}$.
13. $f(x) = \sin\left[\cos\left(x^2 + x\right)\right]$

---

14. For the curve $y = 5x + 3\sin(2x) - 2\cos(3x)$ and point $P(0, -2)$,
    (a) Confirm the point P lies on the curve. (b) Find the equation of the tangent line at P .
15. Find the value(s) of θ for which the curve $f(\theta) = \cos^2\theta - \sin\theta$ has a horizontal tangent line.

3.9 Implicit Differentiation

When we define y by placing y on one side of the equation and $f(x)$ on the other we say y is **explicitly defined**.

Example 3-23

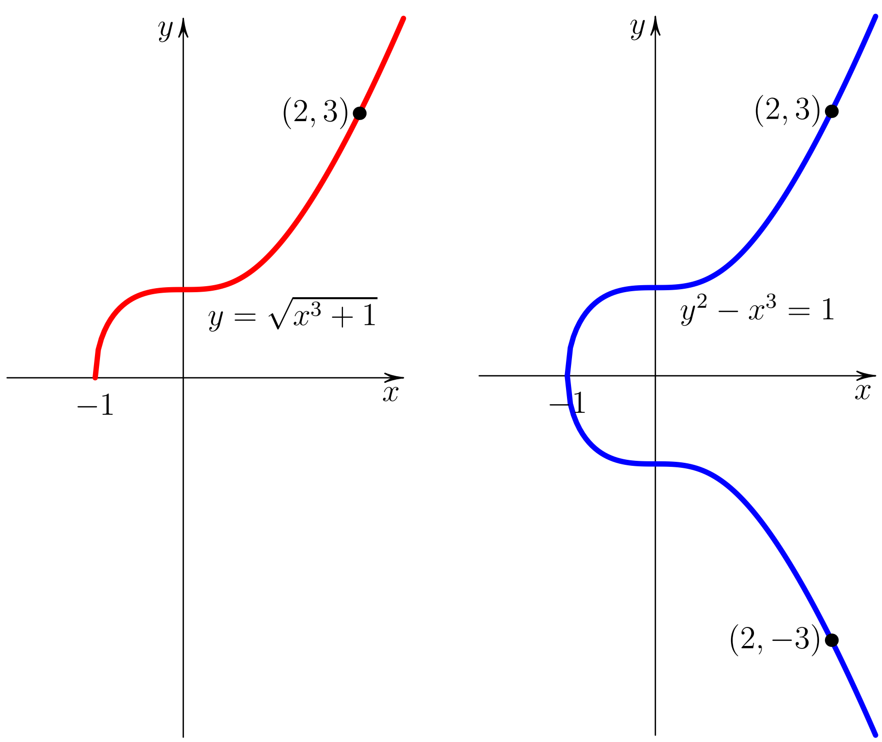
Explicit: $y = \sqrt{x^3+1}$ $\qquad$ $y = x\cos x$

On the other hand y may be defined **implicitly** through any equation involving x and y with y not-isolated on one side or the other:

Example 3-24

Implicit: $y^2 - x^3 = 1$ $\qquad$ $x^2 + y^2 + 2xy = 10$

In the implicit case the curve is defined by finding points (x, y) that satisfy the equation. In the first implicit example the point $(2, 3)$ is on the curve defined by the equation because $3^2 - 2^3 = 1$. Sometimes one may *solve the equation for y* to rewrite an implicitly defined y as an explicitly defined one, but this is often not possible. An explicitly defined equation will always be a function, whereas a curve defined implicitly will often only be a relation. For instance in our first implicit example the point $(2, -3)$ also lies on the curve so y cannot be described by a function. If we tried to solve it for y we would get our first explicit example, but notice now, since the square root function is always the positive root, only half of the implicitly defined curve is present. ($y = -\sqrt{x^3+1}$ would be a function describing the lower part of the implicit relation.) The situation is depicted below.



Suppose we wanted the tangent slope at $(2,-3)$ for our implicitly defined example. An elegant method to find the derivative at this point is to use **implicit differentiation** which involves the following steps:

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x.
2. Solve the resulting equation for y'.

Differentiating any expression involving only x follows the usual rules. When differentiating y with respect to x we just get $\frac{d}{dx}y = \frac{dy}{dx} = y'$. When differentiating a function of y with respect to x the Chain Rule is required:

$$\frac{d}{dx}f(y) = \frac{df}{dy} \cdot \frac{dy}{dx} = f'(y) \cdot y'$$

In other words, differentiate the function of y as you would x but remember to multiply it by y'.

The following examples illustrate the procedure.

Example 3-25

Use implicit differentiation to find y' of the following functions:

1. $x^3 + y^4 = 9$
2. $y^2 + \tan x = 5$

Solution:

1. Differentiate both sides of equation with respect to x:

$$\frac{d}{dx}\left(x^3 + y^4\right) = \frac{d}{dx}(9)$$
$$3x^2 + 4y^3 y' = 0$$

Solve for y':

$$y' = -\frac{3x^2}{4y^3}$$

2. Differentiate both sides with respect to x:

$$\frac{d}{dx}\left(y^2 + \tan x\right) = \frac{d}{dx}(5)$$
$$2yy' + \sec^2 x = 0$$

Solve for y':

$$2yy' = -\sec^2 x$$
$$y' = -\frac{\sec^2 x}{2y}$$

Further Questions:

Find y' if

1. $y^3 = 3xy - x^3$
2. $x^2y^2 + xy = 6x + 5y$
3. $\cos x + \sin y = xy$
4. $\sin(x + y) = \cos x$
5. $\cos(xy) - \tan x + x^5y = \sin y$

When differentiating implicitly the derivative is usually a function of both the x *and* y-coordinates.

Example 3-26

Find the equation of the tangent line to the curve $x^3 + y^4 = 9$ at $P(2, -1)$.

Solution:

The slope m of the tangent line is the derivative evaluated at the point. From Problem 1 of Example 3-25 the derivative is $y' = -\dfrac{3x^2}{4y^3}$. Inserting $(x, y) = (2, -1)$ yields:

$$y'(2,-1) = -\frac{3x^2}{4y^3}\bigg|_{(x,y)=(2,-1)} = -\frac{3(2)^2}{4(-1)^3} = 3$$

With $m = 3$ and $(x_0, y_0) = (2, -1)$ in the point-slope formula $y = m(x - x_0) + y_0$ the tangent line is

$$\begin{aligned} y &= 3(x-2) - 1 && (\Leftarrow \text{ Point-Slope form}) \\ y &= 3x - 6 - 1 \\ y &= 3x - 7 && (\Leftarrow \text{ Slope-Intercept form}) \end{aligned}$$

Further Question:

Find the equation of the tangent line to the curve $y^2 - x^3 = 1$ at $P(2, -3)$.

Exercise 3-9

Answers: Page 245

1-6: Calculate y' for functions $y = y(x)$ defined implicitly by the following equations. (Any value that is not x or y should be considered a constant.)

1. $x^2 + y^2 = 3x$
2. $xy^2 - 2x^3y + x^3 = 1$; Also find $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,2)}$.
3. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
4. $\sin(xy) = y$
5. $\cos(x + y) + x^2 = \sin y$
6. $\dfrac{\sec y}{x} = \sin x$

---

7. Consider the curve generated by the relation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$.
   (a) Confirm that the point $P(-1, 3\sqrt{3})$ lies on the curve.
   (b) Find the equation of the tangent line to the curve at the point P .

The following exercise requires use of all the differentiation rules we have introduced.

Exercise 3-10

Answers: Page 246

1-12: Differentiate the following functions using any appropriate rules.

1. $f(x) = 4x^5 + 3x^2 + x + 4$
2. $y = x^8 + \dfrac{2}{x^3} - \sqrt{x} + \dfrac{4}{\sqrt{x}} + 10$
3. $g(t) = \sqrt[3]{t^2} + t^4 + \dfrac{6}{t^2}$
4. $f(x) = \dfrac{x+5}{\sqrt{x}}$
5. $g(x) = \left(\sqrt{x} + 3x + 1\right)(x + \pi)$
6. $h(y) = \dfrac{(y+4)^3}{y+5}$
7. $y = \sqrt[4]{x^3 + 2x + 5}$
8. $f(\theta) = \cos 3\theta + \sin^2 \theta$
9. $g(x) = \tan\left(x^2 + 1\right)\cos(x)$
10. $y = \sqrt[4]{(\sin t + 5)^3}$
11. $f(x) = \sqrt[3]{\dfrac{x^4 + 5x - 1}{x^2 - 3}}$
12. $x^3y^4 + y^2 = xy + 6$

3.10 Higher Derivatives

Consider the function $y = f(x)$ with first derivative $f'(x)$. The latter is itself a function, and if we differentiate the derivative we get the **second derivative**, $\boldsymbol{f''(x)}$:

$$f''(x) = \frac{d}{dx}\left(f'(x)\right)$$

Other notation for the second derivative is:

$$y'' \qquad \frac{d^2y}{dx^2} \qquad \frac{d^2f}{dx^2} \qquad \frac{d^2}{dx^2}f \qquad D^2f \qquad D_x^2f$$

Putting the 2 in the numerator before the y allows us to write the second derivative operator $\frac{d^2}{dx^2}$ separately from the function following it upon which it acts.[3]

Similarly we can differentiate again to get the third derivative of a function, $f'''(x)$, also denoted:

$$y''' \qquad \frac{d^3y}{dx^3} \qquad \frac{d^3f}{dx^3} \qquad \frac{d^3}{dx^3}f \qquad D^3f \qquad D_x^3f$$

Example 3-27

Find the requested derivatives for each of the following functions.

1. If $f(x) = 4x^3 + 5x^2 + 6x - 10$, find $f'(x), f''(x), f'''(x)$.
2. If $g(t) = \sqrt{t^2 + \frac{1}{t}} = \sqrt{t^2 + t^{-1}}$, find $g''(1)$.
3. If $y = \sec x + \cos x$, find y''.
4. If $g(t) = \dfrac{t^3+1}{t+4}$, find $\dfrac{d^2g}{dt^2}$.

Solution:

1. $f'(x) = 12x^2 + 10x + 6\,, \quad f''(x) = 24x + 10\,, \quad f'''(x) = 24$
2. $$\begin{aligned} g'(t) &= \left(\sqrt{t^2 + \frac{1}{t}}\right)' = \left[\left(t^2 + t^{-1}\right)^{\frac{1}{2}}\right]' \\ &= \frac{1}{2}\left(t^2 + t^{-1}\right)^{-\frac{1}{2}}\left(2t - t^{-2}\right) \\ g''(t) &= -\frac{1}{4}(t^2 + t^{-1})^{-\frac{3}{2}}(2t - t^{-2})(2t - t^{-2}) + \frac{1}{2}(t^2 + t^{-1})^{-\frac{1}{2}}(2 + 2t^{-3}) \\ &= -\frac{1}{4}\left(t^2 + t^{-1}\right)^{-\frac{3}{2}}\left(2t - \frac{1}{t^2}\right)^2 + \left(t^2 + t^{-1}\right)^{-\frac{1}{2}}\left(1 + \frac{1}{t^3}\right) \\ g''(1) &= -\frac{1}{4}\left(1^2 + 1^{-1}\right)^{-\frac{3}{2}}\left(2(1) - \frac{1}{1^2}\right)^2 + \left(1^2 + 1^{-1}\right)^{-\frac{1}{2}}\left(1 + \frac{1}{1^3}\right) \\ &= -\frac{1}{2^2}2^{-\frac{3}{2}}(1^2) + 2^{-\frac{1}{2}}(2^1) = 2^{-\frac{7}{2}} + 2^{\frac{1}{2}} \end{aligned}$$

[3] The notation also keeps the units straight. For instance, for acceleration, as we will see, which is the second derivative of spatial position (a distance) with respect to time, $a = \frac{d^2s}{dt^2}$, only the time unit is squared, not the distance to give dimensions of acceleration of m/(sec^2). The squaring of the denominator unit (and not the numerator which remains as is) is reflected correctly in the second derivative notation.

3. $y' = \sec x \tan x - \sin x$
$$y'' = (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x) - \cos x$$
$$= \sec x \tan^2 x + \sec^3 x - \cos x$$

4. $$\frac{dg}{dt} = \frac{3t^2(t+4) - (t^3+1)(1)}{(t+4)^2}$$
$$= \frac{3t^3 + 12t^2 - t^3 - 1}{(t+4)^2} = \frac{2t^3 + 12t^2 - 1}{(t+4)^2}$$
$$\frac{d^2g}{dt^2} = \frac{(6t^2+24t)(t+4)^2 - (2t^3+12t^2-1)2(t+4)}{(t+4)^4} = \frac{(6t^2+24t)(t+4) - 2(2t^3+12t^2-1)}{(t+4)^3}$$
$$= \frac{6t^3 + 24t^2 + 24t^2 + 96t - 4t^3 - 24t^2 + 2}{(t+4)^3} = \frac{2t^3 + 24t^2 + 96t + 2}{(t+4)^3}$$

Further Questions:

Find the requested derivatives.

1. If $y = 3x^4 - 5x^3 + 10x$, find y', y'', and y''' .
2. If $y = \sqrt{t^2 + 12t}$, find $\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dt^2}$.
3. If $f(x) = x^3 + \tan x$, find $f'(x)$ and $f''(x)$.
4. Find $\dfrac{d^2}{dx^2}\left(\dfrac{x^2}{x+1}\right)$.

It is also possible to use implicit differentiation to find higher derivatives. First find the first derivative, y', as usual. Then differentiate both sides of that function to find y''. One can replace any y''s which show in the y'' formula by your expression for y' thereby writing y'' entirely in terms of x and y only.

Example 3-28

Use implicit differentiation to determine y'' of the functions from Example 3-25:

1. $x^3 + y^4 = 9$
2. $y^2 + \tan x = 5$

Solution:

1. From the earlier example it was found that
$$y' = -\frac{3x^2}{4y^3}.$$
Differentiate both sides of this equation with respect to x:
$$y'' = -\frac{3}{4} \cdot \frac{2xy^3 - 3x^2 3y^2 y'}{16y^6}$$

Substitute expression for y' above to get y'' in terms of x and y alone:

$$\begin{aligned} y'' &= -\frac{3}{4}\cdot\frac{2xy^3-9x^2y^2\left(-\frac{3x^2}{4y^3}\right)}{16y^6} = -\frac{3}{4}\cdot\frac{2xy^3-9x^2y^2\left(-\frac{3x^2}{4y^3}\right)}{16y^6}\cdot\frac{4y^3}{4y^3} \\ &= -\frac{3}{4}\cdot\frac{8xy^6+27x^4y^2}{64y^9} = -\frac{3}{4}\cdot\frac{y^2\left(8xy^4+27x^4\right)}{64y^9} \\ &= -\frac{3}{4}\cdot\frac{8xy^4+27x^4}{64y^7} \end{aligned}$$

2. From the earlier example it was found that

$$y' = -\frac{\sec^2 x}{2y}\,.$$

Differentiate both sides with respect to x:

$$\begin{aligned} y'' &= -\frac{(2\sec x\sec x\tan x)(2y)-(\sec^2 x)(2y')}{4y^2} = -\frac{2y\sec^2 x\tan x - y'\sec^2 x}{2y^2} \\ &= -\frac{2y\sec^2 x\tan x-\left(-\frac{\sec^2 x}{2y}\right)\sec^2 x}{2y^2}\cdot\frac{2y}{2y} \\ &= -\frac{4y^2\sec^2 x\tan x+\sec^4 x}{4y^3} \end{aligned}$$

Further Questions:

Use implicit differentiation to find y'' for the following:

1. $x^2+y^2=9$
2. $\sin x = xy$

In general upon differentiating $y=f(x)$ n times one gets the n^{th} derivative, $f^{(n)}(x)$, which may also be written:

$$y^{(n)} \qquad \frac{d^n y}{dx^n} \qquad \frac{d^n f}{dx^n} \qquad \frac{d^n}{dx^n}f \qquad D^n f \qquad D^n_x f$$

Example 3-29

Find $f^{11}(x)$ if $f(x)=\sin(2x)$.

Solution:

Differentiating repeatedly we have:

$$\begin{aligned} f^{(0)}(x) &= \sin(2x)\cdot 2^0 \\ f^{(1)}(x) &= \cos(2x)\cdot 2^1 \\ f^{(2)}(x) &= -\sin(2x)\cdot 2^2 \\ f^{(3)}(x) &= -\cos(2x)\cdot 2^3 \\ f^{(4)}(x) &= \sin(2x)\cdot 2^4 \end{aligned}$$

The pattern of sine and cosine repeates every four iterations. Since we want $f^{(11)}(x)$ dividing 11 by 4 gives a remainder of 3 so our first factor will be $-\cos(2x)$, the same as $f^{(3)}(x)$. We must multiply this by 2 raised to the power of the derivative and so:

$$f^{(11)}(x) = -2^{11}\cos(2x)$$

Further Question:

Find $f^{(20)}(x)$ if $f(x) = \dfrac{1}{x^5}$.

Exercise 3-11

Answers: Page 246

1-4: Calculate the second derivative for each of the following functions.

1. $f(x) = \cot x$
2. $f(x) = (x-2)^{10}$; Also find $f''(3)$.
3. $y = x^3 \sec x$
4. $x^2 - y^2 = 16$ (Use implicit differentiation.)

5-6: When one uses derivatives, their simplification becomes important.

5. Show that $f''(x) = \dfrac{32\left(3x^2+16\right)}{\left(x^2-16\right)^3}$ for $f(x) = \dfrac{x^2}{x^2-16}$.
6. Show that $f''(x) = \dfrac{8x+8}{(x-2)^4}$ for $f(x) = \dfrac{x^2}{x^2-4x+4}$.

3.11 Acceleration in One Spatial Dimension

Recall that the average velocity over the time interval from $t = a$ to $t = a + h$ for motion in one dimension is

$$\text{average velocity} = \frac{f(a+h) - f(a)}{h} .$$

while the (instantaneous) velocity at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} ,$$

where here displacement in one spatial dimension was given by the position function of the particle, $s = f(t)$. In other words velocity at a given time t is just the derivative of displacement:

$$v(t) = \left.\frac{ds}{dt}\right|_t = f'(t)$$

The **acceleration** of a particle is just the second derivative of the displacement or the derivative therefore of the velocity:

$$a(t) = v'(t) = f''(t)$$

Example 3-30

The position function of a particle moving along a line is given by $s = f(t) = t^3 + 3t^2 - 9t + 9$ where t is the time in seconds and s is the displacement in metres.

1. Find the velocity and acceleration at time t.
2. What are the velocity and acceleration of the particle after 1 second and after 2 seconds?
3. When is the particle at rest?
4. When is the particle moving in the positive direction?

Solution:

1. $v(t) = f'(t) = 3t^2 + 6t - 9\,,\ a(t) = v'(t) = 6t + 6$
2. $v(1) = 3(1)^2 + 6(1) - 9 = 0\,\frac{\text{m}}{\text{s}} \qquad a(1) = 6(1) + 6 = 12\,\frac{\text{m}}{\text{s}^2}$

   $v(2) = 3(2^2) + 6(2) - 9 = 15\,\frac{\text{m}}{\text{s}} \qquad a(2) = 6(2) + 6 = 18\,\frac{\text{m}}{\text{s}^2}$
3. At rest (instantaneously) means the velocity must be zero:

   $v(t) = 0 \implies 3t^2 + 6t - 9 = 0 \implies t^2 + 2t - 3 = 0 \implies (t+3)(t-1) = 0$

   So $t = -3$ seconds or $t = 1$ second. If the particle only started moving at $t = 0$ seconds (so the position function $s = f(t)$ is only valid for $t \geq 0$) then $t = 1$ second is the only solution.
4. The particle is moving in the positive direction when the velocity is positive ($v(t) > 0$).

   $$t^2 + 2t - 3 > 0 \implies (t+3)(t-1) > 0 \implies \begin{cases} t+3 > 0 & \text{and } t-1 > 0 \\ & \text{or} \\ t+3 < 0 & \text{and } t-1 < 0 \end{cases}$$

   $$\implies \begin{cases} t > -3 & \text{and } t > 1 \\ & \text{or} \\ t < -3 & \text{and } t < 1 \end{cases} \implies \begin{cases} t > 1\,\text{sec} \\ \ \text{or} \\ t < -3\,\text{sec} \end{cases}$$

   Once again we reject $t < -3$ seconds if the position function is only valid for $t \geq 0$.

Further Questions:

1. The position function of a particle moving along a line is given by $s = f(t) = t^3 - 9t^2 + 24t$ where t is the time in seconds and s is the displacement in metres.
   (a) Find the velocity and the acceleration at time t.
   (b) What are the velocity and acceleration of the particle
      i. After 1 second?
      ii. After 3 seconds?
   (c) When is the particle (instantaneously) at rest?
   (d) When is the particle moving in the positive direction?
2. If a ball is thrown vertically upward with an initial velocity of 25 m/sec then its height, due to gravity, after t seconds is approximately $s = 25t - 5t^2$.
   (a) What is the maximum height reached by the ball?
   (b) What is the velocity of the ball when it is 30 m above the ground
      i. On its way up?
      ii. On its way down?

3.12 Related Rates

If a spherical balloon is being blown up both its radius and its volume are changing in time. That is there are two rates $\frac{dr}{dt}$ and $\frac{dV}{dt}$ of interest. These two rates are related to each other, i.e. they are not independent. In a related rates problem a formula is found relating the rates of interest and it is used to solve for an unknown rate under given conditions.

The following steps are typically followed in a related rates problem:

Steps for Solving a Related Rates Problem

1. **Identify rates, Draw Diagram:** Identify the given and unknown rates of the problem, labelling the variables and their time derivatives appropriately. Often a diagram illustrating the variables is helpful at this step. Note the rates of the problem will involve time in their units.
2. **Determine Variable Equation:** Find an equation relating the variables involved in the rates of the problem. If other variables appear in the equation find a constraint to write them in terms of the rate variables and thereby remove them.
3. **Differentiate:** Differentiate both sides of the equation with respect to time t to get the related rates equation. Since the variables are assumed to be functions of time, the **Chain Rule must be used**.
4. **Solve Unknown Rate:** Use the given rate information and other given information in the related rates equation to solve for the unknown rate. Sometimes the equation in Step 2 (or others) must be used to solve for one of the unknown values needed in the related rates equation.

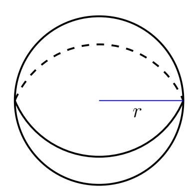
Example 3-31

Solve the following related rates problems:

1. A child is filling a spherical water balloon from the tap at a rate of $\frac{1}{8}\frac{\text{L}}{\text{s}}$. At what rate is the radius of the balloon changing when the diameter D is 10 cm? Hint: Recall $1\,\text{L}=1000\,\text{cm}^3$.

   Solution:

   (i) Identify Rates:

   - Given: $\frac{dV}{dt} = \frac{1}{8}\frac{\text{L}}{\text{s}} = \frac{1}{8}\frac{\text{L}}{\text{s}} \cdot \frac{1000\,\text{cm}^3}{1\,\text{L}} = 125\,\frac{\text{cm}^3}{\text{s}}$
   - Unknown: Find $\frac{dr}{dt}$ when $r = \frac{D}{2} = \frac{10\,\text{cm}}{2} = 5\,\text{cm}$.

   $\Longrightarrow$ Related variables are V and r, both of which are functions of time.

   (ii) Variable Equation:

   Equation for volume of a sphere, $V = \frac{4}{3}\pi r^3$, relates V to r.

   (iii) Differentiate:
   Differentiating both sides of the equation with respect to time gives:
   $$\frac{dV}{dt} = \frac{4}{3}\pi(3)r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$
   Note the use of the Chain Rule since $V = V(t)$ and $r = r(t)$.

(iv) Solve Unknown Rate:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} \implies \left(\frac{dr}{dt}\right)_{r=5\,\text{cm}} = \frac{1}{4\pi(5\,\text{cm})^2}\left(125\,\frac{\text{cm}^3}{\text{s}}\right) = \frac{5}{4\pi}\,\frac{\text{cm}}{\text{s}}$$

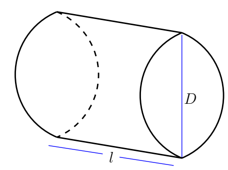
2. Suppose the child in the previous problem is now filling cylindrical balloons of fixed diameter $D = 10\,\text{cm}$. How quickly is the length of the balloon changing if it is being filled at a rate of $\frac{1}{8}\frac{\text{L}}{\text{s}}$?

Solution:

(i) Rates: Given $\dfrac{dV}{dt} = 125\,\dfrac{\text{cm}^3}{\text{s}}$, find $\dfrac{dl}{dt}$. (Related variables are volume V and length l .)

(ii) Equation: Volume of a right circular cylinder is area of base times height.

$$V = \pi r^2 l = \pi\left(\frac{D}{2}\right)^2 l \implies V = \frac{\pi D^2 l}{4}$$



(iii) Differentiate: $\dfrac{dV}{dt} = \dfrac{d}{dt}\left[\left(\dfrac{\pi D^2}{4}\right) l\right] = \dfrac{\pi D^2}{4}\dfrac{dl}{dt}$

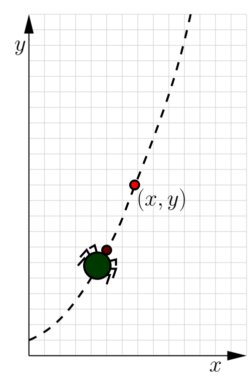
Note in this problem the diameter D is a constant so $\pi D^2/4$ is constant and can be pulled in front of the derivative.

(iv) Solve: $\dfrac{dl}{dt} = \dfrac{4}{\pi D^2}\dfrac{dV}{dt} \implies \dfrac{dl}{dt} = \dfrac{4}{\pi(10\,\text{cm})^2}\left(125\,\dfrac{\text{cm}^3}{\text{s}}\right) = \dfrac{5}{\pi}\,\dfrac{\text{cm}}{\text{s}}$

Here the rate of change of length l is constant while the balloon is filling, unlike the spherical balloon case where $\dfrac{dr}{dt}$ depends upon the radius.

3. As an insect crawls across a piece of graph paper it follows a path described by the curve $y = 3x^2 + x + 1$. If the insect's x-coordinate is changing at a rate of $3\,\frac{\text{units}}{\text{s}}$ when its y-coordinate is 5 units, how fast is its y-coordinate changing at that moment? The graph paper's origin $(0, 0)$ is at the lower left corner of the paper.

Solution:



(i) Rates: $\dfrac{dx}{dt} = 3\,\dfrac{\text{units}}{\text{s}}$, find $\dfrac{dy}{dt}$ when $y = 5$ units.

(ii) Equation: $y = 3x^2 + x + 1$

(iii) Differentiate: $\dfrac{dy}{dt} = 6x\dfrac{dx}{dt} + \dfrac{dx}{dt}$

(iv) Solve: Looking a our related rates equation we need to know x. We know that $y = 5$ units, so use our variable equation to solve for x :

$$y = 5 = 3x^2 + x + 1 \implies 0 = 3x^2 + x - 4$$

$$\implies x = \frac{-1 \pm \sqrt{1^2 - 4(3)(-4)}}{2(3)} = \frac{-1 \pm \sqrt{49}}{6} = \frac{-1 \pm 7}{6}$$

$$\implies x = 1 \text{ or } x = -\frac{4}{3} \text{ (reject)}$$

Hence our solution is

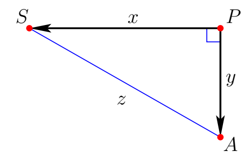
$$\left(\frac{dy}{dt}\right)_{x=1} = 6(1)(3) + 3 = 18 + 3 = 21\,\frac{\text{units}}{\text{s}}$$

4. Sarah starts at a point P and runs west at a rate of $4\frac{\text{m}}{\text{s}}$. After two minutes, Anna starts at P and runs south at a rate of $6\frac{\text{m}}{\text{s}}$. At what rate is the distance between them changing 2 minutes after Anna starts?

Solution:

(i) Rates: Let x be Sarah's distance from P and y be Anna's distance from P and z the distance between Sarah and Anna.

Given $\frac{dx}{dt} = 4\,\frac{\text{m}}{\text{s}}$ and $\frac{dy}{dt} = 6\,\frac{\text{m}}{\text{s}}$, find $\frac{dz}{dt}$ at 2 minutes after Sarah starts.



(ii) Equation: $z^2 = x^2 + y^2$

(iii) Differentiate: $2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

(iv) Solve: $\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$

We need values for x, y, and z. Since dx/dt and dy/dt are constant the distances are just these speeds times time. Sarah runs for $4\,\text{min} = 240\,\text{s}$ and Anna runs for $2\,\text{min} = 120\,\text{s}$ and therefore:

$$\left.\begin{aligned} x &= \left(4\frac{\text{m}}{\text{s}}\right)(240\,\text{s}) = 960\,\text{m} \\ y &= \left(6\frac{\text{m}}{\text{s}}\right)(120\,\text{s}) = 720\,\text{m} \end{aligned}\right\} \implies z = \sqrt{x^2+y^2} = \sqrt{(960\,\text{m})^2 + (720\,\text{m})^2} = 1200\,\text{m}$$

$$\implies \frac{dz}{dt} = \frac{1}{1200\,\text{m}}\left[(960\,\text{m})\left(4\,\frac{\text{m}}{\text{s}}\right) + (720\,\text{m})\left(6\,\frac{\text{m}}{\text{s}}\right)\right] = 6.8\,\frac{\text{m}}{\text{s}}$$

Further Questions:

1. A ladder 5 metres long with one end against a vertical wall and one end on the ground begins to slide. If the bottom of the ladder is sliding away from the wall horizontally at a rate of 0.5 m/sec, what is the vertical velocity of the top of the ladder when the bottom of the ladder is 3 m from the wall?

2. At 2 p.m. Albert is 100 km due west of Beth. If Albert is cycling directly south at 35 km/h while Beth is cycling directly north at 25 km/h, how fast is the distance between the cyclists changing at 6 p.m. ?

3. A stone is thrown into a shallow pool of water creating a circular ripple. If the radius of the ripple increases at a constant 2 m/sec, at what rate is the area enclosed by the ripple increasing at the end of 6 seconds?

4. After a hot air balloon is launched it rises vertically at a constant speed of 2 m/s. When the balloon is at an altitude of 7 m a member of the launch crew drives away on a moped down a straight road at a speed of 5 m/s. How fast is the distance between the driver and the balloon changing 4 seconds later?

5. A two-metre tall man walks toward a lamppost that is 5 metres in height. If the man is walking at a speed of 1.8 m/s, how fast is the length of his shadow changing when he is 3 metres from the lamppost?

6. A model in a fashion show walks down the straight runway at a speed of 0.75 m/s. A person in the audience located 12 metres from the runway keeps a camera focused on the model. At what rate is the camera rotating when the model is 5 metres from the point on the runway closest to the person?

7. The Great Pyramid of Khufu at Giza is a square pyramid with approximate base side length of $B = 230$ m and height $H = 150$ m. If it took approximately 20 years to build, one can use the formula for the volume of a pyramid to estimate the rate of change of volume with time (assumed constant) to be about 360 m^3/day. Before it was finished the construction was a frustum of a pyramid (i.e. a pyramid missing its top) of height h, base side length B, and top side length b. How fast was the height changing when the structure was 100 m high? Use that the volume of a frustum of a pyramid is given by $V = \frac{1}{3}h(A_1 + A_2 + \sqrt{A_1 A_2})$ where A_1 and A_2 are the areas of its square base and top respectively. (Hint: Use constants B and H until the end of your calculation.)[4]

Exercise 3-12

Answers: Page 247

1-7: Solve the following problems involving related rates.

1. An oil spill spreads in a circle whose area is increasing at a constant rate of 10 square kilometres per hour. How fast is the radius of the spill increasing when the area is 18 square kilometres?
2. A spherical balloon is being filled with water at a constant rate of 3 cm^3/s. How fast is the diameter of the balloon changing when it is 5 cm in diameter?
3. An observer who is 3 km from a launchpad watches a rocket that is rising vertically. At a certain point in time the observer measures the angle between the ground and her line of sight of the rocket to be $\pi/3$ radians. If at that moment the angle is increasing at a rate of 1/8 radians per second, how fast is the rocket rising when she made the measurement?
4. A water reservoir in the shape of a cone has height 20 metres and radius 6 metres at the top. Water flows into the tank at a rate of 15 $\mathrm{m}^3/\mathrm{min}$, how fast is the level of the water increasing when the water is 10 m deep? Hint: Use similar triangles. (See Section 1.2.9)
5. At 8 a.m., a car is 50 km west of a truck. The car is traveling south at 50 km/h and the truck is traveling east at 40 km/h. How fast is the distance between the car and the truck changing at noon?
6. A boy is walking away from a 15-metres high building at a rate of 1 m/sec . When the boy is 20 metres from the building, what is the rate of change of his distance from the top of the building?
7. The hypotenuse of a right angle triangle has a constant length of 13 cm. The vertical leg of the triangle increases at the rate of 3 cm/sec. What is the rate of change of the horizontal leg, when the vertical leg is 5 cm long?

---

[4] To check the answer is of the correct order of magnitude note that the average rate of change with height over the time of construction is

$$\frac{\Delta h}{\Delta t} = \frac{150 \text{ m}}{(20 \text{ years})(365.25 \text{ days/year})} = \frac{150 \text{ m}}{7305 \text{ days}} = 0.02 \text{ m/day}$$

Should the answer to the actual problem be higher or lower than this value? Also note that this problem naively assumes that lifting blocks to the top of the pyramid required the same time as placing those on the bottom. In fact it is believed long ramps of earth were created to place higher blocks. This has not been factored into these calculations.

3.13 Differentials

Let $y = f(x)$ be a curve in the xy-plane and let $P\,(x, f(x))$ and $Q\,(x + \Delta x, f(x + \Delta x))$ be points on the curve.

For the given **increment in x**, Δx, the **increment in y** is

$$\Delta y = f(x + \Delta x) - f(x)\,.$$

We define the **differential of y** to be:

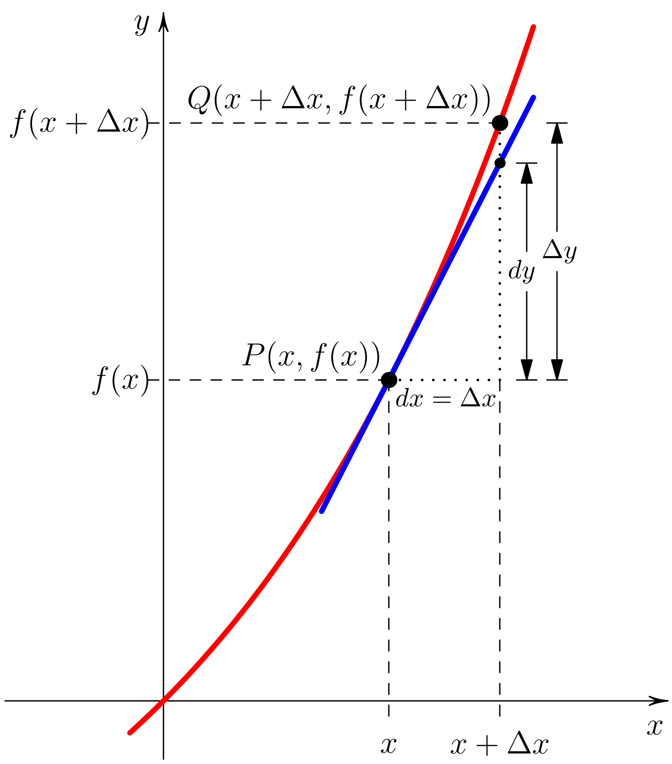
$$dy = f'(x)dx\ .$$

Here dy is considered a single variable which is a function of both x and the differential dx. The definition is motivated by the fact that, in Leibniz notation, $f'(x) = \frac{dy}{dx}$.

Letting $dx = \Delta x$ one has that:

- Δy is the change in the height of the curve $y = f(x)$
- dy is the change in the height of the tangent line to the curve $y = f(x)$ at $P(x, f(x))$.

Diagrammatically we have:



As $dx = \Delta x \to 0$ then

$$\begin{aligned} \Delta y &\approx dy \\ f(x + \Delta x) - f(x) &\approx f'(x)\Delta x \end{aligned}$$

The approximation, as clear from the diagram, is only valid for small Δx.

Example 3-32

Compare Δy and dy if $y = x^3 + 3x^2 + 1$ and x changes from 2.0 to 2.01

Solution:

$$\Delta y = f(2.01) - f(2) = \left[(2.01)^3 + 3(2.01)^2 + 1\right] - [2^3 + 3(2)^2 + 1] = 21.240901 - 21 = 0.240901$$
$$dy = f'(x)dx = (3x^2 + 6x)dx \implies dy = f'(2)(0.01) = [3(2)^2 + 6(2)](0.01) = 24(0.01) = 0.24$$

Further Question:

Compare Δy and dy if $y = f(x) = x^4 + x + 2$ and x changes from 1.0 to 1.01 .

The differential approximation $\Delta y \approx dy$ is particularly useful for approximating the absolute error Δy of a quantity that depends on a variable x with known error Δx. This is because, by assumption, the error Δx is typically small, which is when the differential approximation is valid.

Example 3-33

A circular puddle has radius $20.0 \pm 0.5\,\text{cm}$. What is its area? (Include its absolute error.)

Solution:

The puddle radius is $r \pm \Delta r = 20.0 \pm 0.5\,\text{cm}$ and its area is $A = \pi r^2$.

$A = \pi r^2 = \pi(20\,\text{cm})^2 = 400\pi\,\text{cm}^2$

$dA = A'(x)\,dx = 2\pi r\,dr$

$\Delta A \approx dA = 2\pi(20.0\,\text{cm})(0.5\,\text{cm}) = 20\pi\,\text{cm}^2$

Therefore $A \pm \Delta A = 400\pi \pm 20\pi\,\text{cm}^2 \approx 1256 \pm 63\,\text{cm}^2$.

Further Question:

A square rug is measured to have a side length of 3.00 ± 0.02 metres. What is the area of the rug?

From the differential approximation of Δy it follows that

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x \ .$$

Example 3-34

Use differentials to find an approximation to $\sqrt{1.01}$.

Solution:

We desire $1.01 = x + \Delta x$ for some value of x close to 1.01 and at which one can easily evaluate f and its derivative. Therefore choose $x = 1$ and so $\Delta x = 1.01 - x = 0.01$.

$$f(x) = \sqrt{x} \implies f(1) = \sqrt{1} = 1$$
$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \implies f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$
$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$
$$f(1.01) = f(1 + 0.01) \approx f(1) + f'(1)(0.01) = 1 + \frac{1}{2}(0.01) = 1.005$$

Thus $\sqrt{1.01} \approx 1.005$ which can be compared with the actual value 1.004987 .

Further Question:

Use differentials to find an approximation to $\sqrt[3]{1.02}$.

For $P(a, f(a))$ we have, replacing x in our previous formula with a:

$$f(a + \Delta x) \approx f(a) + f'(a)\Delta x \ .$$

The function at a value x near a will be well-approximated by the tangent at P evaluated at x. Setting $x = a + \Delta x$ in our last formula (so $\Delta x = x - a$) gives:

$$f(x) \approx f(a) + f'(a)(x - a) \ .$$

The right hand side of this equation is just the expression for the tangent line. This is the **linear** or **tangent line approximation** of f at a. The tangent function,

$$L(x) = f(a) + f'(a)(x - a) \ ,$$

is called the **linearization** of f at a.

Example 3-35

Find the linear approximation (linearization) $L(x)$ of the function $f(x) = x^3 + 2x^2 - x - 3$ at $a = 4$ and use it to approximate $f(3.98)$.

Solution:

$$\begin{aligned} f(x) &= x^3 + 2x^2 - x - 3 \implies f(4) = 4^3 + 2(4)^2 - 4 - 3 = 64 + 32 - 7 = 89 \\ f'(x) &= 3x^2 + 4x - 1 \implies f'(4) = 3(4)^2 + 4(4) - 1 = 48 + 16 - 1 = 63 \\ L(x) &= f(a) + f'(a)(x - a) \\ L(x) &= 89 + 63(x - 4) \end{aligned}$$

Since $f(x) \approx L(x)$ for x near $a = 4$ we have

$$f(3.98) \approx L(3.98) = 89 + 63(3.98 - 4) = 89 + 63(-0.02) = 87.74$$

which can be compared with the actual value of $f(3.98) = (3.98)^3 + 2(3.98)^2 - 3.98 - 3 = 87.745592$.

Further Questions:

1. Find the linear approximation (linearization) $L(x)$ of the function $g(x) = \sqrt[3]{1 + x}$ at $a = 0$ and use it to approximate $g(-0.05)$.
2. Find the linearization $L(x)$ of the function $f(x) = \dfrac{1}{\sqrt{2 + x}}$ near $a = 0$.

A better approximation to $f(x)$ near a is the **quadratic approximation** given by:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2$$

Example 3-36

Find the quadratic approximation of $f(x) = \sqrt{x^2 + 5}$ at $a = 2$.

Solution:

$$f(x) = \sqrt{x^2+5} \implies f(2) = \sqrt{4+5} = 3$$
$$f'(x) = \frac{1}{2}(x^2+5)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+5}} \implies f'(2) = \frac{x}{\sqrt{2^2+5}} = \frac{2}{3}$$
$$f''(x) = \left[x\left(x^2+5\right)^{-\frac{1}{2}}\right]' = (x^2+5)^{-\frac{1}{2}} - \frac{1}{2}x(x^2+5)^{-\frac{3}{2}}(2x) = \frac{1}{\sqrt{x^2+5}} - \frac{x^2}{(x^2+5)^{\frac{3}{2}}}$$
$$\implies f''(2) = \frac{1}{\sqrt{2^2+5}} - \frac{4}{(2^2+5)^{\frac{3}{2}}} = \frac{1}{3} - \frac{4}{[(3^2)]^{\frac{3}{2}}} = \frac{1}{3} - \frac{4}{27} = \frac{9-4}{27} = \frac{5}{27}$$

Therefore $f(x) \approx 3 + \frac{2}{3}(x-2) + \frac{5}{54}(x-2)^2$ for x near $a = 2$.

Further Question:

Find the quadratic approximation of $f(x) = \sqrt[3]{x}$ near $a = 8$.

Exercise 3-13

Answers: Page 247

1-2: Find the volume V and the absolute error ΔV of the following objects.

1. A cubical cardboard box with side length measurement of $l = 5.0 \pm 0.2$ cm.
2. A spherical cannonball with measured radius of $r = 6.0 \pm 0.5$ cm.

3-4: Find the linear approximation (linearization) $L(x)$ of the function at the given value of x.

3. $f(x) = \sqrt{2x^3 - 7}$ at x-value $a = 2$.
4. $f(x) = \tan x$ at x-value $a = \pi/4$.

Answers: Page 247

Chapter 3 Review Exercises

1-3: For each function calculate $f'(x)$ using the definition of the derivative.

1. $f(x) = x^3 + 2$
2. $f(x) = \dfrac{4x-3}{x+2}$
3. $f(x) = \sqrt{2x+1}$

4-10: Differentiate the functions.

4. $y = 3x^4 + \sqrt[3]{2x} - \dfrac{5}{\sqrt{x}} + \pi$
5. $g(x) = \left(\sqrt{2x} - 4x + 3\right)(3x + \sin x)$
6. $h(y) = \dfrac{\sqrt{y+5}}{3y+2}$
7. $f(\theta) = \cos^2\theta + 4\cos\left(\theta^2\right)$
8. $g(x) = \sec\left(x^3+4\right)\cos(2x)$
9. $f(x) = \sqrt[5]{\dfrac{x^3-4x+10}{4x^2+5}}$
10. $x^4y^3 + 4y^2 = xy + \sin y$

---

11. Find the equation of the tangent line to the curve $y = 3\sin x - 2\cos(3x)$ at the point $P(\frac{\pi}{2}, 3)$.
12. Find the value(s) of x for which the curve $y = \dfrac{x+1}{x^2+3}$ has a horizontal tangent line.
13. Find the value(s) of θ for which the curve $f(\theta) = \cos(2\theta) - 2\cos\theta$ has a horizontal tangent line.
14. The height h and radius r of a circular cone are increasing at the rate of 3 cm/sec. How fast is the volume of the cone increasing when $h = 8$ cm and $r = 3$ cm?
15. A right triangle has a constant height of 30 cm. If the base of the right triangle is increasing at a rate of 6 cm/sec, how fast is the angle between the hypotenuse and the base changing when the base is 30 cm?
16. If the area of an equilateral triangle is increasing at a rate of 5 cm^2/sec, find the rate at which the length of a side is changing when the area of the triangle is 100 cm^3.

Chapter 4: Derivative Applications

4.1 Maximum and Minimum Values

In practical applications we are often interested in how large or small a function can become. The following definitions make this concept precise.

Definition: A function $f(x)$ has an **absolute maximum** (or **global maximum**) at c if $f(c) \geq f(x)$ for all x in the domain D. The number $f(c)$ is called the **(absolute) maximum value** of f on D.

Definition: A function $f(x)$ has an **absolute minimum** (or **global minimum**) at c if $f(c) \leq f(x)$ for all x in the domain D. The number $f(c)$ is called the **(absolute) minimum value** of f on D.

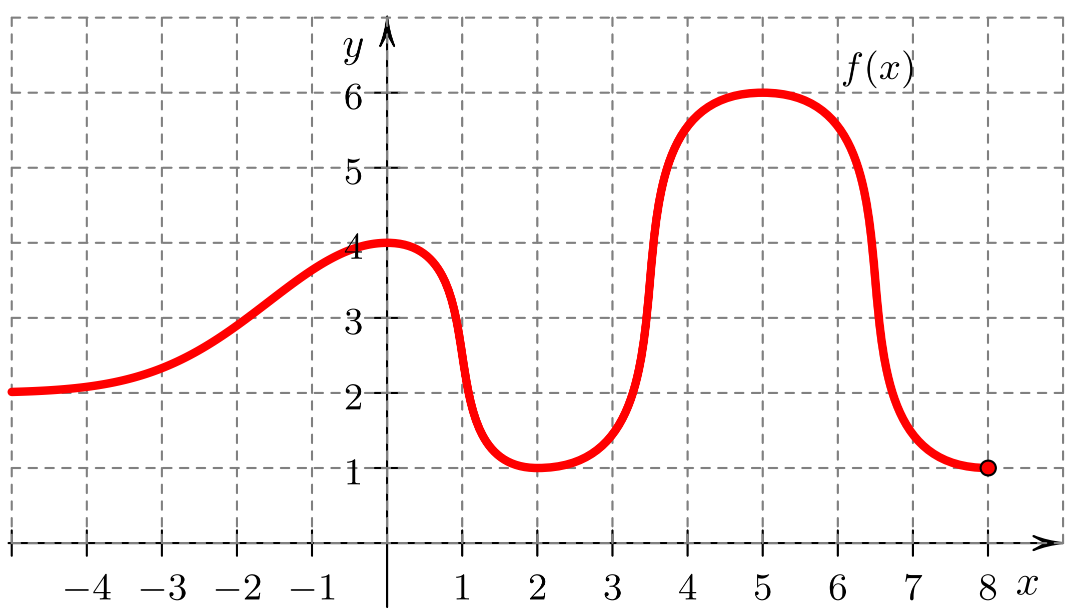
Definition: A function $f(x)$ has a **relative maximum** (or **local maximum**) at c if $f(c) \geq f(x)$ over some open interval (a, b) containing c. The number $f(c)$ is called a **relative** or **local maximum** value of f.

Definition: A function $f(x)$ has a **relative minimum** (or **local minimum**) at c if $f(c) \leq f(x)$ over some open interval (a, b) containing c. The number $f(c)$ is called a **relative** or **local minimum** value of f.

- **Note:** The absolute maximum and absolute minimum values of f are called the **extreme values** of f. Similarly we define the **relative** or **local extreme values** to be the relative maximum and minimum values.

Example 4-1

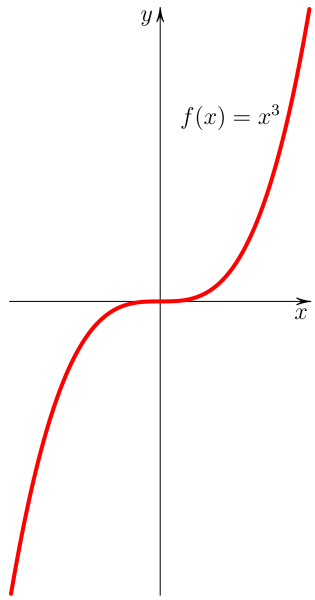
For the graphically defined function:



Find the (absolute) extreme values and relative extreme values as well as their locations.

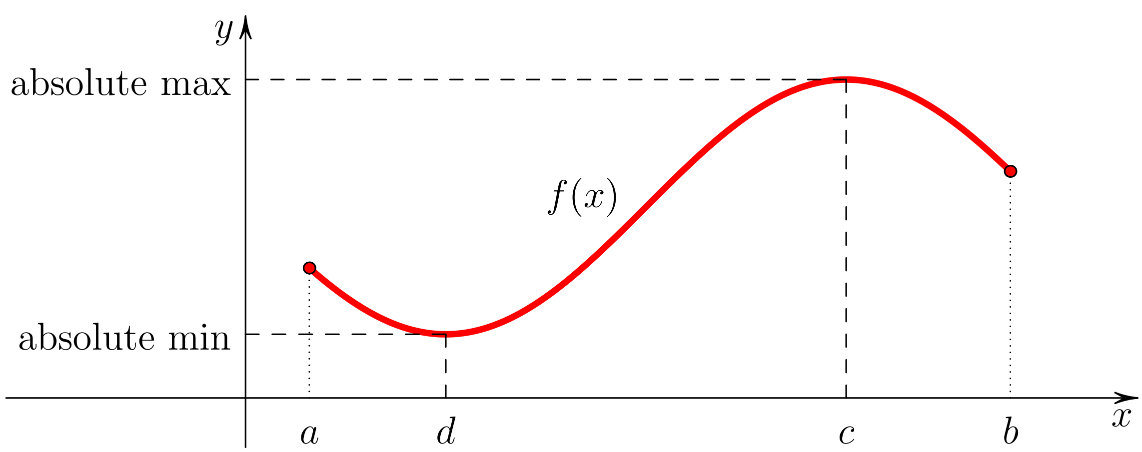
Answer: Absolute maximum of 6 at $x = 5$, Absolute minimum of 1 at $x = 2$ and $x = 8$
Relative maximum of 4 at $x = 0$ and of 6 at $x = 5$, Relative minimum of 1 at $x = 2$.

We note that a function need not have any extreme values, absolute or relative. The function $y = x^3$ is such a function.



The following theorem gives sufficient conditions for absolute extrema to exist.

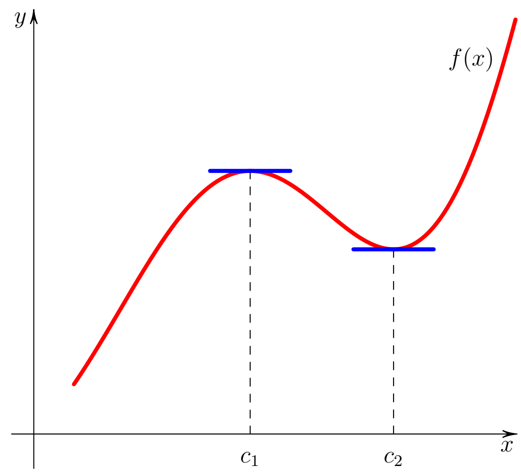
Theorem 4-1: Suppose f is continuous on the closed interval $[a, b]$. Then f has an absolute maximum and an absolute minimum on $[a, b]$. (i.e. There exist numbers c and d in $[a, b]$ such that $f(c)$ is the absolute maximum and $f(d)$ is the absolute minimum of f on $[a, b]$.) This is known as the **Extreme Value Theorem**.



Note:

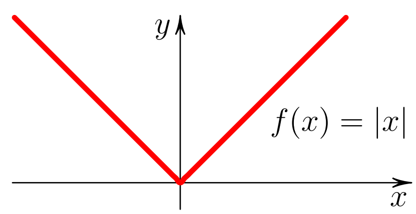
- The absolute extreme values can occur at the endpoints of the interval.
- The assumption that the interval be closed is required.
- The locations of the absolute extreme values are not necessarily unique. (There could be two or more numbers, say, on the interval where the absolute minimum is achieved.)

Theorem 4-2: If a function f has a relative extremum (maximum or minimum) at c and if $f'(c)$ exists, then $f'(c) = 0$. This is known as **Fermat's Theorem**.



Notes:

- The converse is not true: $f'(c) = 0$ does not imply a relative extremum occurs at c. $f(x) = x^3$ has no relative extrema yet $f'(0) = 0$.
- A relative extremum can occur at a number where the derivative f' does not exist. So $f(x) = |x|$ has a relative minimum at $x = 0$ but $f'(0)$ does not exist as is clear from the following graph of the function.



Definition: If c is a number in the domain D of function f such that the derivative vanishes ($f'(c) = 0$) or does not exist, then c is a **critical number** of f.

Example 4-2

Find the critical numbers of the following functions.

1. $f(t) = t^3 + t^2 - 5t + 2$
2. $g(x) = x^2\sqrt[3]{x+1}$
3. $s(t) = \dfrac{t^3}{3t+2}$
4. $g(\theta) = \cos 2\theta + 2\sin\theta$
5. $f(x) = (2x+1)\sqrt{x^2-4}$

Solution:

1. $f(t) = t^3 + t^2 - 5t + 2$ (a polynomial) $\implies D = (-\infty, \infty)$

   $f'(t) = 3t^2 + 2t - 5$

   $f'(t) = 0 \implies 3t^2 + 2t - 5 = 0 \implies t = \dfrac{-2 \pm \sqrt{4 - 4(3)(-5)}}{6} = \dfrac{-2 \pm 8}{6} \implies \begin{cases} t &= 1 \\ t &= -\frac{10}{6} = -\frac{5}{3} \end{cases}$

   Therefore the critical numbers are $t = 1$ and $t = -\frac{5}{3}$.

2. $g(x) = x^2 \sqrt[3]{x+1}$ (odd integer root) $\implies D = (-\infty, \infty)$

   $$g'(x) = 2x\sqrt[3]{x+1} + x^2 \cdot \frac{1}{3}(x+1)^{-\frac{2}{3}} = 2x\sqrt[3]{x+1} + \frac{x^2}{3(x+1)^{\frac{2}{3}}}$$

   $$= 2x\sqrt[3]{x+1} \cdot \frac{3(x+1)^{\frac{2}{3}}}{3(x+1)^{\frac{2}{3}}} + \frac{x^2}{3(x+1)^{\frac{2}{3}}} = \frac{6x(x+1) + x^2}{3(x+1)^{\frac{2}{3}}} = \frac{7x^2 + 6x}{3(x+1)^{\frac{2}{3}}}$$

   $g'(x)$ is not defined when $x = -1$.

   $g'(x) = 0 \implies 7x^2 + 6x = 0 \implies x(7x+6) = 0 \implies \begin{cases} x = 0 \\ x = -\frac{6}{7} \end{cases}$

   Therefore the critical numbers are $x = 0$, $x = -1$, and $x = -\dfrac{6}{7}$.

3. $s(t) = \dfrac{t^3}{3t+2}$ (a rational function) $3t + 2 = 0 \implies t = -\dfrac{2}{3} \implies D = \mathbb{R} - \{-\dfrac{2}{3}\}$

   $$s'(t) = \frac{3t^2(3t+2) - t^3(3)}{(3t+2)^2} = \frac{9t^3 + 6t^2 - 3t^3}{(3t+2)^2} = \frac{6t^3 + 6t^2}{(3t+2)^2}$$

   $s'(t)$ is not defined when $t = -\frac{2}{3}$ but this is not in D and hence not a critical number.

   $s'(t) = 0 \implies 6t^3 + 6t^2 = 0 \implies 6t^2(t+1) = 0 \implies \begin{cases} t &= 0 \\ t &= -1 \end{cases}$

   Critical numbers: $t = 0$, $t = -1$

4. $g(\theta) = \cos 2\theta + 2\sin\theta$ (trigonometric functions defined everywhere) $\implies D = \mathbb{R}$

   $g'(\theta) = -2\sin 2\theta + 2\cos\theta$

   $g'(\theta) = 0 \implies -2\sin 2\theta + 2\cos\theta = 0 \implies -4\sin\theta\cos\theta + 2\cos\theta = 0$

   $\implies 2\cos\theta(-2\sin\theta + 1) = 0$

   $$\implies \begin{cases} \cos\theta = 0 & \implies \theta = \frac{\pi}{2} + n\pi \\ -2\sin\theta + 1 = 0 & \implies \sin\theta = \frac{1}{2} \implies \begin{cases} \theta &= \frac{\pi}{6} + 2n\pi \\ \theta &= \frac{5\pi}{6} + 2n\pi \end{cases} \end{cases}$$

   (See Appendix B on solving trigonometric equations.)

   Critical numbers:
   $\left\{\frac{\pi}{2} + n\pi \,|\, n \text{ an integer}\right\} \cup \left\{\frac{\pi}{6} + 2n\pi \,|\, n \text{ an integer}\right\} \cup \left\{\frac{5\pi}{6} + 2n\pi \,|\, n \text{ an integer}\right\}$

5. $f(x) = (2x+1)\sqrt{x^2 - 4}$

   $$x^2 - 4 \geq 0 \implies (x-2)(x+2) \geq 0 \implies \begin{cases} x - 2 \geq 0 \quad \text{and } x + 2 \geq 0 \\ \qquad\text{or} \\ x - 2 \leq 0 \quad \text{and } x + 2 \leq 0 \end{cases}$$

   $$\implies \begin{cases} x \geq 2 \quad \text{and } x \geq -2 \\ \qquad\text{or} \\ x \leq 2 \quad \text{and } x \leq -2 \end{cases} \implies \begin{cases} x \geq 2 \\ \text{or} \\ x \leq -2 \end{cases} \implies D = (-\infty, -2] \cup [2, \infty)$$

$$f'(x) = 2\sqrt{x^2-4} + (2x+1)\frac{1}{2}(x^2-4)^{-\frac{1}{2}}(2x) = 2\sqrt{x^2-4} + \frac{x(2x+1)}{\sqrt{x^2-4}}$$
$$= \frac{2(x^2-4)+x(2x+1)}{\sqrt{x^2-4}} = \frac{4x^2+x-8}{\sqrt{x^2-4}}$$

$f'(x)$ is not defined when $x = 2$ or $x = -2$ (and these both are in the domain D).

$$f'(x) = 0 \implies 4x^2+x-8 = 0 \implies x = \frac{-1 \pm \sqrt{1-(4)(4)(-8)}}{8} = \frac{-1 \pm \sqrt{129}}{8}$$

Critical numbers: $x = 2$, $x = -2$, $x = \dfrac{-1 \pm \sqrt{129}}{8}$

Further Questions:

Find the critical numbers of the following functions:

1. $f(x) = x^3 - 2x^2$
2. $g(t) = t(t-2)^2$
3. $f(x) = \sqrt[3]{x}(4-x)$
4. $g(t) = \dfrac{t^2+5}{t-2}$
5. $f(\theta) = \cos(2\theta) + \theta$
6. $h(z) = \sqrt{z}\left(z^2+2\right)$
7. $g(t) = 4\sin^3 t + 3\sqrt{2}\cos^2 t$

The next theorem narrows down the values where relative extrema can potentially occur.

Theorem 4-3: If f has a relative extremum (minimum or maximum) at c, then c is a critical number of f.

Knowing that an absolute extrema of a continuous function f on $[a, b]$ will either occur in (a, b) at a relative extrema or at an endpoint, we can now construct a method to find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$.

Method for Finding Absolute Extrema of continuous f on $[a, b]$

1. Find all critical numbers in (a, b).
2. Evaluate f at the critical numbers.
3. Evaluate $f(a)$ and $f(b)$.
4. The largest of the values from 2. and 3. is the absolute maximum value and the smallest of these values is the absolute minimum value.

Example 4-3

Find the absolute maximum and minimum of the following functions on the given closed interval. Also indicate at what value(s) of x they are located.

1. $f(x) = 2 - x^{\frac{5}{3}}$ on $[-1, 8]$
2. $g(t) = t^4 - 4t^2 + 1$ on $[-1, 2]$
3. $f(\theta) = \sin 2\theta + \cos 2\theta$ on $\left[0, \dfrac{\pi}{2}\right]$

Solution:

1. $f(x) = 2 - x^{\frac{5}{3}}$; $[-1, 8]$

   Find critical number(s):

   $f'(x) = -\frac{5}{3}x^{\frac{2}{3}}$

   $f'(x) = 0 \implies -\frac{5}{3}x^{\frac{2}{3}} = 0 \implies x^{\frac{2}{3}} = 0 \implies x = 0$ (critical number)

   Evaluate function at critical numbers in $[-1, 8]$ and interval endpoints:

   $f(-1) = 2 - (-1)^{\frac{5}{3}} = 2 - (-1)^5 = 2 + 1 = 3$

   $f(0) = 2 - 0 = 2$

   $f(8) = 2 - (8)^{\frac{5}{3}} = 2 - (2)^5 = 2 - 32 = -30$

   Absolute minimum is -30 located at $x = 8$.
   Absolute maximum is 3 located at $x = -1$.
   Note a compact way to write this is to say the absolute minimum is $f(8) = -30$ and the absolute maximum is $f(-1) = 3$ thereby identifying the value of the absolute extrema and its location at the same time.

2. $g(t) = t^4 - 4t^2 + 1$; $[-1, 2]$

   $g'(t) = 4t^3 - 8t$

   $g'(t) = 0 \implies 4t^3 - 8t = 0 \implies 4t(t^2 - 2) = 0 \implies \begin{cases} t &= 0 \\ t &= \pm\sqrt{2} \end{cases}$

   Note that while $t = 0$, $t = -\sqrt{2}$, and $t = \sqrt{2}$ are all critical numbers of $g(t)$, $-\sqrt{2} \approx -1.41$ does not lie in $[-1, 2]$ and is not considered.

   $g(-1) = (-1)^4 - 4(-1)^2 + 1 = 1 - 4 + 1 = -2$

   $g(0) = 0 - 0 + 1 = 1$

   $g(\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 + 1 = 4 - 8 + 1 = 5 - 8 = -3$

   $g(2) = 2^4 - 4(2)^2 + 1 = 16 - 16 + 1 = 1$

   Absolute minimum is $g\left(\sqrt{2}\right) = -3$.
   Absolute maximum is $g(0) = g(2) = 1$. (i.e. located at both $x = 0$ and $x = 2$)

3. $f(\theta) = \sin 2\theta + \cos 2\theta$; $\left[0, \frac{\pi}{2}\right]$

   $f'(\theta) = 2\cos 2\theta - 2\sin 2\theta$

   $f'(\theta) = 0 \implies 2\cos 2\theta - 2\sin 2\theta = 0 \implies \sin 2\theta = \cos 2\theta \implies \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = 1$

   $\implies 2\theta = \frac{\pi}{4} + 2n\pi$ or $2\theta = \frac{5\pi}{4} + 2n\pi \implies \theta = \frac{\pi}{8} + n\pi$ or $\theta = \frac{5\pi}{8} + n\pi$

   $\implies \theta = \frac{\pi}{8}$ (only critical number in interval)

   (See Appendix B on solving trigonometric equations.)

   $f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$

   $f(\frac{\pi}{8}) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2} \approx 1.41$

   $f(\frac{\pi}{2}) = \sin\pi + \cos\pi = 0 - 1 = -1$

   Absolute minimum is $f(\frac{\pi}{2}) = -1$
   Absolute maximum is $f(\frac{\pi}{8}) = \sqrt{2}$

The student is encouraged is to plot the graphs of these functions using computer software or otherwise to confirm these results.

Further Questions:

Find the absolute maximum and minimum of the following functions on the given closed intervals:

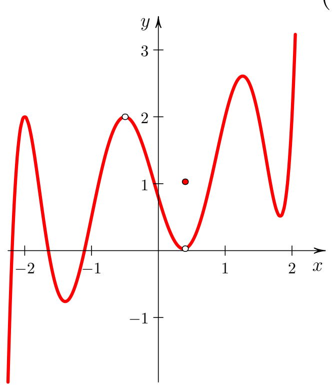
1. $f(x) = 4x^3 - 15x^2 + 12x + 7$ on $[0, 3]$
2. $f(x) = (x^2 - 4)^2$ on $[-3, 1]$
3. $f(x) = 1 - x^{\frac{2}{3}}$ on $[-1, 8]$

Answers: Page 248

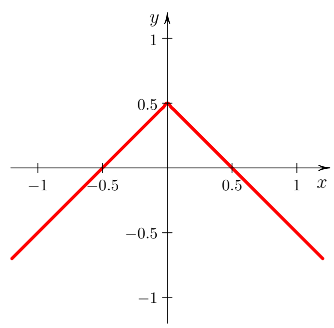
Exercise 4-1

1. Identify the relative maxima, relative minima, absolute maxima, and absolute minima on each of the following graphs.

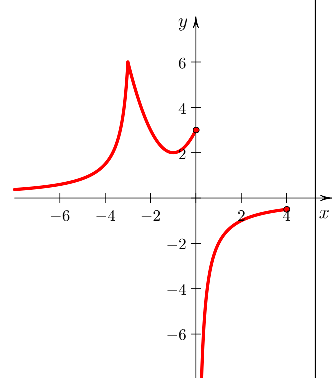
(a)



(b)



(c)



2-10: Find the critical numbers of the given function.

2. $f(x) = x^3 - 9x^2 + 24x - 15$
3. $h(x) = |x| + 1$
4. $f(s) = \dfrac{s}{s^2 + 6}$ on the interval $[0, 10]$.
5. $f(x) = x^3 + 5x^2 + 3x + 1$
6. $g(t) = \dfrac{1}{4}t^4 + 2t^2 - 5t + 6$
7. $H(x) = \dfrac{x+3}{x-5}$
8. $f(t) = \sqrt{t^2 - 4}$
9. $g(x) = \sqrt[3]{x^2 - 5}$
10. $F(\theta) = 2\sin(\theta) - \theta$

11-15: Find the absolute maximum and absolute minimum values and their locations for the given function on the closed interval.

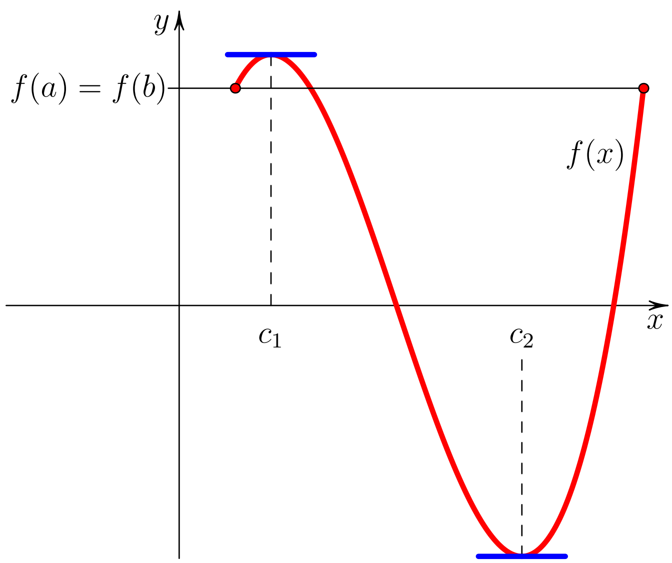
11. $f(x) = x^4 - x^2 + 1$ on $[-2, 2]$
12. $f(x) = x^3 + 5x^2 + 3x + 1$ on $[-1, 0]$.
13. $g(t) = \sqrt{t}\,(t - 2)$ on $[0, 1]$.
14. $H(x) = x^{\frac{1}{3}} - 3$ on $[-1, 8]$.
15. $f(x) = \sin x \cos x$ on $[0, 2\pi]$

4.2 The Mean Value and Other Derivative Theorems

We now consider several theorems whose results are used in proving deeper calculus theorems.

Theorem 4-4: Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). If $f(a) = f(b)$ then there exists a number c in (a, b) such that $f'(c) = 0$. (i.e. The tangent line at the point $(c, f(c))$ is horizontal.) This is known as **Rolle's Theorem**.

The situation is illustrated below.



Notes:

- For the trivial case where $f(x)$ is a constant function we can take any c in (a, b) to prove the theorem.
- The proof of the theorem for the non-trivial case hinges on the observation that if $f(a) = f(b)$ then at least one absolute extremum (which exists by the Extreme Value Theorem) must occur at c within (a, b) and hence be a *relative* extremum which by Fermat's Theorem has $f'(c) = 0$.
- There may be more than one c with $f'(c) = 0$ as illustrated in the diagram.

Example 4-4

Verify that Rolle's Theorem applies for $f(x) = x^3 + x^2 - 2x + 3$ on the interval $[-2, 1]$. Next find a value c it predicts must exist.

Solution:

Rolle's Theorem requirements:

- $f(x) = x^3 + x^2 - 2x + 3$ is a polynomial and hence continuous on closed interval $[-2, 1]$.
- $f'(x) = 3x^2 + 2x - 2$ so $f(x)$ is therefore differentiable on the open interval $(-2, 1)$.
- $f(-2) = (-2)^3 + (-2)^2 - 2(-2) + 3 = -8 + 4 + 4 + 3 = 3$ and
  $$f(1) = (1)^3 + 2^2 - 2(1) + 3 = 3$$
  $$\implies f(-2) = f(1)$$

Then by Rolle's Theorem there exists a number c in $(-2, 1)$ such that $f'(c) = 0$. We can find c by solving for it directly:

$$0 = f'(c) = 3c^2 + 2c - 2 \implies c = \frac{-2 \pm \sqrt{(2)^2 - 4(3)(-2)}}{(2)(3)} = \frac{-2 \pm \sqrt{28}}{6} = \frac{-2 \pm \sqrt{4}\sqrt{7}}{6}$$

$$c = \frac{-1 \pm \sqrt{7}}{3} = \begin{cases} \frac{-1+\sqrt{7}}{3} \approx 0.55 \\ \frac{-1-\sqrt{7}}{3} \approx -1.22 \end{cases}$$

Therefore $c = \frac{-1\pm\sqrt{7}}{3}$, both of which lie in $(-2, 1)$, thereby confirming the prediction of Rolle's Theorem.

Further Question:

Verify that Rolle's Theorem applies for $f(x) = x\sqrt{x+6} + 1$ on the interval $[-6, 0]$. Next find a c it predicts must exist.

Example 4-5

Show that the equation $1 - x^3 - 2x = 0$ has exactly one real root (solution) on $[-1, 1]$.

Solution:

Defining $f(x) = 1 - x^3 - 2x$ we have that f is continuous on $[-1, 1]$ and $f(-1) = 1 - (-1) + 2 = 4$ and $f(1) = 1 - 1 - 2 = -2$. Since f changes sign on the interval it follows by the Intermediate Value Theorem that there exists at least one c in $(-1, 1)$ such that $0 = f(c) = 1 - c^3 - 2c$. Thus we have at least one solution c on $(-1, 1)$.

To prove that there is *exactly* one solution we can use **proof by contradiction**. We will assume there is more than one solution and show this leads to a logical inconsistency. If there are two or more solutions in $(-1, 1)$ let a and b be any two such distinct solutions. Then $f(a) = f(b) = 0$. But f is continuous on $[a, b]$ and $f'(x) = -3x^2 - 2$ shows f is differentiable on (a, b). Then by Rolle's Theorem there is a c in (a, b) with $f'(c) = 0$. However $f'(x) = -(3x^2 + 2)$ is strictly less than zero as $3x^2 \geq 0$ and $2 > 0$. This shows $f'(c) < 0$, a contradiction. Therefore our assumption of there being more than one solution in $(-1, 1)$ led to a contradiction and hence there is exactly one solution in $(-1, 1)$. Since f at the endpoints -1 and 1 is non-zero these x values are not solutions so there is exactly one solution on $[-1, 1]$.

Further Question:

Show that the equation $1 + 2x + x^3 + 4x^5 = 0$ has exactly one real root (solution) on $[-1, 0]$.

Rolle's Theorem is used to prove the more general Mean Value Theorem to which we now turn.

Theorem 4-5: Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists a number c in (a, b) such that

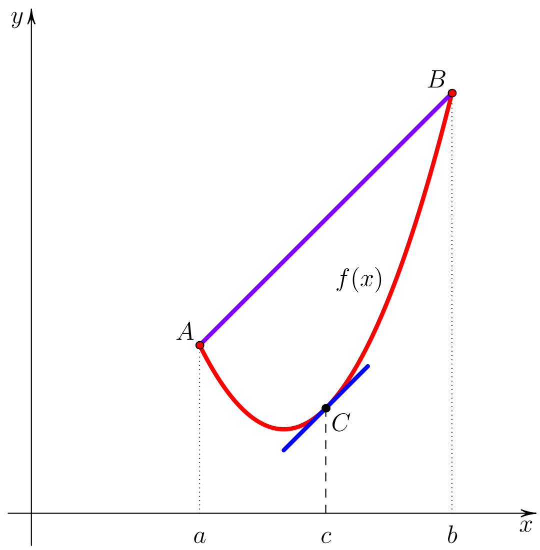
$$f'(c) = \frac{f(b) - f(a)}{b - a} \ .$$

This is known as the **Mean Value Theorem**.

Geometrically the Mean Value Theorem states that the slope of the secant line between the endpoints of the curve $A(a, f(a))$ and $B(b, f(b))$,

$$m = \frac{f(b) - f(a)}{b - a} \ ,$$

is equal to the slope of the tangent line, $f'(c)$, at at least one point $C(c, f(c))$ along the curve (not an endpoint).



If we consider one dimensional motion then the Mean Value Theorem states that the average velocity for the trip over the time interval $[a, b]$ will equal the (instantaneous) velocity at at least one time c in (a, b) during the trip.

To prove the Mean Value Theorem one shows that the new function $g(x)$ defined to be the difference between $f(x)$ and the secant line between $(a, f(a))$ and $(b, f(b))$, i.e.

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right] ,$$

satisfies Rolle's Theorem. The c guaranteed for $g(x)$ by the latter can be shown to be the c required for $f(x)$ for the Mean Value Theorem. Also note that Rolle's Theorem itself follows immediately from the Mean Value Theorem for if $f(b) = f(a)$ then $f'(c) = 0$ in the Mean Value Theorem.

Example 4-6

Verify that Mean Value Theorem applies for $f(x) = \dfrac{x + 3}{x + 1}$ on the interval $[1, 3]$. Next find a value c it predicts must exist.

Solution:

Mean Value Theorem requirements:

- $f(x) = \frac{x+3}{x+1}$ is a rational function and hence continuous on its domain $D = \mathbb{R} - \{-1\}$ and hence f is continuous on the closed interval $[1, 3]$.
- $f'(x) = \frac{(1)(x+1)-(x+3)(1)}{(x+1)^2} = -\frac{2}{(x+1)^2}$ so f is therefore differentiable on the open interval $(1, 3)$.

Then

$$\begin{array}{l} f(a) = f(1) = \dfrac{1+3}{1+1} = \dfrac{4}{2} = 2 \\ f(b) = f(3) = \dfrac{3+3}{3+1} = \dfrac{6}{4} = \dfrac{3}{2} \end{array} \implies \frac{f(b) - f(a)}{b - a} = \frac{2 - (3/2)}{3 - 1} = -\frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{4}$$

so by the Mean Value Theorem there exists a number c in $(1, 3)$ such that $f'(c) = -\frac{1}{4}$.

We can find c by solving for it directly:

$$-\frac{1}{4} = f'(c) = -\frac{2}{(c+1)^2} \implies (c+1)^2 = 8 \implies c+1 = \pm\sqrt{8} = \pm\sqrt{4}\sqrt{2} = \pm 2\sqrt{2}$$

$$\implies c = \pm 2\sqrt{2} - 1 = \begin{cases} 2\sqrt{2} - 1 \approx 1.83 \\ -2\sqrt{2} - 1 \approx -3.83 \text{ (reject)} \end{cases}$$

Therefore $c = \sqrt{2} - 1$ lies in $(1,3)$ with $f'(c) = -1/4$ thereby confirming the MVT prediction.

Further Questions:

Verify that the Mean Value Theorem applies for the following functions on the given closed intervals and find a value of c it predicts must exist.

1. $f(x) = 2x^3 + x^2 - x - 1$ on $[0,2]$
2. $f(x) = x - 3x^{\frac{1}{3}}$ on $[0,8]$

Two corollaries of the Mean Value Theorem are the following:

Theorem 4-6: Suppose $f'(x) = 0$ for all x in (a,b), then f is constant on the interval.

Theorem 4-7: Suppose $f'(x) = g'(x)$ for all x in (a,b), then $f(x) = g(x) + C$ for some constant C.

The former corollary results by applying the M.V.T. to the interval $[x_1, x_2]$ generated by any two values x_1 and x_2 in (a,b). Since the resulting c has $f'(c) = 0$ by assumption, this implies $f(x_2) - f(x_1)$ must vanish. (i.e. $f(x_2) = f(x_1)$.) Since x_1 and x_2 are arbitrarily chosen it follows that all values of $f(x)$ must equal each other and hence some constant C.

The second corollary follows from the first by applying it to the function $h(x) = f(x) - g(x)$ whose derivative $h'(x) = f'(x) - g'(x)$ will identically vanish. Graphically it means that if two functions have the same derivative they differ at most by a vertical shift.

Answers: Page 248

Exercise 4-2

1. Using Rolle's Theorem, show that the graph of $f(x) = x^3 - 2x^2 - 7x - 2$ has a horizontal tangent line at a point with x-coordinate between -1 and 4. Next find the value $x = c$ guaranteed by the theorem at which this occurs.
2. By Rolle's Theorem it follows that if f is a function defined on $[0,1]$ with the following properties:
   (a) f is continuous on $[0,1]$
   (b) f is differentiable on $(0,1)$
   (c) $f(0) = f(1)$

   then there exists a least one value c in $(0,1)$ with $f'(c) = 0$. Show that each condition is required for the conclusion to follow by giving a counterexample in the case that (a), (b), or (c) is not required.
3. Suppose that f is continuous on $[-3,4]$, differentiable on $(-3,4)$, and that $f(-3) = 5$ and $f(4) = -2$. Show there is a c in $(-3,4)$ with $f'(c) = -1$.
4. Verify the Mean Value Theorem for the function $f(x) = x^3 + 2x - 2$ on the interval $[-1,2]$.

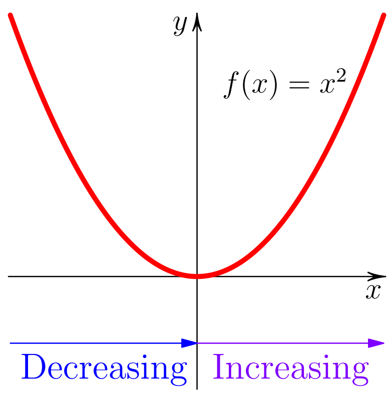
4.3 Increasing and Decreasing

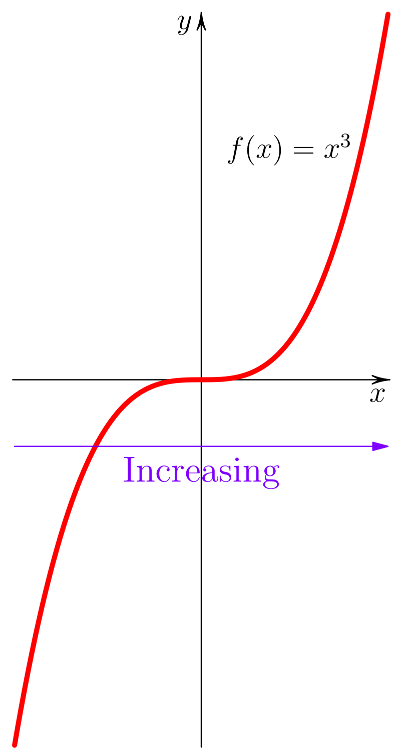
We now consider how the knowledge of the properties a function and its derivatives can help us in understanding the behaviour of its graph, in particular finding where the function is *interesting*. This analysis will allow us to sketch a function with limited functional evaluation being necessary.

We first consider what it means for a function to be increasing or decreasing. The following definitions make the intuitive idea of increase and decrease clear:

Definition: A function $f(x)$ is **increasing** on an interval I if its graph continuously rises as x goes from left to right through the interval. (i.e. if x_1 and x_2 are in the interval and $x_1 < x_2$, then $f(x_1) < f(x_2)$.)

Definition: A function $f(x)$ is **decreasing** on an interval I if its graph continuously falls as x goes from left to right through the interval. (i.e. if x_1 and x_2 are in the interval and $x_1 < x_2$, then $f(x_1) > f(x_2)$.)

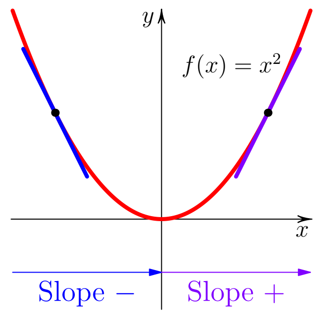




Example 4-7

From the diagrams, $y = x^2$ decreases on $(-\infty, 0]$ and increases on $[0, \infty)$ while $y = x^3$ increases on the interval $(-\infty, \infty)$.

Where a differentiable function increases or decreases can be determined by the **first derivative** of a function since whether a function increases or decreases is related to its tangent slope.



Test for Increasing/Decreasing[1]
Suppose f is differentiable on (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b).
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).

Example 4-8

One may verify for the above examples that $\frac{d}{dx}x^2 = 2x$ is negative when $x < 0$ ($y = x^2$ decreasing on $(-\infty, 0)$) and positive when $x > 0$ ($y = x^2$ increasing on $(0, \infty)$). On the other hand $\frac{d}{dx}x^3 = 3x^2$ is positive for $x \neq 0$ so by the test $y = x^3$ is increasing on $(-\infty, 0) \cup (0, \infty)$. For an interval containing $x = 0$ where the derivative is zero the function is increasing by using the definition and so x^3 is increasing on $(-\infty, \infty)$.

Critical numbers usually separate intervals of increase or decrease.[2] Analysis of whether a function is increasing or decreasing on each side of a critical number allows us to determine whether a critical number is a relative extremum and if so, of which kind (maximum or minimum).

First Derivative Test for a Relative Extremum
Let $I = (a, b)$ be an interval containing critical number c of function f and further let f be continuous on I and differentiable on I (except perhaps at c). Let $L = (a, c)$ and $R = (c, b)$ be the subintervals to the left and right of c.

1. If f' is positive on L and negative on R, then f has a relative maximum at c.
2. If f' is negative on L and positive on R, then f has a relative minimum at c.
3. If f' has the same sign on L and R (either both positive or negative), then f has no relative extremum at c.

[1] To prove this result assume $f' > 0$ on (a, b) and let x_1 and x_2 be arbitrary points in (a, b) with $x_1 < x_2$. Since f is differentiable on (a, b) it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the Mean Value Theorem there exists c in (x_1, x_2) with $f'(c) = [f(x_2) - f(x_1)]/(x_2 - x_1)$. But c lies in (a, b) so $f'(c) = [f(x_2) - f(x_1)]/(x_2 - x_1)$ is positive. Since $x_2 - x_1 > 0$ it follows that $f(x_2) - f(x_1) > 0$ or $f(x_2) > f(x_1)$. Since x_1 and x_2 were arbitrarily chosen, f is increasing on (a, b). A similar argument shows that if $f' < 0$ on the interval the function is decreasing there.

[2] Though not always, consider $y = \tan^2 x$ at $x = \pi/2$.

Example 4-9

The function $y = x^2$ has a single critical number of $c = 0$ where the derivative vanishes. Since the function decreases to the left of 0 and increases to the right of it, a relative minimum occurs at $x = 0$ which is readily verified from the graph of the function. $y = x^3$ also has a critical number of $c = 0$ but in this case the function is increasing on the intervals to the left and right of it so there is no relative extremum at 0.

Example 4-10

Find the open intervals upon which the following functions are increasing or decreasing. Also find any relative minima and maxima and their locations.

1. $g(t) = \dfrac{t^2 + t + 2}{t^2 + 3}$
2. $f(x) = \sqrt{x}(2x - 3)$

Solution:

1. $g(t) = \dfrac{t^2 + t + 2}{t^2 + 3}\ , \quad t^2 + 3 > 0 \implies D = \mathbb{R}$

$$g'(t) = \frac{(2t+1)(t^2+3) - (t^2+t+2)(2t)}{(t^2+3)^2} = \frac{2t^3 + 6t + t^2 + 3 - 2t^3 - 2t^2 - 4t}{(t^2+3)^2} = \frac{-t^2 + 2t + 3}{(t^2+3)^2}$$
$$= -\frac{t^2 - 2t - 3}{(t^2+3)^2} = -\frac{(t-3)(t+1)}{(t^2+3)^2}$$

$$g'(t) = 0 \implies (t-3)(t+1) = 0 \implies \begin{cases} t &= 3 \\ t &= -1 \end{cases}$$

Due to the continuity of g', the critical numbers $t = -1$ and $t = 3$ partition the real axis into intervals over which the function either increases or decreases. This may be determined by analyzing the sign of g' on the intervals. (See Appendix A .)

Interval	$g'(t) = -\frac{(t-3)(t+1)}{(t^2+3)^2}$	g
$x < -1$	$-\frac{(-)(-)}{(+)^2} = (-)$	decreasing
$-1 < x < 3$	$-\frac{(-)(+)}{(+)^2} = (+)$	increasing
$3 < x$	$-\frac{(-)(-)}{(+)^2} = (-)$	decreasing

Therefore $g(t)$ is increasing on $(-1, 3)$ and decreasing on $(-\infty, -1) \cup (3, \infty)$.
Since one goes from decreasing to increasing at $x = -1$ and from increasing to decreasing at $x = 3$ we have, by the First Derivative Test, that $x = -1$ is the location of a relative minimum and $x = 3$ is the location of a relative maximum.

$$g(-1) = \frac{(-1)^2 + (-1) + 2}{(-1)^2 + 3} = \frac{2}{4} = \frac{1}{2}$$
$$g(3) = \frac{3^2 + 3 + 2}{3^2 + 3} = \frac{14}{12} = \frac{7}{6}$$

Thus $g(-1) = \frac{1}{2}$ is a relative minimum and $g(3) = \frac{7}{6}$ is a relative maximum.

2. $f(x) = \sqrt{x}(2x-3)$, $x \geq 0 \implies D = [0, \infty)$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(2x-3) + \sqrt{x}(2) = \frac{1}{2\sqrt{x}}(2x-3) + 2\sqrt{x} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} = \frac{2x-3+4x}{2\sqrt{x}} = \frac{6x-3}{2\sqrt{x}}$$

$f'(x)$ is not defined when $x = 0$.

$$f'(x) = 0 \implies 6x - 3 = 0 \implies x = \frac{3}{6} \implies x = \frac{1}{2}$$

critical numbers partition the domain as follows:

Interval	$f'(x) = \frac{6(x-\frac{1}{2})}{2\sqrt{x}}$	f
$0 < x < \frac{1}{2}$	$\frac{(+)(-)}{(+)(+)} = (-)$	decreasing
$\frac{1}{2} < x < 3$	$\frac{(+)(+)}{(+)(+)} = (+)$	increasing

Function f is increasing on $(\frac{1}{2}, \infty)$ and decreasing on $(0, \frac{1}{2})$.
The critical number $x = 0$ is not contained within an open interval within D and so is not a relative extrema. Since f goes from decreasing to increasing at $x = 1/2$ the First Derivative Test shows that value is the location of a relative minimum.

$f(\frac{1}{2}) = \sqrt{(\frac{1}{2})}\left[2(\frac{1}{2}) - 3\right] = \frac{1}{\sqrt{2}}(-2) = -\frac{1}{\sqrt{2}}(\sqrt{2})^2 = -\sqrt{2}$

Thus $f(\frac{1}{2}) = -\sqrt{2}$ is a relative minimum. There are no relative maxima.

Further Question:

Find the open intervals upon which the function

$$f(x) = \frac{x^2 - 2x + 1}{x}$$

is increasing or decreasing. Also find any relative minima and maxima and their locations.

Appendix A reviewing inequalities may be helpful in determining when f' is positive and negative.

Answers: Page 249

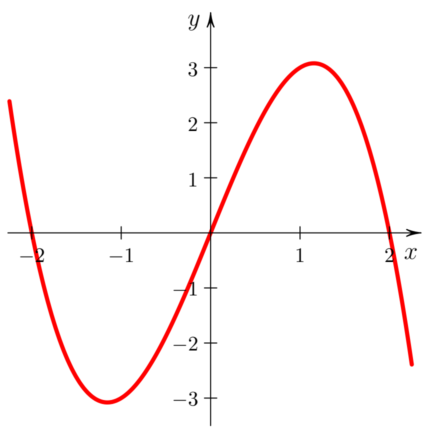
Exercise 4-3

1-5: Find the open intervals upon which the following functions are increasing or decreasing. Also find any relative minima and maxima and their locations.

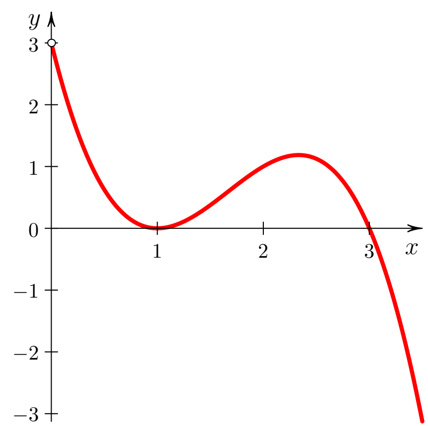
1. $f(x) = x^2 + 2x + 1$
2. $f(x) = \dfrac{x^2 - 3x + 1}{x - 1}$
3. $f(x) = 4\cos x - 2x$ on $[0, 2\pi]$
4. $f(x) = |x|$
5. $f(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

6-7: Each graph below is a graph of the derivative f' of a function f. In each case use the graph of f' to sketch a possible graph of f.

6.



7.

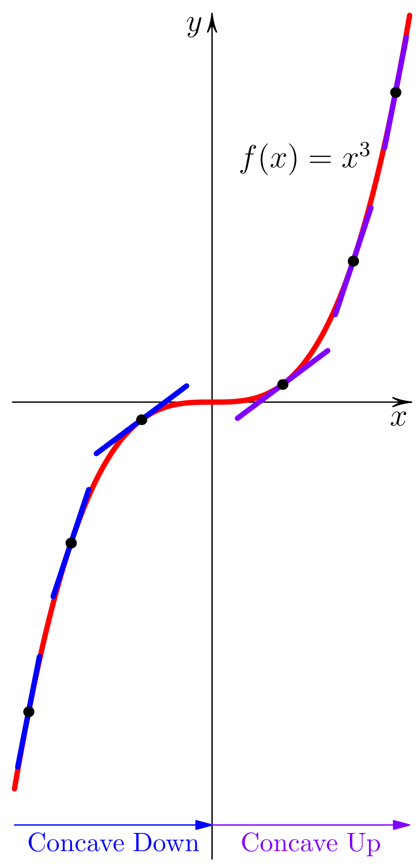


4.4 Concavity

Like increasing and decreasing, concavity is a property of intervals on a graph.

Definition: The graph of function $f(x)$ is **concave upward** on (a, b) if f' is an increasing function on (a, b). Graphically, the curve $y = f(x)$ lies above all its tangent lines on (a, b).

Definition: The graph of function $f(x)$ is **concave downward** on (a, b) if f' is a decreasing function on (a, b). Graphically the curve $y = f(x)$ lies below all its tangent lines on (a, b).



Example 4-11

For $y = x^3$ the graph is concave upward on $(0, \infty)$ because the curve lies above the tangent lines at any point on the interval. Alternatively one sees that as we move from left to right on the interval the tangent slopes are increasing. Similarly $y = x^3$ is concave down on $(-\infty, 0)$ as the curve lies below the tangent line at any point in the interval. Alternatively the tangent slopes are decreasing as one moves from left to right through the interval.

An easy way to remember concavity is to note that if the interval looks like part of the mouth of a happy face, ☺, then it is concave upward. If it is part of the mouth of a sad face, ☹, then it is concave downward.

Since the second derivative $f''(x)$ tells us whether the derivative (and hence tangent slope) is increasing or decreasing we can determine the concavity on an interval using it as follows.

Test for Determining Concavity
Suppose function f is twice differentiable on an interval (a, b).

1. If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave upward on (a, b).
2. If $f''(x) < 0$ for all x in (a, b), then the graph of f is concave downward on (a, b).

To remember this note that f'' is positive when the function is concave upward (☺) and negative when the function is concave downward (☹).

Example 4-12

For $y = f(x) = x^3$ we have $f''(x) = 6x$ which is positive for $x > 0$ and negative for $x < 0$ which indicates the curve is concave upward and concave downward on those intervals as was already observed.

Just as intervals of increasing and decreasing are often punctuated by locations of relative maxima and minima, regions of concavity for a graph are often broken up by inflection points:

Definition: An **inflection point** is a point P on the graph of a function at which the function is continuous and the graph changes from concave upward to concave downward or vice versa.

Due to the concavity test, possible inflection points are located at numbers c in the domain of f where $f''(c) = 0$ or $f''(c)$ does not exist.

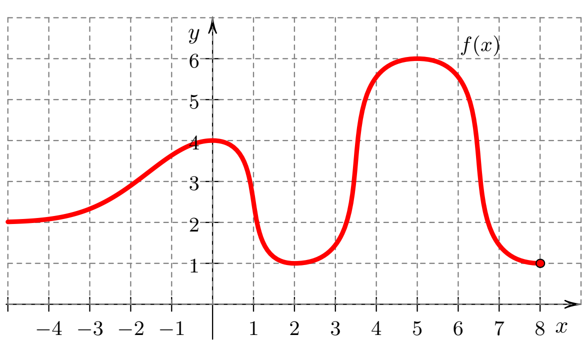
Note that an inflection point is a point P and is therefore reported as $(c, f(c))$.

Example 4-13

For our function $y = x^3$ the point $(0, 0)$ is an inflection point as the function is continuous there and the concavity changes.

Example 4-14

For the following graphically defined function find intervals where the function is increasing, decreasing, concave upward, and downward. Identify any inflection points.



Answer: Increasing: $(-\infty, 0) \cup (2, 5)$
Decreasing: $(0, 2) \cup (5, 8)$
CC Upward: $(-\infty, -2) \cup (1, 3.5) \cup (6.5, 8)$
CC Downward: $(-2, 1) \cup (3.5, 6.5)$
Inf. Points: $(-2, 3)$, $(1, 2.5)$, $(3.5, 3.5)$, $(6.5, 3.5)$

Example 4-15

Find the open intervals upon which the function $f(x) = x^3 - 12x^2$ is concave upward or concave downward. Also find any inflection points.

Solution:

$$f(x) = x^3 - 12x^2\,,\; f \text{ a polynomial} \implies D = (-\infty, \infty)$$
$$f'(x) = 3x^2 - 24x$$
$$f''(x) = 6x - 24 = 6(x-4)$$
$$f''(x) = 0 \implies 6(x-4) = 0 \implies x = 4$$

Since $f''(x)$ is continuous, the value $x = 4$ partitions the real axis into intervals that are either concave upward or downward. The concavity may be determined by analyzing the sign of $f''(x)$:

Interval	$f''(x) = 6(x-4)$	f
$x < 4$	$(+)(-) = (-)$	concave downward
$4 < x$	$(+)(+) = (+)$	concave upward

Therefore $f(x)$ is concave downward on $(-\infty, 4)$ and $f(x)$ is concave upward on $(4, \infty)$. Since concavity changes sign at $x = 4$, and that value lies in the domain of f, we have that $x = 4$ is the x-component of an inflection point.

$y = f(x) = 4^3 - 12(4)^2 = -128$

Thus $(4, -128)$ is an inflection point of function f.

Further Question:

Find the open intervals upon which the function $f(x) = x^4 - x^3$ is concave upward or concave downward. Also find any inflection points.

Example 4-16

Find the intercepts, maxima and minima, intervals of increase and decrease, intervals of concave upward and downward, and inflection points of the following functions. Then sketch their graphs.

1. $f(x) = 3x^5 - 10x^3$
2. $f(x) = x - 2\sin x$ on $[0, 2\pi]$

Solution:

1. $f(x) = 3x^5 - 10x^3 = x^3(3x^2 - 10)$

   f a polynomial $\implies D = \mathbb{R}$

   y-intercept: $f(0) = 0 - 0 = 0 \implies y = 0$

   x-intercept(s): $f(x) = 0 \implies x^3(3x^2 - 10) = 0 \implies \begin{cases} x &= 0 \\ x &= \pm\sqrt{\frac{10}{3}} \end{cases}$

   $f'(x) = 15x^4 - 30x^2 = 15x^2(x^2 - 2)$

   critical numbers: $f'(x) = 0 \implies 15x^2(x^2 - 2) = 0 \implies \begin{cases} x &= 0 \\ x &= \pm\sqrt{2} \end{cases}$

   $f''(x) = 60x^3 - 60x = 60x(x^2 - 1)$

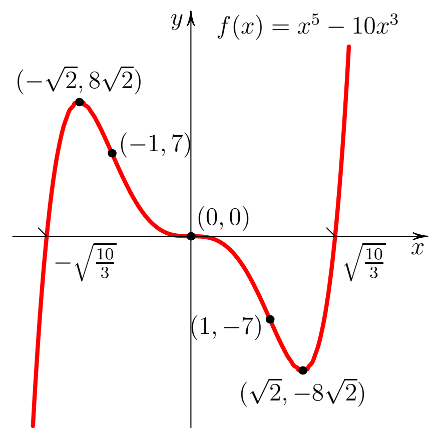
   potential concavity change: $f''(x) = 0 \implies 60x(x^2 - 1) = 0 \implies \begin{cases} x &= 0 \\ x &= \pm 1 \end{cases}$

To analyze the intervals of increase/decrease and concavity we can partition the real axis by the critical numbers and the values where $f''(x) = 0$. Since both the first and second derivatives are continuous, each such sub-interval will either increase or decrease and be either concave upward or downward.

Interval	$f'(x) =$ $15x^2[x-(-\sqrt{2})](x-\sqrt{2})$	$f''(x) =$ $60x[x-(-1)](x-1)$	f
$x < -\sqrt{2}$	$(+)(-)^2(-)(-) = (+)$	$(+)(-)(-)(-) = (-)$	inc, concave down
$-\sqrt{2} < x < -1$	$(+)(-)^2(+)(-) = (-)$	$(+)(-)(-)(-) = (-)$	dec, concave down
$-1 < x < 0$	$(+)(-)^2(+)(-) = (-)$	$(+)(-)(+)(-) = (+)$	dec, concave up
$0 < x < 1$	$(+)(+)^2(+)(-) = (-)$	$(+)(+)(+)(-) = (-)$	dec, concave down
$1 < x < \sqrt{2}$	$(+)(+)^2(+)(-) = (-)$	$(+)(+)(+)(+) = (+)$	dec, concave up
$\sqrt{2} < x$	$(+)(+)^2(+)(+) = (+)$	$(+)(+)(+)(+) = (+)$	inc, concave up

Evaluate f at useful x values:

$$f(-\sqrt{2}) = 3(-\sqrt{2})^5 - 10(-\sqrt{2})^3 = -12\sqrt{2} + 20\sqrt{2} = 8\sqrt{2}$$
$$f(-1) = 3(-1)^5 - 10(-1)^3 = -3 + 10 = 7$$
$$f(1) = 3(1)^5 - 10(1)^3 = 3 - 10 = -7$$
$$f(\sqrt{2}) = 3(\sqrt{2})^5 - 10(\sqrt{2})^3 = 12\sqrt{2} - 20\sqrt{2} = -8\sqrt{2}$$



The function f is increasing on $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ and decreasing on $(-\sqrt{2}, \sqrt{2})$. Analysis of critical numbers using the First Derivative Test shows:

- $f(-\sqrt{2}) = 8\sqrt{2}$ is a relative maximum.
- $f(\sqrt{2}) = -8\sqrt{2}$ is a relative minimum.

The function f is concave upward on $(-1, 0) \cup (1, \infty)$ and concave downward on $(-\infty, -1) \cup (0, 1)$. Analysis of locations where $f''(x) = 0$ shows that points $(-1, 7)$, $(0, 0)$, and $(1, -7)$ are all inflection points.

Plotting intercepts, relative extrema, inflection points and then connecting these dots using the information on increasing/decreasing and concavity makes a sketch of the graph straightforward.

2. $f(x) = x - 2\sin x$ on $[0, 2\pi]$

f a polynomial$(D = \mathbb{R})$ + sine$(D = \mathbb{R})$ $\implies$ f defined on entire interval $\implies D = [0, 2\pi]$

y-intercept: $f(0) = 0 - 2\sin 0 = 0 - 0 = 0 \implies y = 0$

x-intercept(s): $f(x) = 0 \implies x - 2\sin x = 0 \implies \begin{cases} x = 0 \text{ (by inspection)} \\ x = 1.895494267 \approx 1.9 \end{cases}$

(The second solution must be found numerically, for example, by the Bisection Method. A root exists by the Intermediate Value Theorem as $f(0.1) \approx -0.1 < 0$ and $f(2\pi) = 2\pi > 0$.)

$f'(x) = 1 - 2\cos x$

critical numbers: $f'(x) = 0 \implies 1 - 2\cos x = 0 \implies \cos x = \dfrac{1}{2} \implies x = \dfrac{\pi}{3} + 2n\pi$ or $x = \dfrac{5\pi}{3} + 2n\pi$

$$\implies x = \frac{\pi}{3} \text{ or } x = \frac{5\pi}{3} \text{ in } [0, 2\pi]$$

$f''(x) = 2\sin x$

potential concavity change: $f''(x) = 0 \implies \sin x = 0 \implies x = n\pi$

$$\implies x = 0 \text{ or } x = \pi \text{ or } x = 2\pi \text{ in } [0, 2\pi]$$

To evaluate the sign of f' and f'' it is of value to use test points in the intervals. (See Appendix A.) One can choose arbitrary values in the interval and use a calculator or judiciously choose angles when possible for which the trigonometric values may be derived.

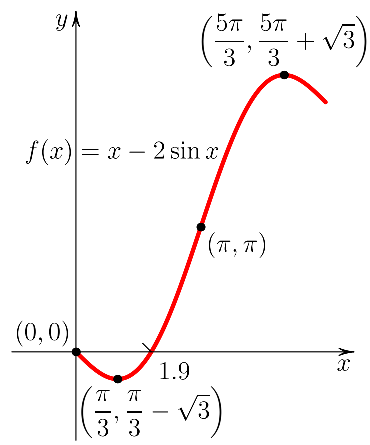
Interval	Test Value	$f'(x) = 1 - 2\cos x$	$f''(x) = 2\sin x$	f
$0 < x < \frac{\pi}{3}$	$\frac{\pi}{4}$	$1 - \sqrt{2} = (-)$	$\sqrt{2} = (+)$	decreasing, concave upward
$\frac{\pi}{3} < x < \pi$	$\frac{\pi}{2}$	$1 = (+)$	$2 = (+)$	increasing, concave upward
$\pi < x < \frac{5\pi}{3}$	$\frac{5\pi}{4}$	$1 + \sqrt{2} = (+)$	$-\sqrt{2} = (-)$	increasing, concave downward
$\frac{5\pi}{3} < x < 2\pi$	6	$\approx -.92 = (-)$	$\approx -.56 = (-)$	decreasing, concave downward

Evaluate f at useful x values:

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2\sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} - \sqrt{3}$$

$$f(\pi) = \pi - 2\sin\pi = \pi + 0 = \pi$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\sin\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + 2\sin\left(\frac{\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$$



The function f is increasing on $(\pi/3, 5\pi/3)$ and decreasing on $(0, \pi/3) \cup (5\pi/3, 2\pi)$. Analysis of critical numbers using the First Derivative Test shows:

- $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}$ is a relative minimum.
- $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$ is a relative maximum.

The function f is concave upward on $(0, \pi)$ and concave downward on $(\pi, 2\pi)$. Analysis of locations where $f''(x) = 0$ shows that the point (π, π) is an inflection point. A sketch is shown. Values involving π and $\sqrt{3}$ need to be evaluated or estimated to get the relative point positions correct in the sketch.

Further Questions:

Find the intercepts, relative maxima and minima, intervals of increase and decrease, intervals of concave upward and downward, and inflection points of the following functions. Then sketch their graphs.

1. $f(x) = x^4 - 6x^2$
2. $f(x) = x - 3x^{\frac{1}{3}}$
3. $f(x) = y = 3\sin x - \sin^3 x$ on $[0, 2\pi]$

Example 4-17

Sketch the graph of a continuous function on $\mathbb{R}$ satisfying all of the following:

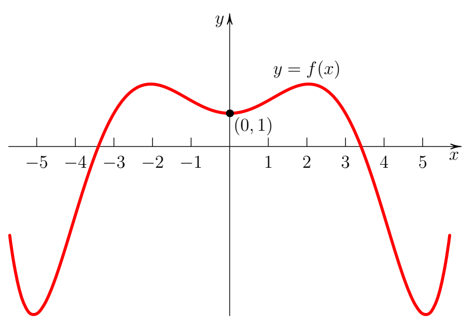
- $f(0) = 1$
- $f'(2) = f'(5) = 0$
- $f'(x) > 0$ if $0 < x < 2$ or $x > 5$
- $f'(x) < 0$ if $2 < x < 5$
- $f''(x) > 0$ if $0 < x < 1$ or $x > 4$
- $f''(x) < 0$ if $1 < x < 4$
- $f(-x) = f(x)$

Solution:

We see that $x = 2$ and $x = 5$ are critical numbers. Summarizing the information for f' and f'' on a table gives:

Interval	$f'(x)$	$f''(x)$	f
$0 < x < 1$	$(+)$	$(+)$	increasing, concave up
$1 < x < 2$	$(+)$	$(-)$	increasing, concave down
$2 < x < 4$	$(-)$	$(-)$	decreasing, concave down
$4 < x < 5$	$(-)$	$(+)$	decreasing, concave up
$5 < x$	$(+)$	$(+)$	increasing, concave up

By the First Derivative Test, $x = 2$ will be the location of a relative maximum and $x = 5$ will be the location of a relative minimum. Since f is continuous, $x = 1$ and $x = 4$ will be the x-coordinates of points of inflection. Since $f(0) = 1$, we have a y-intercept of $y = 1$. Start there and sketch to the right on the graph using the above information. For the graph for $x < 0$ note that $f(-x) = f(x)$ implies that f is an even function. As such once the function is drawn for $x \geq 0$ this only needs to be reflected across the y-axis to generate the graph of f for $x < 0$. One possible graph that meets all of these criteria is as follows:



Further Question:

Sketch the graph of a function satisfying all of the following:

- $f(0) = 0$
- $f'(1) = f'(3) = f'(5) = 0$
- $f'(x) > 0$ if $0 < x < 1$ or $3 < x < 5$
- $f'(x) < 0$ if $1 < x < 3$ or $x > 5$
- $f''(x) > 0$ if $2 < x < 4$
- $f''(x) < 0$ if $0 < x < 2$ or $x > 4$
- $f(-x) = -f(x)$

The second derivative can also be used to evaluate whether critical numbers are the locations of relative extrema and, if so, whether they are maxima or minima:

Second Derivative Test for a Relative Extremum
Suppose f'' is continuous on an open interval that contains c and that $f'(c) = 0$.

1. If $f''(c) < 0$, then f has a relative maximum at c .
2. If $f''(c) > 0$, then f has a relative minimum at c .
3. If $f''(c) = 0$, then the test is inconclusive.

This result follows from the fact that a negative second derivative at c means the graph of the function is concave downward (☹) at that value (and hence will have a relative maximum), while a positive second derivative means the graph of the function is concave upward (☺) at that value (and hence will have a relative minimum).

Example 4-18

- $y = x^2$ has $f'(0) = 0$ and $f''(0) = 2$. Since $f''(0)$ is greater than zero the function is concave upward at $x = 0$ and hence has a relative minimum at that critical number.
- $y = x^3$ has $f'(0) = 0$. However $f''(0) = 0$ as well so the Second Derivative Test is inconclusive. We need to use a different test (First Derivative Test) to determine that no extrema occurs at 0.
- $y = x^4$ has $f'(0) = 0$ and $f''(0) = 0$ so the Second Derivative Test is also inconclusive here. However in this case the First Derivative Test confirms a relative minimum occurs at 0.

Example 4-19

Find the relative maxima and minima (and their locations) of the questions in Example 4-16 :

1. $f(x) = 3x^5 - 10x^3$
2. $f(x) = x - 2\sin x$ on $[0, 2\pi]$

Use the Second Derivative Test where possible.

Solution:

1. From Example 4-16 we found:

$$f(x) = 3x^5 - 10x^3 = x^3(3x^2 - 10)$$
$$f'(x) = 15x^4 - 30x^2 = 15x^2(x^2 - 2) \implies \text{critical numbers: } x = 0,\ \pm\sqrt{2}$$
$$f''(x) = 60x^3 - 60x = 60x(x^2 - 1)$$

Evaluating the second derivative at the critical numbers yields:

$$f''(-\sqrt{2}) = 60(-\sqrt{2})\left[\left(-\sqrt{2}\right)^2 - 1\right] = -60\sqrt{2} < 0 \quad ☹$$
$$\implies x = -\sqrt{2} \text{ is the location of a relative maximum.}$$

$f''(0) = 60(0)\left[(0)^2 - 1\right] = 0$

$\implies$ Second Derivative Test is inconclusive.

$\implies$ Apply First Derivative Test as before to conclude no extrema at $x = 0$.

$f''(\sqrt{2}) = 60(\sqrt{2})\left[(\sqrt{2})^2 - 1\right] = 60\sqrt{2} > 0$ ☺

$\implies x = \sqrt{2}$ is the location of a relative minimum.

As before evaluate f at these locations to conclude $f(-\sqrt{2}) = 8\sqrt{2}$ is a relative maximum and $f(\sqrt{2}) = -8\sqrt{2}$ is a relative minimum.

2. From Example 4-16 we found:

   $f(x) = x - 2\sin x$ on $[0, 2\pi]$

   $f'(x) = 1 - 2\cos x \implies$ critical numbers: $x = \dfrac{\pi}{3}$ or $x = \dfrac{5\pi}{3}$

   $f''(x) = 2\sin x$

   Evaluating the second derivative at the critical numbers yields:

   $f''\left(\frac{\pi}{3}\right) = 2\sin\left(\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3} > 0$ ☺

   $\implies x = \dfrac{\pi}{3}$ is the location of a relative minimum.

   $f''\left(\frac{5\pi}{3}\right) = 2\sin\left(\frac{5\pi}{3}\right) = -2\sin\left(\frac{\pi}{3}\right) = -2\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3} < 0$ ☹

   $\implies x = \dfrac{5\pi}{3}$ is the location of a relative maximum.

   As before evaluate f at these locations to conclude $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}$ is a relative minimum and $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$ is a relative maximum.

Further Questions:

Find the relative maxima and minima (and their locations) of

1. $f(x) = x^3 - 2x^2 + x$
2. $B(x) = 3x^{\frac{2}{3}} - x$

Use the Second Derivative Test where possible.

Answers: Page 250

Exercise 4-4

1-2: Show that the following functions have no inflection points:

1. $f(x) = 2x^4$
2. $f(x) = \dfrac{1}{x}$

---

3. Find the relative extrema (and their locations), the intervals of concavity, and the inflection points of the function $f(x) = x^5 - 15x^3 + 1$.

4-5: Determine the relative maxima and minima of the following functions and their locations. Use the Second Derivative Test.

4. $f(x) = x^3 - 12x + 1$
5. $f(x) = \cos(2x) - 4\sin(x)$ on the interval $(-\pi, \pi)$

---

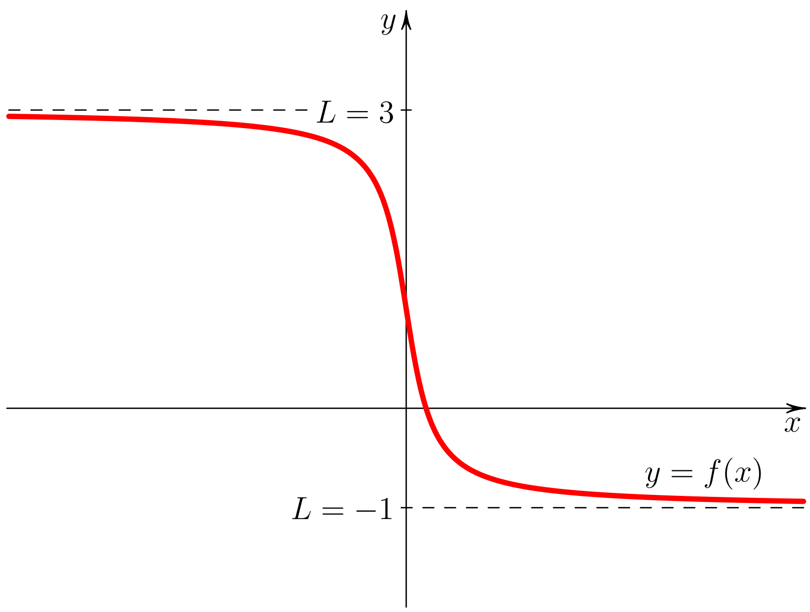
6. Can you use the Second Derivative Test to categorize the critical number $x = 0$ of the function $f(x) = \sin^4 x$? Explain why or why not.

7-9: Find the domain, intercepts, intervals of increase and decrease, relative maxima and minima, intervals of concave upward and downward, and inflection points of the given functions and then sketch their graphs.

7. $f(x) = x^3 - 6x^2$
8. $g(t) = t^{\frac{3}{2}} - 3t^{\frac{1}{2}}$
9. $F(x) = \sqrt{x^2 + 9}$

4.5 Limits at Infinity and Horizontal Asymptotes

We have seen how y-values of $y = f(x)$ may approach infinity at a vertical asymptote. We now consider the behaviour of a function as x approaches infinity.



Definition: If the values of $f(x)$ can be made arbitrarily close to the value L as x is made sufficiently large then we write:[3]

$$\lim_{x\to\infty} f(x) = L \ .$$

Definition: If the values of $f(x)$ can be made arbitrarily close to the value L as x is made sufficiently large negatively then we write:

$$\lim_{x\to-\infty} f(x) = L \ .$$

Example 4-20

For the above graphically defined example one has:

$$\lim_{x\to\infty} f(x) = -1$$

and

$$\lim_{x\to-\infty} f(x) = 3$$

[3] A rigorous definition of this limit can be made as follows. Suppose function f is defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ if and only if for any $\epsilon > 0$ there exists $c \in (a, \infty)$ such that $|f(x) - L| < \epsilon$ whenever $x > c$. A similar definition $\lim_{x\to-\infty} f(x) = L$ may also be formulated.

The following theorem, in conjunction with our other limit theorems, allows us to evaluate many limits as $x \to \infty$ or $-\infty$.

Theorem 4-8: Let $r > 0$ be a rational number. Then

- $\displaystyle\lim_{x\to\infty} \frac{1}{x^r} = 0$
- $\displaystyle\lim_{x\to-\infty} \frac{1}{x^r} = 0$

In the second case ($x \to -\infty$), r must be such that x^r is defined.[4]

When one tries to evaluate limits where the magnitude of x approaches infinity, direct substitution of "$x = \infty$" may lead to the indeterminate form $\dfrac{\infty}{\infty}$. As a general strategy one divides the numerator and denominator in an expression by the highest power term found in either to evaluate the limit.

Example 4-21

Evaluate the given limits.

1. $\displaystyle\lim_{x\to\infty} \frac{2x^3+4x-1}{3x^3-5x^2+2}$
2. $\displaystyle\lim_{x\to\infty} \frac{4x^3+5}{x^2+1}$
3. $\displaystyle\lim_{x\to\infty} \frac{5x^4+3x+2}{6x^5+2x^3+1}$
4. $\displaystyle\lim_{x\to-\infty} \left(\sqrt{x^2+3x}+x\right)$

Solution:

1. $\displaystyle\lim_{x\to\infty} \frac{2x^3+4x-1}{3x^3-5x^2+2} = \lim_{x\to\infty} \frac{x^2\left(2+\frac{4}{x^2}-\frac{1}{x^3}\right)}{x^2\left(3-\frac{5}{x}+\frac{2}{x^3}\right)} = \lim_{x\to\infty} \frac{x^2}{x^2}\cdot\frac{2+\frac{4}{x^2}-\frac{1}{x^3}}{3-\frac{5}{x}+\frac{2}{x^3}} = \lim_{x\to\infty} 1\cdot\frac{2+0-0}{3-0+0} = \frac{2}{3}$
2. $\displaystyle\lim_{x\to\infty} \frac{4x^3+5}{x^2+1} = \lim_{x\to\infty} \frac{x^3}{x^2}\cdot\frac{4+\frac{5}{x^3}}{1+\frac{1}{x^2}} = \lim_{x\to\infty} x\cdot\frac{4+\frac{5}{x^3}}{1+\frac{1}{x^2}} = (+\infty)\frac{4+0}{1+0} = \infty$
3. $\displaystyle\lim_{x\to\infty} \frac{5x^4+3x+2}{6x^5+2x^3+1} = \lim_{x\to\infty} \frac{x^4}{x^5}\cdot\frac{5+\frac{3}{x^3}+\frac{2}{x^4}}{6+\frac{2}{x^2}+\frac{1}{x^5}} = \lim_{x\to\infty} \frac{1}{x}\cdot\frac{5+\frac{3}{x^3}+\frac{2}{x^4}}{6+\frac{2}{x^2}+\frac{1}{x^5}} = 0\cdot\frac{5+0+0}{6+0+0} = 0$
4. Unlike the previous limits that all had the indeterminate form $\frac{\infty}{\infty}$ this limit has the form $\infty - \infty$ since the square root is always positive and the second term is approaching $-\infty$. Resolving the limit may be accomplished by multiplying by $1 = \frac{\text{conjugate}}{\text{conjugate}}$:

$$\begin{aligned}\lim_{x\to-\infty}\left(\sqrt{x^2+3x}+x\right) &= \lim_{x\to-\infty}\frac{\left(\sqrt{x^2+3x}+x\right)\left(\sqrt{x^2+3x}-x\right)}{\sqrt{x^2+3x}-x} = \lim_{x\to-\infty}\frac{x^2+3x-x^2}{\sqrt{x^2+3x}-x}\\ &= \lim_{x\to-\infty}\frac{3x}{\sqrt{(x^2)\left(1+\frac{3}{x}\right)}-x} = \lim_{x\to-\infty}\frac{3x}{\sqrt{x^2}\sqrt{1+\frac{3}{x}}-x}\\ &= \lim_{x\to-\infty}\frac{3x}{|x|\sqrt{1+\frac{3}{x}}-x} = \lim_{x\to-\infty}\frac{3x}{(-x)\sqrt{1+\frac{3}{x}}-x}\\ &= \lim_{x\to-\infty}\frac{x}{x}\cdot\frac{3}{-\sqrt{1+\frac{3}{x}}-1} = 1\cdot\frac{3}{-\sqrt{1+0}-1} = \frac{3}{-1-1} = -\frac{3}{2}\end{aligned}$$

[4] We note in passing that the theorem is also valid for $0 < r < 1$, not just $r \geq 1$. So, for instance, $\displaystyle\lim_{x\to\infty}\frac{1}{x^{1/2}} = \lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0$ despite the fact that for large x, $\sqrt{x} < x$. The point is that $\sqrt{x}$ increases without bound as x does so the limit vanishes.

Note that critical to the correct evaluation of this limit is to recall that $\sqrt{x^2} = |x|$ and that $|x| = -x$ since, as $x \to -\infty$, x is negative.

The reader can check all of these limits by evaluating the function at either a large positive value of x (for $x \to +\infty$ limit) or a large negative value of x (for $x \to -\infty$ limit). If this is done on a calculator it is a good idea to store the value in a variable when evaluating the function.

Further Questions:

Evaluate the following:

1. $\lim\limits_{x\to\infty} \dfrac{2x^2 - 5x + 7}{7x^2 + 3x + 2}$
2. $\lim\limits_{x\to\infty} \dfrac{x^3 - 1}{x^4 + 1}$
3. $\lim\limits_{x\to\infty} \dfrac{x^5 + 3x + 5}{x^2 - 2x + 1}$
4. $\lim\limits_{x\to-\infty} \dfrac{x^3 + 5x + 1}{2x^3 - 4x^2 + 3}$
5. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2 + 3}}{x + 2}$
6. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2 + 1}}{x - 3}$
7. $\lim\limits_{x\to\infty} \left(\sqrt{x^2 + 3x + 1} - x\right)$
8. $\lim\limits_{x\to\infty} \left(x - \sqrt{4x^2 + 5x - 3}\right)$

Horizontal Asymptotes

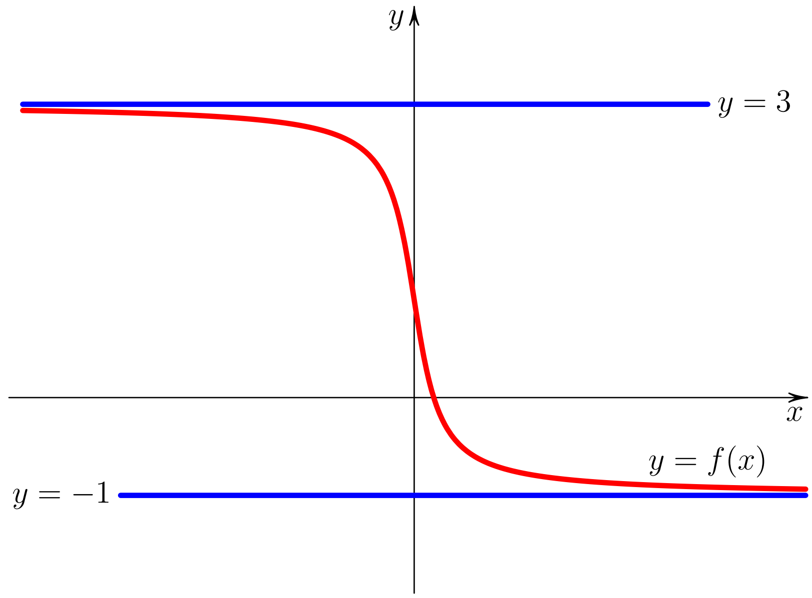
Recall that a vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \pm\infty \qquad \lim_{x\to a^+} f(x) = \pm\infty \qquad \lim_{x\to a^-} f(x) = \pm\infty$$

Definition: The horizontal line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either of the following statements is true:

$$\lim_{x\to\infty} f(x) = L \qquad \lim_{x\to-\infty} f(x) = L$$

The following function has horizontal asymptotes $y = -1$ and $y = 1$.



Example 4-22

In the above graphically defined example the horizontal asymptotes are the lines $y = -1$ and $y = 3$.

Example 4-23

Find the horizontal and vertical asymptotes of the following functions.

1. $f(x) = \dfrac{x^2 - 4x}{x^2 - 16}$

2. $f(x) = \dfrac{\sqrt{9 + x^2}}{3 - x}$

Solution:

1. $f(x) = \dfrac{x^2 - 4x}{x^2 - 16}$

   horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2 - 3x}{x^2 - 16} = \lim_{x\to\infty} \frac{1 - \frac{3}{x}}{1 - \frac{16}{x^2}} = \frac{1-0}{1-0} = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2 - 3x}{x^2 - 16x} = \lim_{x\to-\infty} \frac{1 - \frac{3}{x}}{1 - \frac{16}{x^2}} = \frac{1-0}{1-0} = 1$$

   vertical asymptotes:

$$x^2 - 16 = 0 \implies (x-4)(x+4) = 0 \implies x = \pm 4$$

$$\lim_{x\to4} \frac{x^2 - 4x}{x^2 - 16} = \lim_{x\to4} \frac{x(x-4)}{(x+4)(x-4)} = \lim_{x\to4} \frac{x}{x+4} = \frac{4}{4+4} = \frac{4}{8} = \frac{1}{2} \quad \text{(no asymptote)}$$

$$\lim_{x\to-4} \frac{x^2 - 4x}{x^2 - 16} : \frac{(-4)^2 - 4(-4)}{(-4^2) + 16} = \frac{16+16}{16-16} = \frac{32}{0}$$

   Therefore f has horizontal asymptote $y = 1$ (in both directions) and vertical asymptote $x = -4$.

Note that near $x = -4$ we have:

$$\lim_{x\to -4^-} f(x) = \lim_{x\to -4^-} \frac{x(x-4)}{(x+4)(x-4)} = \lim_{x\to -4^-} \frac{x}{x+4} = \frac{-4}{0^-} = +\infty$$
$$\lim_{x\to -4^+} f(x) = \lim_{x\to -4^+} \frac{x(x-4)}{(x+4)(x-4)} = \lim_{x\to -4^+} \frac{x}{x+4} = \frac{-4}{0^+} = -\infty$$

which is useful information when graphing the function near the vertical asymptote.

2. $f(x) = \dfrac{\sqrt{9+x^2}}{3-x}$

   horizontal asymptotes:

$$\lim_{x\to\infty} \frac{\sqrt{9+x^2}}{3-x} = \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{\frac{9}{x^2}+1}}{3-x} = \lim_{x\to\infty} \frac{|x|\sqrt{\frac{9}{x^2}+1}}{3-x} = \lim_{x\to\infty} \frac{x\sqrt{\frac{9}{x^2}+1}}{x\left(\frac{3}{x}-1\right)}$$
$$= \lim_{x\to\infty} \frac{\sqrt{\frac{9}{x^2}+1}}{\frac{3}{x}-1} = \frac{\sqrt{0+1}}{0-1} = \frac{1}{-1} = -1$$
$$\lim_{x\to-\infty} \frac{\sqrt{9+x^2}}{3-x} = \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{\frac{9}{x^2}+1}}{3-x} = \lim_{x\to-\infty} \frac{|x|\sqrt{\frac{9}{x^2}+1}}{3-x} = \lim_{x\to-\infty} \frac{-x\sqrt{\frac{9}{x^2}+1}}{x\left(\frac{3}{x}-1\right)}$$
$$= \lim_{x\to-\infty} \frac{-\sqrt{\frac{9}{x^2}+1}}{\frac{3}{x}-1} = \frac{-\sqrt{0+1}}{0-1} = \frac{-1}{-1} = 1$$

   vertical asymptotes:

$$\lim_{x\to 3} \frac{\sqrt{9+x^2}}{3-x} : \frac{\sqrt{9+3^2}}{3-3} = \frac{\sqrt{18}}{0}$$

   Therefore f has horizontal asymptotes $y = 1$ and $y = -1$ and vertical asymptote $x = 3$.
   (Near $x = 3$ one finds $\lim_{x\to 3^-} f(x) = \frac{\sqrt{18}}{0^+} = \infty$ and $\lim_{x\to 3^+} f(x) = \frac{\sqrt{18}}{0^-} = -\infty$.)

Further Questions:

Find the horizontal and vertical asymptotes of the following functions.

1. $y = \dfrac{x+3}{x^2-9}$

2. $f(x) = \dfrac{\sqrt{4x^2+1}}{x+1}$

Answers: Page 251

Exercise 4-5

1-13: Determine the following limits. For any limit that does not exist, identify if it has an infinite trend (∞ or $-\infty$).

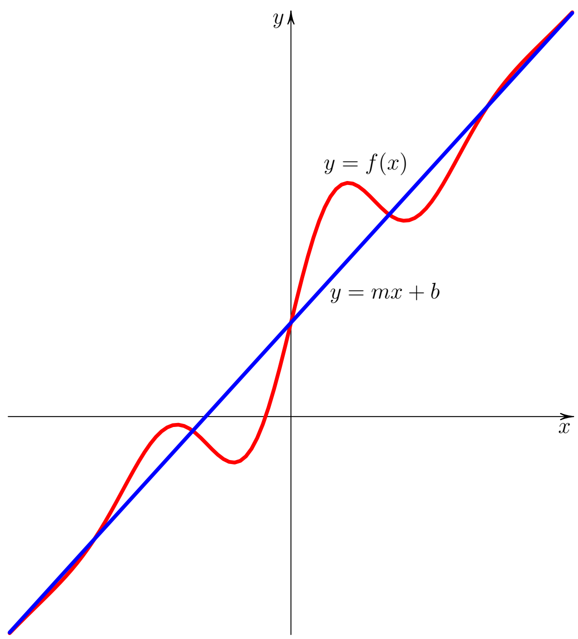
1. $\lim_{x\to-\infty} \frac{18x^2-3x}{3x^5-3x^2+2}$
2. $\lim_{x\to\infty} \frac{x^2}{\sqrt{x^3+2}}$
3. $\lim_{x\to-\infty} \frac{\sqrt{x^2+2x}}{6x+3}$
4. $\lim_{x\to\infty} \left[\sec\left(\frac{1}{x}\right)+1\right]$
5. $\lim_{x\to-\infty} \sqrt{x^2+3x+5}+x$
6. $\lim_{x\to\infty} \frac{x^2+5x+4}{3x^2+2}$
7. $\lim_{x\to\infty} \frac{4x^5-3x^2+6}{x^4+7}$
8. $\lim_{x\to\infty} \frac{3x^4+6x-7}{x^5+10}$
9. $\lim_{x\to\infty} \frac{\sqrt{x^2+10}}{x+3}$
10. $\lim_{x\to-\infty} \frac{\sqrt{2x^2+3}}{x-2}$
11. $\lim_{x\to-\infty} \frac{5x+\sqrt{x^2+1}}{x+5}$
12. $\lim_{x\to\infty} \left(\sqrt{x^2+4x+1}-x\right)$
13. $\lim_{x\to-\infty} \left(x+\sqrt{x^2-6x+5}\right)$

14-21: Find the horizontal asymptotes of the following functions.

14. $f(x) = \frac{3x+3}{2x-4}$
15. $f(x) = x^3+5x+2$
16. $g(t) = \frac{\sqrt{t^2+3}}{t-2}$
17. $f(x) = \frac{x^2-2x+1}{2x^2-2x-12}$
18. $f(x) = \frac{\cos x}{x}$
19. $y = \frac{5x^2-3x+1}{x^2-16}$
20. $f(x) = \frac{x^3+1}{x^3+x^2}$
21. $F(x) = \frac{x}{\sqrt{4x^2+1}}$

4.6 Slant Asymptotes

A function has a **slant asymptote** if it approaches an oblique (slanted) line as x approaches infinity as shown in the following graph.



A slant asymptote will occur if

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0 \text{ or } \lim_{x\to-\infty} [f(x) - (mx + b)] = 0$$

for some constants m, and b, the slope and y-intercept of the asymptote, respectively. The equation of the slant asymptote is therefore:

$$y = mx + b \ .$$

A rational function $f(x) = P(x)/Q(x)$ will have a slant asymptote if the degree of the numerator $P(x)$ is one more than the denominator $Q(x)$. In this case use long division to find the equation of the slant asymptote. While a function may have a different slant asymptote as $x \to \infty$ and $x \to -\infty$, for the rational functions under consideration here these will be the same.

Example 4-24

Find the slant asymptote (if any) of the rational function $y = f(x) = \dfrac{x^3 + x^2 + 7}{2x^2 - 2x}$.

Solution:

The degree of the numerator is 3 which is one more than the degree of the denominator which is 2 so the rational function will have a slant asymptote. Using **polynomial long division** one finds $x^3 + x^2 + 7$ divided by $2x^2 - 2x$ is $\frac{1}{2}x + 1$ with a remainder of $2x + 7$. In symbols

$$\frac{x^3 + x^2 + 7}{2x^2 - 2x} = \frac{1}{2}x + 1 + \frac{2x + 7}{2x^2 - 2x} \ ,$$

since the remainder is yet to be divided by $2x^2 - 2x$.

The slant asymptote is therefore the line $y = \frac{1}{2}x + 1$ since

$$\lim_{x\to\pm\infty}\left[f(x) - \left(\frac{1}{2}x + 1\right)\right] = \lim_{x\to\pm\infty}\frac{2x+7}{2x^2-2x} = \lim_{x\to\pm\infty}\frac{x}{x^2}\cdot\frac{2+\frac{7}{x}}{2-\frac{2}{x}} = \lim_{x\to\pm\infty}\frac{1}{x}\cdot\frac{2+\frac{7}{x}}{2-\frac{2}{x}} = 0\cdot\frac{2+0}{2-0} = 0$$

Further Questions:

Find the slant asymptotes (if any) of the following:

1. $f(x) = \dfrac{x^3}{x^2+2}$
2. $y = \dfrac{6x^4+5}{2x^3-4}$
3. $y = \dfrac{2x^4 - x^3 + x - 3}{x^3 - 2}$

Answers: Page 252

Exercise 4-6

1-3: Find any slant asymptotes of the graphs of the following functions.

1. $f(x) = \dfrac{3x^2-4x}{x+2}$
2. $g(x) = \dfrac{x^3+2x+1}{x^4+5x-7}$
3. $y = \dfrac{x^3-2x^2+1}{x^2+2}$

4.7 Curve Sketching

We have now developed many tools to analyze a function that enable us to accurately sketch a curve.

Sketching the Graph of $y = f(x)$

Domain: Determine D, the set of x-values for which the function is defined.

Intercepts: The y-intercept is $y = f(0)$ while the x-intercepts are $x = c$ where c are the solutions of $f(c) = 0$.

Symmetry: Is the function even, $f(-x) = f(x)$, odd, $f(-x) = -f(x)$, or periodic, $f(x+p) = f(x)$?

Asymptotes: Find any vertical $\left(x = a \text{ where } \lim_{x\to a^{(\pm)}} f(x) = \pm\infty\right)$,
horizontal $\left(y = L \text{ where } \lim_{x\to\pm\infty} f(x) = L\right)$,
or slant $\left(y = mx + b \text{ where } \lim_{x\to\pm\infty} [f(x) - (mx+b)] = 0\right)$ asymptotes.

Intervals: Partition D into open intervals using locations of any vertical asymptotes, critical numbers (c in D where $f'(c) = 0$ or does not exist), and numbers c in D where $f''(c) = 0$ or does not exist.

Increasing/Decreasing: Use increasing/decreasing test to determine intervals of increase ($f'(x) > 0$) or decrease ($f'(x) < 0$).

Relative Extrema: For any critical numbers determine relative extrema. Evaluate them using the First Derivative Test (f' goes from negative to positive at $c \Rightarrow$ relative minimum, positive to negative $\Rightarrow$ relative maximum, no change $\Rightarrow$ no extremum) or Second Derivative Test ($f''(c) > 0 \Rightarrow$ relative minimum, $f''(c) < 0 \Rightarrow$ relative maximum, $f''(c) = 0 \Rightarrow$ inconclusive). Evaluate $f(c)$ to find the value of any relative extrema.

Concavity: Use the concavity test ($f''(x) > 0 \Rightarrow$ concave upward, $f''(x) < 0 \Rightarrow$ concave downward) to find intervals of concavity.

Inflection Points: Consider x-values c where $f''(c) = 0$ or does not exist but f is continuous. Inflection points $(c, f(c))$ occur where concavity changes (positive to negative or vice versa).

Sketch: Sketch the above information on a graph choosing axis limits that will include the interesting regions of the graph.

Example 4-25

Apply calculus techniques to identify all important features of the graph of each function and then sketch it.

1. $f(x) = \dfrac{x^2}{x^2 - 3}$
2. $f(x) = x(x+1)^{\frac{3}{5}}$

Solution:

1. $f(x) = \dfrac{x^2}{x^2-3}$

   domain: $x^2-3=0 \implies x=\pm\sqrt{3} \implies D=\mathbb{R}-\left\{-\sqrt{3},\sqrt{3}\right\}$

   y-intercept: $f(0)=\dfrac{0^2}{0^2-3}=0 \implies y=0$

   x-intercept(s): $f(x)=0 \implies x^2=0 \implies x=0$

   symmetry: $f(-x)=\dfrac{(-x)^2}{(-x)^2-3}=\dfrac{x^2}{x^2-3}=f(x) \implies f$ is even.

   horizontal asymptotes:

   $\lim\limits_{x\to\infty}\dfrac{x^2}{x^2-3}=\lim\limits_{x\to\infty}\dfrac{1}{1-\frac{3}{x^2}}=\dfrac{1}{1-0}=1 \implies y=1$ (Similarly $\lim\limits_{x\to-\infty}f(x)=1 \implies y=1$.)

   vertical asymptotes:

   $\lim\limits_{x\to\sqrt{3}}\dfrac{x^2}{x^2-3}:\dfrac{3}{3-3}=\dfrac{3}{0} \implies x=\sqrt{3}$ $\quad$ $\lim\limits_{x\to-\sqrt{3}}\dfrac{x^2}{x^2-3}:\dfrac{3}{3-3}=\dfrac{3}{0} \implies x=-\sqrt{3}$

   slant asymptotes: None since the degree of the numerator and denominator are both equal to 2.

   $$f'(x)=\frac{2x(x^2-3)-x^2(2x)}{(x^2-3)^2}=\frac{-6x}{(x^2-3)^2}$$

   critical numbers: $f'(x)=0 \implies -6x=0 \implies x=0$

   $$f''(x)=\frac{-6(x^2-3)^2-(-6x)(2)(x^2-3)(2x)}{(x^2-3)^4}=\frac{-6(x^2-3)+24x^2}{(x^2-3)^3}=\frac{18x^2+18}{(x^2-3)^3}=\frac{18(x^2+1)}{(x^2-3)^3}$$

   potential concavity change: $f''(x)=0 \implies x^2+1=0 \implies$ no solution

   Note that $f'(x)$ and $f''(x)$ are not defined if $x=\pm\sqrt{3}$, however these values are not in the domain D. As such they cannot be a location for an extrema nor an inflection point. However the sign of f' and f'' can change at such a discontinuity so these values also must be used when partitioning the real axis into intervals. In general one must remember to consider vertical asymptotes in addition to critical values and locations where $f''(x)=0$ or is undefined when doing interval analysis.

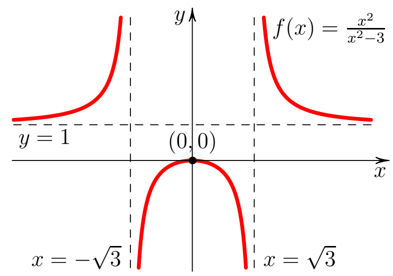
Interval	$f'(x)=\frac{(-6)x}{(x^2-3)^2}$	$f''(x)=\frac{18(x^2+1)}{[x-(-\sqrt{3})]^3(x-\sqrt{3})^3}$	f
$x<-\sqrt{3}$	$\frac{(-)(-)}{(+)}=(+)$	$\frac{(+)(+)}{(-)^3(-)^3}=(+)$	increasing, concave upward
$-\sqrt{3}<x<0$	$\frac{(-)(-)}{(+)}=(+)$	$\frac{(+)(+)}{(+)^3(-)^3}=(-)$	increasing, concave downward
$0<x<\sqrt{3}$	$\frac{(-)(+)}{(+)}=(-)$	$\frac{(+)(+)}{(+)^3(-)^3}=(-)$	decreasing, concave downward
$\sqrt{3}<x$	$\frac{(-)(+)}{(+)}=(-)$	$\frac{(+)(+)}{(+)^3(+)^3}=(+)$	decreasing, concave upward

   The function f is increasing on $(-\infty,-\sqrt{3})\cup(-\sqrt{3},0)$ and decreasing on $(0,\sqrt{3})\cup(\sqrt{3},\infty)$. (Note that the vertical asymptotes at $x=\pm\sqrt{3}$ mean we cannot combine these intervals.) Analysis of the critical number at $x=0$ using the First Derivative Test and the table or the Second Derivative Test ($f''(0)=-\frac{2}{3}<0$) shows that

   - $f(0)=0$ is a relative maximum.

The function f is concave upward on $(-\infty,-\sqrt{3})\cup(\sqrt{3},\infty)$ and concave downward on $(-\sqrt{3},\sqrt{3})$. There are no inflection points. (Concavity does change at $x=\pm\sqrt{3}$ but these are not in the domain.)

A sketch of the graph is below.



2. $f(x)=x(x+1)^{\frac{3}{5}}$

domain: Since $(x+1)^{\frac{3}{5}}=\left[(x+1)^3\right]^{\frac{1}{5}}=\sqrt[5]{(x+1)^3}$ (odd root) $\implies D=\mathbb{R}$

y-intercept: $f(0)=0(1)^{\frac{3}{5}}=0 \implies y=0$

x-intercept(s): $f(x)=0 \implies x=0$

$$\text{or } (x+1)^{\frac{3}{5}}=0 \implies \left[(x+1)^{\frac{3}{5}}\right]^{\frac{5}{3}}=0^{\frac{5}{3}} \implies x+1=0 \implies x=-1$$

symmetry: $f(-x)=(-x)(-x+1)^{\frac{3}{5}}=(-x)(-1)^{\frac{3}{5}}(x-1)^{\frac{3}{5}}=x(x-1)^{\frac{3}{5}} \implies$ No symmetry

horizontal asymptotes: None since:

$$\lim_{x\to\infty} x(x+1)^{\frac{3}{5}}=\lim_{x\to\infty} x^1(x)^{\frac{3}{5}}\left(1+\frac{1}{x}\right)^{\frac{3}{5}}=\lim_{x\to\infty} x^{\frac{8}{5}}\left(1+\frac{1}{x}\right)^{\frac{3}{5}}=(\infty)\left(1+0\right)^{\frac{3}{5}}=\infty$$

$$\lim_{x\to-\infty} x(x+1)^{\frac{3}{5}}=\lim_{x\to-\infty} x^{\frac{8}{5}}\left(1+\frac{1}{x}\right)^{\frac{3}{5}}=\lim_{x\to-\infty}\left(x^{\frac{1}{5}}\right)^8\left(1+\frac{1}{x}\right)^{\frac{3}{5}}=(+\infty)\left(1+0\right)^{\frac{3}{5}}=\infty$$

vertical asymptotes: None

slant asymptotes: None

$$f'(x)=(x+1)^{\frac{3}{5}}+\frac{3}{5}x(x+1)^{-\frac{2}{5}}=(x+1)^{\frac{3}{5}}\cdot\frac{5(x+1)^{\frac{2}{5}}}{5(x+1)^{\frac{2}{5}}}+\frac{3x}{5(x+1)^{\frac{2}{5}}}=\frac{5(x+1)+3x}{5(x+1)^{\frac{2}{5}}}$$
$$=\frac{8x+5}{5(x+1)^{\frac{2}{5}}}$$

critical numbers: $f'(x)$ is not defined at $x=-1$

$$f'(x)=0 \implies 8x+5=0 \implies x=-\frac{5}{8}$$

$$f''(x)=\frac{3}{5}(x+1)^{-\frac{2}{5}}+\frac{3}{5}(x+1)^{-\frac{2}{5}}-\frac{6}{25}x(x+1)^{-\frac{7}{5}}=\frac{6}{5(x+1)^{\frac{2}{5}}}-\frac{6x}{25(x+1)^{\frac{7}{5}}}$$
$$=\frac{6}{5(x+1)^{\frac{2}{5}}}\cdot\frac{5(x+1)^{\frac{5}{5}}}{5(x+1)^{\frac{5}{5}}}-\frac{6x}{25(x+1)^{\frac{7}{5}}}=\frac{30(x+1)-6x}{25(x+1)^{\frac{7}{5}}}=\frac{24x+30}{25(x+1)^{\frac{7}{5}}}=\frac{6(4x+5)}{25(x+1)^{\frac{7}{5}}}$$

potential concavity change: $f''(x)$ is not defined at $x=-1$

$$f''(x)=0 \implies 4x+5=0 \implies x=-\frac{5}{4}$$

When analyzing the sign of $f'(x)$ we note that $(x+1)^{\frac{2}{5}}=\left[(x+1)^{\frac{1}{5}}\right]^2=(\sqrt[5]{x+1})^2\geq 0$ for all x and when analyzing the sign of $f''(x)$ that $(x+1)^{\frac{7}{5}}=(\sqrt[5]{x+1})^7$ will have the same sign as $x+1$ since the fifth root and the seventh power being odd will preserve the sign.

Interval	$f'(x) = \frac{8[x-(-\frac{5}{8})]}{5(x+1)^{\frac{2}{5}}}$	$f''(x) = \frac{24[x-(-\frac{5}{4})]}{25[x-(-1)]^{\frac{7}{5}}}$	f
$x < -\frac{5}{4}$	$\frac{(+)(-)}{(+)(+)} = (-)$	$\frac{(+)(-)}{(+)(-)} = (+)$	decreasing, concave upward
$-\frac{5}{4} < x < -1$	$\frac{(+)(-)}{(+)(+)} = (-)$	$\frac{(+)(+)}{(+)(-)} = (-)$	decreasing, concave downward
$-1 < x < -\frac{5}{8}$	$\frac{(+)(-)}{(+)(+)} = (-)$	$\frac{(+)(+)}{(+)(+)} = (+)$	decreasing, concave upward
$-\frac{5}{8} < x$	$\frac{(+)(+)}{(+)(+)} = (+)$	$\frac{(+)(+)}{(+)(+)} = (+)$	increasing, concave upward

Evaluate f at useful x values:

$$f(-\tfrac{5}{4}) = -\frac{5}{4}\left(-\frac{5}{4}+1\right)^{\frac{3}{5}} = -\frac{5}{4}\left(-\frac{1}{4}\right)^{\frac{3}{5}} = -\frac{5}{4}\cdot\frac{(-1)^{\frac{3}{5}}}{(2^2)^{\frac{3}{5}}} = -\frac{5}{(2)^2}\cdot\frac{-1}{(2)^{\frac{6}{5}}} = \frac{5}{2^{\frac{16}{5}}} \approx 0.54$$

$$f(-1) = (-1)(-1+1)^{\frac{3}{5}} = (-1)(0)^{\frac{3}{5}} = 0$$

$$f(-\tfrac{5}{8}) = -\frac{5}{8}\left(-\frac{5}{8}+1\right)^{\frac{3}{5}} = -\frac{5}{8}\left(\frac{3}{8}\right)^{\frac{3}{5}} = -\frac{5}{8}\cdot\frac{(3)^{\frac{3}{5}}}{(2^3)^{\frac{3}{5}}} = -\frac{5}{(2)^3}\cdot\frac{(3)^{\frac{3}{5}}}{(2)^{\frac{9}{5}}} = -\frac{5(3)^{\frac{3}{5}}}{2^{\frac{24}{5}}} \approx -0.35$$

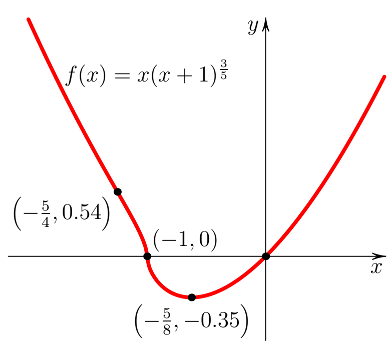
The function f is increasing on $(-\frac{5}{8},\infty)$ and decreasing on $(-\infty,-\frac{5}{8})$. Analysis of the critical numbers $x=-1$ and $x=-\frac{5}{8}$ using the First Derivative Test and the table shows that

- $f(-\frac{5}{8}) = -\frac{5(3)^{\frac{3}{5}}}{2^{\frac{24}{5}}} \approx -0.35$ is a relative minimum.

The function f is concave upward on $(-\infty,-\frac{5}{4})\cup(-1,\infty)$ and concave downward on $(-\frac{5}{4},-1)$. Analysis of the concavity on both sides of those locations where $f''(x)$ vanished or was undefined (-1 and $-\frac{5}{4}$) shows that the following are inflection points:

- $(-1,0)$
- $\left(-\frac{5}{4},\frac{5}{2^{\frac{16}{5}}}\right) \approx (-\frac{5}{4},0.54)$

A sketch of the graph is below.



Further Question:

Sketch the graph of $f(x) = \dfrac{2x^2+2x-3}{x^2+x-2}$.

Exercise 4-7

Answers: Page 252

1-3: Apply calculus techniques to identify all important features of the graph of each function and then sketch it.

1. $f(x) = x^3 - 3x - 2$
2. $y = \dfrac{3x^2}{x^2 - 1}$
3. $f(x) = \dfrac{x^2}{x^2 + 2x + 1}$

4.8 Optimization

A particularly useful application of calculus is optimization. We have already seen how we can find absolute and relative extrema of functions. When these functions represent physical quantities solutions of extrema problems give optimal settings of the independent variable that will result in the lowest or highest possible value of the dependent variable.

So in business one may be interested in minimizing cost or maximizing profit by setting selecting the value of a design specification in its production or determining the sale price of an item. Environmentally one may be interested in minimizing the production of some toxic compound whose formation depends on an alterable temperature at production. Often we are interested in minimizing distance or time.

As with any word problem, the challenge with an optimization problem is to convert this into a calculus problem. Once this is accomplished one follows the steps already discussed for finding the absolute extremum of interest. In summary do the following:

Optimization Problem Steps

1) Identify Variables: Identify the dependent variable to be maximized or minimized and the independent variable to be varied. (If the problem is geometrical in nature try sketching a diagram.) Determine the physically valid domain D of the independent variable.

2) Determine Function: Write the dependent variable as a function of the independent variable. (If another variable appears in the function, try to find a constraint involving the independent variable and it so that the latter may be removed from the function.[5])

3) Differentiate: Take the derivative of the dependent variable with respect to the independent one.

4) Critical Numbers: Find any critical numbers of the independent variable (where the derivative vanishes or does not exist).

5) Evaluate Dependent Variable: If seeking an absolute maximum, check which critical numbers are relative maxima (using the Second or First Derivative Test) and evaluate the dependent variable at those points. Also evaluate the dependent variable at the endpoints of D. The largest of these is the absolute maximum of the dependent variable while its location is the optimal setting of the independent variable. Do similarly if seeking an absolute minimum.[6]

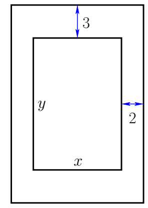
Example 4-26

Solve the following optimization problems.

1. A page in a children's book is to have a total area of 600 cm^2. Its top and bottom margins are to be 3 cm while its side margins are to be 2 cm each. What should the dimensions of the page be to maximize the printed area between the margins?

[5] If no such constraint exists so that the other variable is actually independent, then multivariate calculus would be required.

[6] As a final step, note that plotting your function to confirm the location and value of the extremum is a good idea in practice.



Solution:

(i) Identify Variables:
- independent: x =inner width (y =inner height)
- dependent: maximize printed area A

What are the allowed values of x? Clearly x, as a length, must be positive. The maximum x possible occurs when the height y vanishes:

$$y = 0 \implies (2 + x + 2)(3 + 0 + 3) = 600 \implies x + 4 = 100 \implies x = 96\,\text{cm}$$

So the actual x must lie in the interval $[0, 96]$.[7]

(ii) Determine Function: The printed area $A = xy$. We can eliminate the height y using the constraint that the total area must be $600\,\text{cm}^2$ which implies (see diagram):

$$(4 + x)(6 + y) = 600 \implies 6 + y = \frac{600}{4 + x} \implies y = \frac{600}{4 + x} - 6$$

Inserting this value in $A = xy$ gives A as a function of x alone: $A(x) = \dfrac{600x}{4 + x} - 6x$

(iii) Differentiate: $\dfrac{dA}{dx} = \dfrac{600(4 + x) - 600x(1)}{(4 + x)^2} - 6 \implies \dfrac{dA}{dx} = \dfrac{2400}{(4 + x)^2} - 6$

(iv) Critical Numbers:

$$0 = A' = \frac{2400}{(4 + x)^2} - 6 \implies 0 = 2400 - 6(4 + x)^2 \implies (4 + x)^2 = \frac{2400}{6} = 400$$
$$\implies \sqrt{(4 + x)^2} = \sqrt{400} \implies |4 + x| = 20$$
$$\implies 4 + x = -20 \text{ or } 4 + x = 20 \implies x = -24\,\text{cm or } x = 16\,\text{cm}$$

We will reject $x = -24\,\text{cm}$ as it does not lie in our interval. So $x = 16\,\text{cm}$.

(v) Evaluate Dependent Variable:
We know on the closed interval $[0, 96]$ the function $A(x) = \frac{600x}{4+x} - 6x$ is continuous and hence achieves its absolute maximum at either a critical number or an endpoint. Evaluating at these locations gives:

$$A(0) = \frac{600(0)}{4 + 0} - 6(0) = 0\,\text{cm}^2$$
$$A(16) = \frac{600(16)}{4 + 16} - 6(16) = 480 - 960 = 384\,\text{cm}^2$$
$$A(96) = \frac{600(96)}{4 + 96} - 6(96) = 0\,\text{cm}^2$$

That $A(0) = A(96) = 0\,\text{cm}^2$ makes sense since either the x or y dimension is 0 in these cases. When $x = 16\,\text{cm}$ we find, using our equation for y found above, that

$$y = \frac{600}{4 + 16} - 6 = 30 - 6 = 24\,\text{cm}\,.$$

Hence a maximum printed area of $384\,\text{cm}^2$ occurs when the inner area dimensions are width of $x = 16\,\text{cm}$ by height of $y = 24\,\text{cm}$. The actual page dimensions (what the problem asked for) are therefore:

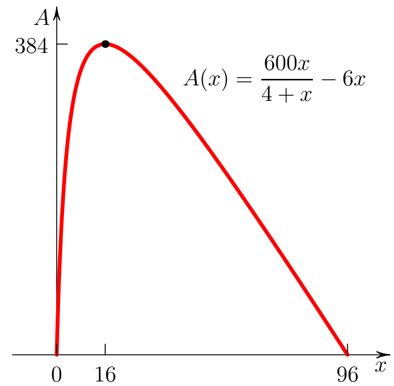
$$\text{page width} = x + 4 = 20\,\text{cm}$$
$$\text{page height} = y + 6 = 30\,\text{cm}$$

Final notes:
One could have shown that $x = 16\,\text{cm}$ was the location of a relative maximum of A using

the Second Derivative Test:

$$A''(x) = -4800(4+x)^{-3} \implies A''(16) = -\frac{4800}{20^3} = -0.6 < 0 \ (\frown)$$

A relative maximum located at $x = 16$ does not guarantee that it is the location of the absolute maximum but it is a useful check nonetheless. We can visualize the optimization problem by graphing $A(x)$ to get:

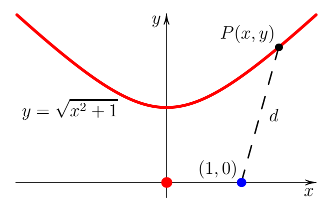


2. Earth is located at the fixed point $(1, 0)$ in a coordinate system superimposed on the plane of the solar system with the sun at the origin. Here measurements are in Astronomical Units (AU).[8] A spaceship follows a hyperbolic trajectory $y = \sqrt{x^2+1}$ through the plane. At what point is the spaceship closest to Earth and how far away is it at that point?

Solution:

(i) Identify Variables:

- independent: Let x be the x-coordinate of the point P on the hyperbola (and y its y-coordinate).
- dependent: Minimize distance d between $P(x, y)$ and $(0, 1)$, the location of the Earth. To simplify our work, it is actually easier to minimize the distance-squared, $D = d^2$. (The location of a minimum of $D = d^2$ will also clearly be the location of a minimum of d.)



Since the x-coordinate can get arbitrarily large in either direction, x can lie anywhere in $(-\infty, \infty)$.

(ii) Determine Function: The distance formula between two points (x_1, y_1) and (x_2, y_2) by the Pythagorean Theorem is just

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \implies D = d^2 = (x_2-x_1)^2 + (y_2-y_1)^2$$

So between the point (x, y) of the spaceship and $(1, 0)$ of Earth we have:

$$D = (x-1)^2 + (y-0)^2 = x^2 - 2x + 1 + y^2$$

To remove the dependency of D on y we use the constraint that the point (x, y) lies on the hyperbola ($y = \sqrt{x^2+1}$) to get

$$D(x) = x^2 - 2x + 1 + \left(\sqrt{x^2+1}\right)^2 = x^2 - 2x + 1 + x^2 + 1$$
$$\implies D(x) = 2x^2 - 2x + 2$$

(iii) Differentiate: $\frac{dD}{dx} = 4x - 2$

(iv) Critical Numbers: $0 = D' = 4x - 2 \implies x = \dfrac{2}{4} = \dfrac{1}{2}$

(v) Evaluate Dependent Variable:

$D\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + 2 = \frac{1-2+4}{2} = \frac{3}{2}\,\mathrm{AU}^2$

To see this is a relative minimum, note that

$D''(x) = 4 \implies D''\left(\frac{1}{2}\right) = 4 > 0$ (☺),

so by the Second Derivative Test $D\left(\frac{1}{2}\right) = \frac{3}{2}\,\mathrm{AU}^2$ is a relative minimum.

The function $D(x)$ is defined on $(-\infty, \infty)$ which is not a closed interval. However we can consider the behaviour at the infinite "endpoints" by taking the limits:

$$\lim_{x\to\infty} D(x) = \lim_{x\to\infty}(2x^2 - 2x + 2) = \lim_{x\to\infty} 2x^2\left(1 - \frac{1}{x} + \frac{1}{x^2}\right) = (+\infty)(1 - 0 + 0) = \infty$$

$$\lim_{x\to-\infty} D(x) = \lim_{x\to-\infty}(2x^2 - 2x + 2) = \lim_{x\to-\infty} 2x^2\left(1 - \frac{1}{x} + \frac{1}{x^2}\right) = (+\infty)(1 - 0 + 0) = \infty$$

This suggests that $D\left(\frac{1}{2}\right) = \frac{3}{2}\,\mathrm{AU}^2$ is an absolute minimum.

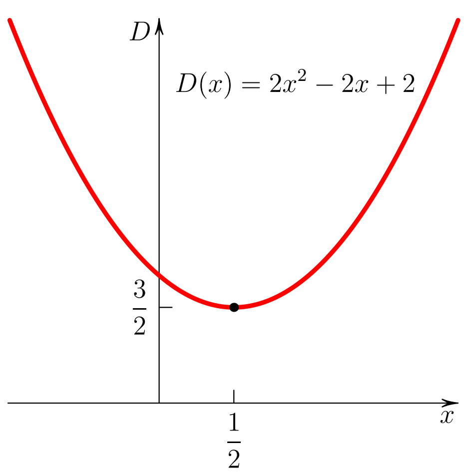
Finally we also need the y-coordinate for the closest point P. Using the constraint equation gives

$$y = \sqrt{\left(\frac{1}{2}\right)^2 + 1} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}\,\mathrm{AU}\,.$$

Thus the point of closet approach is $P\left(\frac{1}{2}, \frac{\sqrt{5}}{2}\right)$ with a closest distance of (since $D = d^2$) $d = \sqrt{\frac{3}{2}}\,\mathrm{AU}$.

Further Notes:

The reader is welcome to do the problem optimizing the distance $d(x) = \sqrt{2x^2 - 2x + 2}$ directly. Also note that a graph of $D(x)$ looks as follows:



Further Questions:

Solve the following optimization problems.

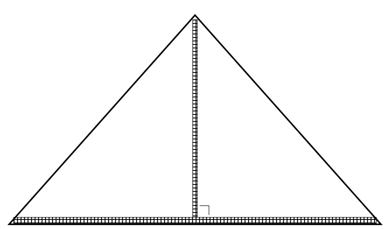
1. A farmer intends to put a rectangular garden along one side of a barn. The remaining three sides of the garden will be fenced. Find the dimensions and area of the largest garden that can be enclosed using 40 metres of fencing material.
2. Find the dimensions of the rectangle with fixed area a having the smallest perimeter possible.
3. A polygonal window is composed of a rectangle topped with an equilateral triangle. Find the dimensions of the window that will maximize the area if the perimeter of the window must be 6 m.
4. In a certain type of cheese production the cheese is pressed into a cylindrical mold with a circular base and no top. If the total area of the mold is to be 1600 cm^2, find the radius and height of the mold that will maximize the volume it contains.
5. A plastic rectangular box for transporting vegetables needs to be designed to have a volume of 72000 cm^3. The box is to have an open top. The cost of material for the base is 0.25 cents/cm^2 while for the sides it is only 0.2 cents/cm^2. If the length of the base is desired to be 1.5 times its width, find the dimensions and cost for the least expensive box meeting these requirements.
6. A semicircle has a radius of 2 cm. What are the dimensions of the rectangle with largest area that can be inscribed within it?

Answers: Page 253

Exercise 4-8

1-10: Solve the following optimization problems following the steps outlined in the text.

1. The entrance to a tent is in the shape of an isosceles triangle as shown below . Zippers run vertically along the middle of the triangle and horizontally along the bottom of it. If the designers of the tent want to have a total zipper length of 5 metres, find the dimensions of the tent that will maximize the area of the entrance. Also find this maximum area.



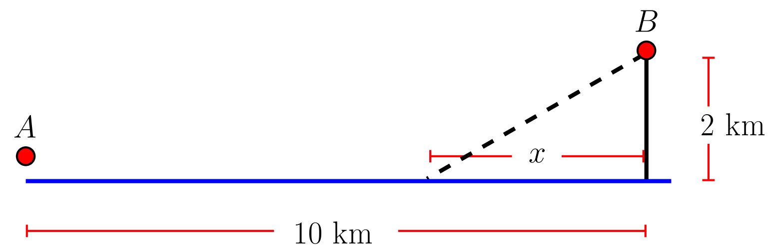
2. Find the point on the line $y = 2x + 2$ closest to the point $(3, 2)$ by
   (a) Using optimization.
   (b) Finding the intersection of the original line and a line perpendicular to it that goes through $(3, 2)$.

[8] One Astronomical Unit equals the average distance from the earth to the sun, approximately 150 million kilometres.

3. The product of two positive numbers is 50. Find the two numbers so that the sum of the first number and two times the second number is as small as possible.

4. A metal cylindrical can is to be constructed to hold 10 cm^3 of liquid. What is the height and the radius of the can that minimize the amount of material needed?

5. A pair of campers wish to travel from their campsite along the river (location A) to visit friends 10 km downstream staying in a cabin that is 2 km from the river (location B) as shown in the following diagram:



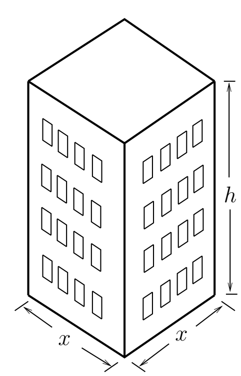
(a) If the pair can travel at 8 km/h in the river downstream by canoe and 1 km/h carrying their canoe by land, at what distance x (see diagram) should they depart from the river to minimize the total time t it takes for their trip?

(b) On the way back from the cabin they can only travel at 4 km/h in their canoe because they are travelling upstream. What distance x will minimize their travel time in this direction?

(c) Using the symbolic constants a for the downstream distance, b for the perpendicular land distance, w for the water speed and v for the land speed, find a general expression for optimal distance x. Verify your results for parts (a) and (b) of this problem by substituting the appropriate constant values.

(d) Does your general result from part (c) depend at all on the downstream distance a? Discuss.

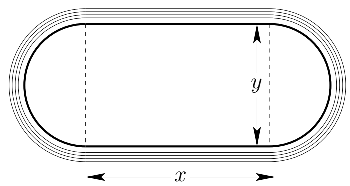
6. A construction company desires to build an apartment building in the shape of a rectangular parallelpiped (shown) with fixed volume of 32000 m^3. The building is to have a square base. In order to minimise heat loss, the total above ground surface area (the area of the four sides and the roof) is to be minimised. Find the optimal dimensions (base length and height) of the building.



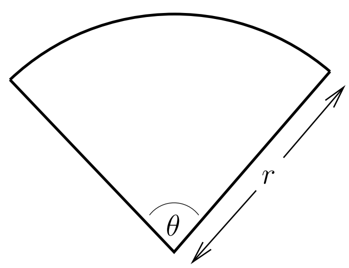
7. A rectangular field is to be enclosed and then divided into three equal parts using 32 metres of fencing. What are the dimensions of the field that maximize the total area?

8. Two power transmission lines travel in a parallel direction (north-south) 10 km apart. Each produces electromagnetic interference (EMI) with the one to the west producing twice the EMI of that of the one to the east due to the greater current the former carries. An amateur radio astronomer wishes to set up his telescope between the two power lines in such a way that the total electromagnetic interference at the location of the telescope is minimized. If the intensity of the interference from each line falls off as 1/distance, how far should the telescope be positioned from the stronger transmission line?

9. An athletic field consists of a rectangular region with a semicircular region at each end. The perimeter of the entire athletic field has to be 400 metres. Find the dimensions that maximize the area of the rectangular region.



10. An antenna is to be created by bending a wire of length 16 cm into the shape of a sector of a circle as shown. In order to maximize the electric flux through the wire, the area of the sector is to be maximized. Find the dimensions r and θ that will maximize the area. Also find the maximum area.



Answers: Page 254

Chapter 4 Review Exercises

1-3: Find the critical numbers of the given functions.

1. $f(x) = \dfrac{x+2}{x^2-3}$

2. $g(t) = \sqrt{t^2-3t}$

3. $F(\theta) = \cos(2\theta) + 2\sin\theta$

4-5: Find the absolute maximum and absolute minimum values and their locations for the given function on the closed interval.

4. $f(x) = \dfrac{x}{x^2+16}$ on the closed interval $[-1,\ 1]$.

5. $g(t) = t\sqrt{8-t^2}$ on the closed interval $[0, 1]$.

6-7: Find the domain, intercepts, asymptotes, relative maxima and minima, intervals of increase and decrease, intervals of concave upward and downward, and inflections points. Then sketch the graph of the given functions.

6. $f(x) = 8x^{\frac{1}{3}} + x^{\frac{4}{3}}$

7. $g(x) = \dfrac{x^2}{x-2}$

8-11: Evaluate the given limits.

8. $\displaystyle\lim_{x\to\infty} \frac{3x^2-4x-5}{2x^5+3}$

9. $\displaystyle\lim_{x\to-\infty} \frac{5x^4-3x+1}{x^4+7}$

10. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{5x^2-4}}{2x+1}$

11. $\displaystyle\lim_{x\to-\infty} \left(2x + \sqrt{x^2+4x+2}\right)$

12-13: Find the horizontal and vertical asymptotes of the given functions.

12. $y = \dfrac{2x^2+7x+3}{x^2+x-6}$

13. $g(t) = \dfrac{\sqrt{9+4t^2}}{2t+3}$

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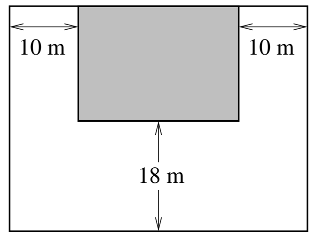
14. A store with a rectangular floorplan is to sit in the middle of one side of a larger rectangular lot with parking on three sides as shown.



If the narrow strips of parking on the side are to be 10 m wide while the parking in the front is to be 18 m wide, find the optimal dimensions of the store that will minimize the total lot area if the store itself must have an area of 1000 m^2. What is the total lot area in this case?

15. A farmer wants to enclose a rectangular garden on one side by a brick wall costing \$20/m and on the other three sides by a metal fence costing \$5/m. Find the dimensions of the garden that minimize the cost if the area of the garden is 250 m^2.

16. A window shaped like a Roman arch consists of a rectangle surmounted by a semicircle. Find the dimensions of the window that will allow the maximum amount of light if the perimeter of the window is 10 m.

Chapter 5: Integration

5.1 Antiderivatives

We know how to take a derivative of a function. If we reverse the procedure and ask which function when differentiated results in a given function we are finding an **antiderivative**.

Definition: If $F'(x) = f(x)$ for all x in an interval I then the function F is called an **antiderivative** of f on I.

Example 5-1

1. $F(x) = \dfrac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$ because $F'(x) = \dfrac{d}{dx}\dfrac{1}{3}x^3 = x^2 = f(x)$.
2. $F(x) = \dfrac{1}{3}x^3 + 10$ is also an antiderivative of $f(x) = x^2$ because $\dfrac{d}{dx}\left(\dfrac{1}{3}x^3 + 10\right) = x^2$.
3. $F(x) = \dfrac{1}{4}x^4 + 2x$ is an antiderivative of $f(x) = x^3 + 2$ since $F'(x) = x^3 + 2$.

The first two questions of the last example illustrate the following theorem:

Theorem 5-1: Let C be an arbitrary constant. If F is an antiderivative of f on an interval I then $F(x) + C$ is also an antiderivative of f on I. Moreover, all antiderivatives of f on I have this form.

That $F(x) + C$ is also an antiderivative given $F(x)$ an antiderivative is due to $\dfrac{d}{dx}C = 0$. Proof that all antiderivatives can be written $F(x) + C$ follows from Theorem 4-7 . As such the general antiderivative of a function f is a family of curves $F(x) + C$.

Consideration of our derivative formulae results in the following table of antiderivatives (up to the additive constant C). Differentiate each answer on the right to confirm the result.

Function	Antiderivative
x^n	$\frac{1}{n+1}x^{n+1}$ where $n \neq -1$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$cf(x)$	$cF(x)$
$f(x) \pm g(x)$	$F(x) \pm G(x)$

For the last two general formulae we are assuming F and G are antiderivatives of f and g respectively and that c is an arbitrary constant. These last two general formulae show that finding an antiderivative is a linear operation. When finding antiderivatives of a sum of terms we can focus on each term separately and pull any constant multipliers out front.

Example 5-2

Find the antiderivative of the given functions.

1. $f(x) = x^3 + 5x^2 + 6$
2. $g(t) = \dfrac{\sqrt{t} + 5t}{2\sqrt{t}}$
3. $f(\theta) = 2\sin\theta + \sec\theta\tan\theta$

Solution:

1. $f(x) = x^3 + 5x^2 + 6 = x^3 + 5x^2 + 6x^0$
   $F(x) = \frac{1}{3+1}x^{(3+1)} + 5\left(\frac{1}{2+1}\right)x^{(2+1)} + 6\left(\frac{1}{0+1}\right)x^{(0+1)} + C = \frac{1}{4}x^4 + \frac{5}{3}x^3 + 6x + C$

2. $g(t) = \frac{\sqrt{t}+5t}{2\sqrt{t}} = \frac{\sqrt{t}}{2\sqrt{t}} + \frac{5t}{2\sqrt{t}} = \frac{1}{2} + \frac{5}{2}t^{\frac{1}{2}}$
   $G(t) = \frac{1}{2}t + \frac{5}{2} \cdot \frac{1}{\frac{3}{2}}t^{\frac{3}{2}} + C = \frac{1}{2}t + \frac{5}{2} \cdot \frac{2}{3}t^{\frac{3}{2}} + C = \frac{1}{2}t + \frac{5}{3}t^{\frac{3}{2}} + C$

3. $f(\theta) = 2\sin\theta + \sec\theta\tan\theta$
   $F(\theta) = -2\cos\theta + \sec\theta + C$

The reader is encouraged to check each antiderivative by differentiating it to see that one arrives at the original function.

Further Questions:

Find the antiderivatives of:

1. $f(x) = x^3 + 2\cos x$
2. $f(x) = \sin x + \tan x \sec x$
3. $f(x) = \left(\sqrt{x} + x\right)^2$
4. $f(x) = \frac{2 - 3\sin x}{\cos^2 x}$
5. $f(x) = \frac{(x-1)^2}{x^4}$

Note that a function f is an antiderivative of its own derivative f'. Similarly, since f'' is the derivative of f', f' is an antiderivative of f''.

Example 5-3

Find all functions f such that $f'(\theta) = (\sec\theta + \tan\theta)^2$.

Solution:

If the derivative of $f(\theta)$ is $f'(\theta)$ then it follows that f is just the antiderivative of $f'(\theta)$. Before antidifferentiating we expand f' and use an identity as shown.

$$\begin{aligned}
f'(\theta) &= (\sec\theta + \tan\theta)^2 = (\sec\theta + \tan\theta)(\sec\theta + \tan\theta) \\
&= \sec^2\theta + 2\sec\theta\tan\theta + \tan^2\theta \qquad \Leftarrow \text{Use identity } \tan^2\theta = \sec^2\theta - 1 \\
&= \sec^2\theta + 2\sec\theta\tan\theta + (\sec^2\theta - 1) \\
&= 2\sec^2\theta + 2\sec\theta\tan\theta - 1 \\
\implies f(\theta) &= 2\tan\theta + 2\sec\theta - \theta + C
\end{aligned}$$

Further Question:

Find all functions g such that $g'(x) = 4\sin x - 3x^5 + 6\sqrt[4]{x}$.

Example 5-4

If $g'(x) = x^3 - 4$ and $g(2) = 3$, find $g(x)$.

Solution:

Since $g'(x) = x^3 - 4$ is the derivative of $g(x)$, it follows that $g(x)$ is the antiderivative of $x^3 - 4$:

$$g(x) = \frac{1}{4}x^4 - 4x + C$$

At this point the answer is a family of functions, one for each value of C. However the extra information $g(2) = 3$ allows us to determine a value of C and hence a specific function:

$$\begin{aligned} 3 = g(2) &= \frac{1}{4}(2)^4 - 4(2) + C = 4 - 8 + C \\ \implies 3 &= -4 + C \implies C = 7 \end{aligned}$$

Therefore $g(x) = \frac{1}{4}x^4 - 4x + 7$.

Further Question:

If $f''(x) = 2 + \frac{4}{x^3}$, $f(1) = 0$, and $f(2) = 3$, find $f(x)$.

Example 5-5

An object falling in gravity at the surface of the Earth undergoes constant vertical acceleration given by $a(t) = -g$ where $g = 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$ is constant. If the initial velocity at time $t = 0$ is v_0 and the initial height at that time is y_0, find $y(t)$, the height of the object as a function of time.

Solution:

Since acceleration $a(t)$ is the derivative of velocity $v(t)$, we have that $v(t)$ is the antiderivative of $a(t)$. Then

$$\begin{aligned} a(t) &= -g = -g(1) = -gt^0 \\ \implies v(t) &= -g\frac{1}{0+1}t^{(0+1)} + C \\ &= -gt + C \end{aligned}$$

We can find the constant C since we know $v(0) = v_0$:

$$\begin{aligned} v_0 &= v(0) = (-g)(0) + C = 0 + C \implies C = v_0 \\ \implies v(t) &= -gt + v_0 \end{aligned}$$

Next since velocity $v(t)$ is the derivative of position $y(t)$ it follows that $y(t)$ is the antiderivative of $v(t)$:

$$\begin{aligned} v(t) &= -gt + v_0 = -gt^1 + v_0t^0 \\ \implies y(t) &= -g\left(\frac{1}{2}\right)t^2 + v_0\frac{1}{1}t^1 + D \\ &= -\frac{g}{2}t^2 + v_0t + D \end{aligned}$$

The second constant D can be determined by using that $y(0) = y_0$:

$$y_0 = y(0) = -\frac{g}{2}(0)^2 + (v_0)(0) + D = 0 + 0 + D \implies D = y_0$$
$$\implies y(t) = -\frac{g}{2}t^2 + v_0 t + y_0$$

Further Question:

An object with mass 2 kg moving in a straight line feels a force (in Newtons) given by $F(t) = 12t + 16$ N . If its initial displacement is $s(0) = 3$ m and its initial velocity is $v(0) = -2$ m/s, find its position function $s(t)$. (Note that according to Newton's 2$^{\text{nd}}$ Law of Motion $F = ma$.)

It will be shown that antidifferentiation is intimately connected with the calculus concept of integration to which we presently turn.

Exercise 5-1

Answers: Page 255

1. Why are $F_1(x) = \frac{1}{4}x^4$ and $F_2(x) = \frac{1}{4}(x^4 + 2)$ both antiderivatives of $f(x) = x^3$?

2-6: Find the antiderivative of the given functions.

2. $f(x) = 3x^2 - 5x + 6$
3. $f(x) = \dfrac{x^3 + 4}{x^2}$
4. $g(t) = \sqrt{t} + \dfrac{2}{\sqrt{t}}$
5. $h(x) = \sqrt[3]{x^2} - 4x^6 + \pi$
6. $f(\theta) = 2\cos\theta - \sin\theta + \sec^2\theta$

7-9: Find the function(s) f satisfying the following.

7. $f''(x) = 2x^3 - 10x + 3$
8. $f''(t) = \sqrt{t} + 6t, \quad f(1) = 1, \quad f'(1) = 2$
9. $f''(\theta) = 3\sin\theta + \cos\theta + 5, \quad f(0) = 3, \quad f'(0) = -1$

---

10. Suppose f is a function with $f'''(x) = 0$ for all x. Show that f has no points of inflection.

5.2 Series

Definition: A **sequence** is an ordered list of numbers (called **terms**).

Example 5-6

- $1, 4, 9, 16, 25, \ldots, 100$
- $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

As the examples show a sequence may have a finite or infinite number of terms.

A similar concept to a sequence is a series.

Definition: A **series** is a **sum** of terms in a sequence.

Example 5-7

- $1 + 4 + 9 + 16 + 25 + \ldots + 100$ (a finite series)
- $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ (an infinite series)

Sigma Notation

When a pattern exists in the terms of a series it can be abbreviated by writing the term as a function of an index.

Example 5-8

In our first series one notices:

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + 10^2 = \sum_{i=1}^{10} i^2$$

Here the symbol $\sum$ is *sigma*, the Greek capital S, representing a sum. The i is the index, related to the position in the series. The limits of 1 and 10 indicate the set of values i ranges over $(1, 2, \ldots, 10)$. Just as the x when writing $f(x)$ is arbitrary ($f(u) = u^2$ is the same function as $f(x) = x^2$), we could have written the series as:

$$\sum_{j=1}^{10} j^2$$

In general we have the following:

Definition: Let a_m, a_{m+1}, $\ldots$, a_n be a sequence of real numbers (so m, n are integers with $m \leq n$) then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n$$

is the **sigma notation** for the series. Here i is called the **index of the summation**. Often we can write the term a_i as a function of the index, $a_i = f(i)$.

Example 5-9

Write the following in sigma notation.

1. $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \ldots + \frac{98}{100}$
2. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \ldots$
3. $\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots + \frac{1}{3^n}$

Solution:

1. $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \ldots + \frac{98}{100} = \frac{1}{1+2} + \frac{2}{2+2} + \frac{3}{3+2} + \frac{4}{4+2} + \ldots + \frac{i}{i+2} + \ldots + \frac{98}{98+2}$
   $= \sum_{i=1}^{98} \frac{i}{i+2}$

   Sigma notation for a series is not unique. This series can also be written $\sum_{i=3}^{100} \frac{i-2}{i}$.

2. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \ldots = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \ldots + \frac{1}{i^3} + \ldots = \sum_{i=1}^{\infty} \frac{1}{i^3}$

3. $\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots + \frac{1}{3^n} = \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \ldots + \frac{1}{3^i} + \ldots + \frac{1}{3^n} = \sum_{i=2}^{n} \frac{1}{3^i}$

Further Questions:

Write the following in sigma notation:

1. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{256}$
2. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$
3. $2^3 + 3^3 + \ldots + n^3$

Often the goal is to evaluate the sum of a series.

Example 5-10

Evaluate the sums of the following series.

1. $\displaystyle\sum_{k=0}^{4} \frac{1}{2^k}$
2. $\displaystyle\sum_{i=0}^{4} \frac{i^2 - i}{i+1}$

Solution:

1. $\displaystyle\sum_{k=0}^{4} \frac{1}{2^k} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{16+8+4+2+1}{16} = \frac{31}{16}$
2. $$\begin{aligned}\sum_{i=0}^{4} \frac{i^2 - i}{i+1} &= \frac{0^2-0}{0+1} + \frac{1^2-1}{1+1} + \frac{2^2-2}{2+1} + \frac{3^2-3}{3+1} + \frac{4^2-4}{4+1} = 0 + 0 + \frac{2}{3} + \frac{6}{4} + \frac{12}{5} \\ &= \frac{20+45+72}{30} = \frac{137}{30}\end{aligned}$$

Further Questions:

Evaluate the sums of the following series:

1. $\displaystyle\sum_{j=0}^{5} 2^j$
2. $\displaystyle\sum_{i=1}^{3} \frac{i-1}{i^2+3}$

Theorem 5-2: The following series sums can be evaluated once and for all for positive integer n:

1. $\displaystyle\sum_{i=1}^{n} 1 = n$
2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$
3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$
4. $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^4+2n^3+n^2}{4}$
5. $\displaystyle\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = \frac{6n^5+15n^4+10n^3-n}{30}$

For general sums we have the following theorem:

Theorem 5-3: If c is any constant (i.e. independent of i) and $\sum_{i=m}^{n} a_i$ and $\sum_{i=m}^{n} b_i$ are arbitrary sums (with the same range of indices) then

1. $\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$
2. $\sum_{i=m}^{n} (a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$

The theorem shows summation is a linear process like differentiation and antidifferentiation.

Example 5-11

The following useful result is a consequence of the previous theorems:

$$\sum_{i=1}^{n} c = nc \ ,$$

since $\sum_{i=1}^{n} c = \sum_{i=1}^{n} c \cdot 1 = c\sum_{i=1}^{n} 1 = cn$.

More complicated series can be evaluated using the previous results.

Example 5-12

Evaluate the sum of the series $\sum_{i=1}^{n} \frac{i^3 + 3i}{n^4}$ using the previous theorems.

Solution:

With respect to the sum the factor of $\frac{1}{n^4}$ is a constant; it does not depend on the summation index i and so it can be pulled out of the sum.

$$\begin{aligned}\sum_{i=1}^{n} \frac{i^3 + 3i}{n^4} &= \frac{1}{n^4}\sum_{i=1}^{n}\left[i^3 + 3i\right] = \frac{1}{n^4}\left[\sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i\right] = \frac{1}{n^4}\left[\left(\frac{n(n+1)}{2}\right)^2 + 3\frac{n(n+1)}{2}\right] \\ &= \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4} + \frac{3}{2}(n)(n+1)\right] = \frac{1}{4}\frac{(n+1)^2}{n^2} + \frac{3}{2}\frac{(n+1)}{n^3}\end{aligned}$$

Further Questions:

Evaluate the following sums using the previous theorems.

1. $\sum_{i=1}^{n} (3 + 2i)^2$
2. $\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^3 + 2\right]$

Sometimes we will want to take the limit of a series as $n \to \infty$. Though technically n is only allowed to take integer values, we evaluate the limit in the same way we do when $x \to \infty$ for a real variable getting arbitrarily large.

Example 5-13

Evaluate the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + \frac{3i}{n} + 5\right]$.

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + \frac{3i}{n} + 5\right] &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i^2}{n^2} + \frac{3i}{n} + 5\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i^2}{n^3} + \frac{3i}{n^2} + \frac{5}{n}\right) \\
&= \lim_{n\to\infty} \left[\frac{1}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n^2}\sum_{i=1}^{n} i + \frac{5}{n}\sum_{i=1}^{n}(1)\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{n^3}\cdot\frac{2n^3+3n^2+n}{6} + \frac{3}{n^2}\cdot\frac{n^2+n}{2} + \frac{5}{n}\cdot n\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{6}\cdot\frac{2+\frac{3}{n}+\frac{1}{n^2}}{1} + \frac{3}{2}\cdot\frac{1+\frac{1}{n}}{1} + 5\right] \\
&= \frac{1}{6}(2+0+0) + \frac{3}{2}(1+0) + 5 = \frac{1}{3} + \frac{3}{2} + 5 = \frac{2+9+30}{6} = \frac{41}{6}
\end{aligned}$$

Further Questions:

Evaluate the following limits:

1. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^3 + 2\right]$
2. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^4$
3. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(1 + \frac{2}{n}i + \frac{3}{n^2}i^2\right)$

Note in the above limits we cannot simply write $\sum_{i=1}^{\infty}$ because the index upper limit n actually appears in the terms themselves!

Answers: Page 255

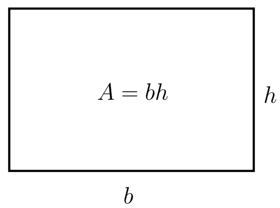
Exercise 5-2

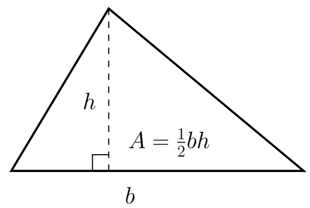
1-4: Evaluate the sums of the following series. (Any value that is not an index being summed over should be treated as a postive integer constant.)

1. $\sum_{i=2}^{5} \frac{i+2}{i-1}$
2. $\sum_{k=1}^{4} 6k$
3. $\sum_{i=1}^{n} \frac{i^2+1}{n^3}$
4. $\sum_{i=1}^{n} i(i-3)$

5.3 Area Under a Curve

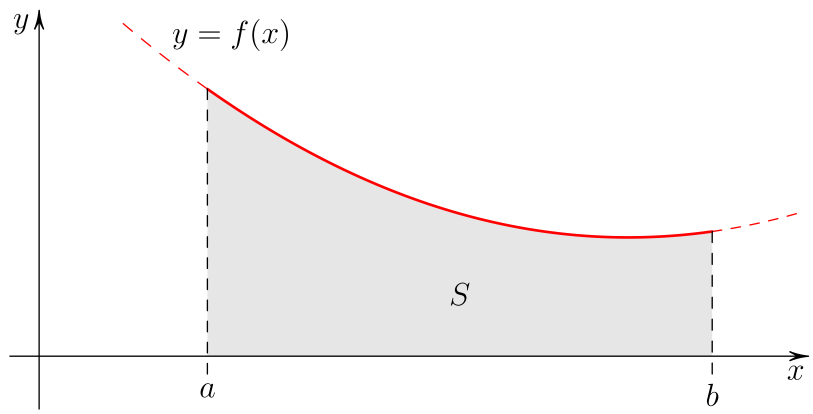
We know how to find the areas of simple shapes like the rectangle and triangle shown below:



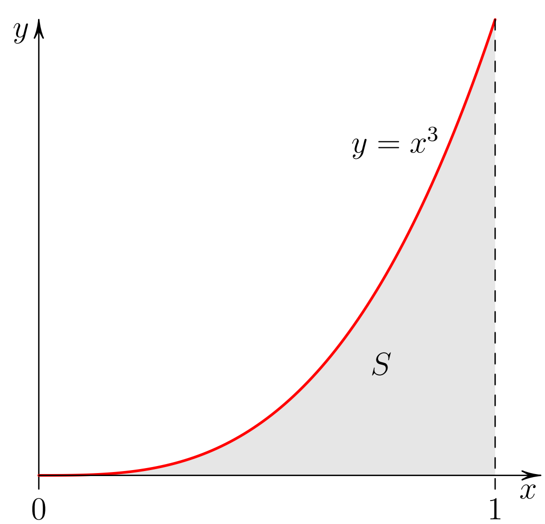


More complicated polygons with straight edges could be subdivided into triangles to find their areas.

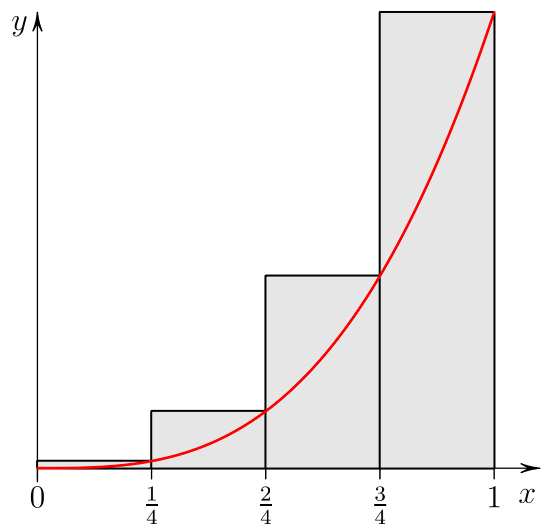
An area bounded by a smooth curve can be resolved using a limiting approach. Consider the general problem of finding the area S of the region that lies under the curve $y = f(x)$ between $x = a$ and $x = b$. By *under the curve* we mean the area between the curve and the x-axis. The situation is illustrated in the following diagram:



We are going to approximate the area S with the sum of the areas of n thin rectangles. Letting the number of rectangles increase without bound ($n \to \infty$) will give us an exact result. For a concrete example consider finding the area under the curve $y = x^3$ from $x = 0$ to $x = 1$ shown below.



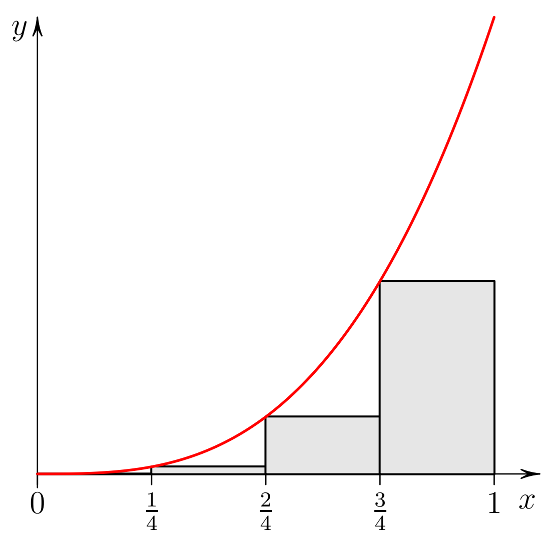
Consider dividing the interval $[0, 1]$ into 4 equal subintervals each of length $\Delta x = 1/4$. The rectangles have bases that are these subintervals and their heights have been chosen to have the values of the function at the **right endpoints** of the subintervals. Diagrammatically they are as follows:



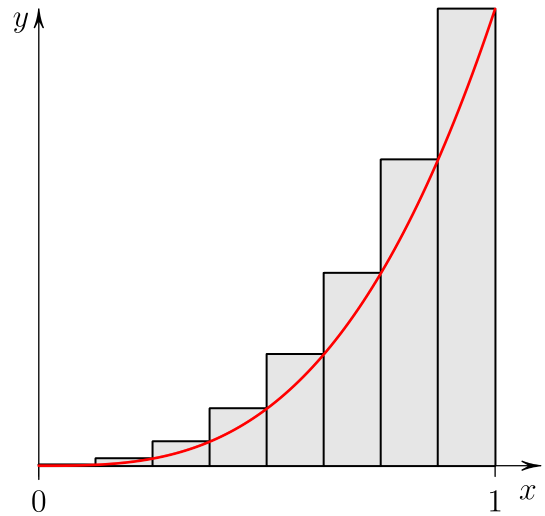
Let S_4 be the sum of the areas of the rectangles. Then an approximation of the desired area is

$$S_4 = A_1 + A_2 + A_3 + A_4 = \frac{1}{4}\left(\frac{1}{4}\right)^3 + \frac{1}{4}\left(\frac{2}{4}\right)^3 + \frac{1}{4}\left(\frac{3}{4}\right)^3 + \frac{1}{4}\left(\frac{4}{4}\right)^3 = \left(\frac{1}{4}\right)\left(\frac{100}{64}\right) = \frac{25}{64} \approx 0.3906\,.$$

Clearly, from the diagram, this will be an overestimate of the area S. If we had, instead used left endpoints for the heights we would have underestimated S:



To get a more precise estimate divide $[0,1]$ into 8 subintervals of length $\Delta x = 1/8$ and let S_8 be the sum of their areas, once again using the function evaluated at the right endpoints for the rectangle height:



The area estimate of S is

$$\begin{aligned} S_8 &= A_1 + A_2 + \ldots + A_8 \\ &= \frac{1}{8}\left[\left(\frac{1}{8}\right)^3 + \left(\frac{2}{8}\right)^3 + \left(\frac{3}{8}\right)^3 + \left(\frac{4}{8}\right)^3 + \left(\frac{5}{8}\right)^3 + \left(\frac{6}{8}\right)^3 + \left(\frac{7}{8}\right)^3 + \left(\frac{8}{8}\right)^3\right] \\ &= \left(\frac{1}{8}\right)\left(\frac{1296}{4096}\right) = \frac{81}{256} \approx 0.3164 \end{aligned}$$

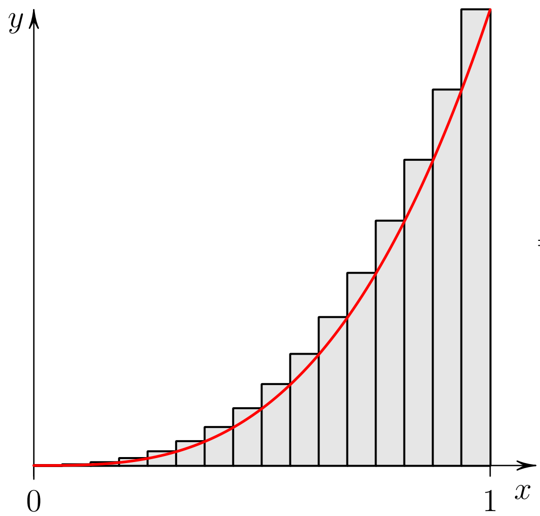
Once again this will be an overestimate of S but we expect, based on the graph, that it has less error.

Let us find S_n for any number of intervals n. Divide the interval $[0,1]$ into n equal subintervals of length $\Delta x = 1/n$. That is $[0, 1/n]$, $[1/n, 2/n]$,...,$[\frac{n-1}{n}, 1]$. Then

$$\begin{aligned} S_n &= A_1 + A_2 + \ldots + A_n \\ &= \frac{1}{n}\left[\left(\frac{1}{n}\right)^3 + \left(\frac{2}{n}\right)^3 + \ldots + \left(\frac{n}{n}\right)^3\right] \\ &= \left(\frac{1}{n}\right)\left(\frac{1}{n^3}\right)\left[1^3 + 2^3 + \ldots + n^3\right] \\ &= \frac{1}{n^4}\sum_{i=1}^{n} i^3 \\ &= \frac{1}{n^4}\frac{n^4 + 2n^3 + n^2}{4} \\ &= \frac{n^2 + 2n + 1}{4n^2} \end{aligned}$$

As a check note that if $n = 4$ we get $S_4 = 0.3906$ and for $n = 8$, $S_8 = 0.3164$.

Now for a larger number of intervals we expect S_n to be a better estimate of S. If we choose $n = 16$ intervals for instance we have:



$$\Rightarrow \quad S_{16} = \frac{289}{1024} = 0.2822$$

So let us let the number of intervals go to infinity ($n \to \infty$):

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n^2 + 2n + 1}{4n^2} = \frac{1}{4} = 0.25$$

As n increases, then S_n becomes a better and better approximation of the area under the curve. Therefore:

$$\text{Area} = S = \lim_{n\to\infty} S_n = \frac{1}{4}$$

In general we want to evaluate the area A under the curve $y = f(x)$, $f(x) \geq 0$ from $x = a$ to b. To do so, divide $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. Label the endpoints $x_0 = a$, x_1, x_2, $\ldots$, x_{n-1}, $x_n = b$. Let x_i^* be any point on the subinterval $[x_{i-1}, x_i]$. The area A of the region is the sum of the n rectangles as $n \to \infty$ ($\Delta x \to 0$):

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x \ .$$

Since for a given n the Δx is constant, it may be pulled outside of the sum and we get:

$$A = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*) \ .$$

The formula can be made more precise by specifying the position of x_i^* within the interval:

right endpoint: If x_i^* is the right endpoint of the interval $[x_{i-1}, x_i]$, then $x_i^* = a + i\frac{b-a}{n}$ and the area is given by

$$A = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$$

midpoint: If x_i^* is the midpoint of the interval $[x_{i-1}, x_i]$, then $x_i^* = a + \left(i - \frac{1}{2}\right)\frac{b-a}{n}$ and the area is given by

$$A = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{n}\right)$$

left endpoint: If x_i^* is the left endpoint of the interval $[x_{i-1}, x_i]$, then $x_i^* = a + (i-1)\dfrac{b-a}{n}$ and the area is given by

$$A = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right)$$

Example 5-14

1. Find the area under the curve $y = f(x) = 3x^2 + 5x - 1$ from $a = 1$ to $b = 3$. Take x_i to be at the right endpoint of the i^{th} interval.

   Solution:

$$\begin{aligned}
A &= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) = \lim_{n\to\infty} \frac{3-1}{n} \sum_{i=1}^{n} f\left(1 + \frac{3-1}{n}i\right) \\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} f\left(1 + \frac{2}{n}i\right) = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[3\left(1 + \frac{2}{n}i\right)^2 + 5\left(1 + \frac{2}{n}i\right) - 1\right] \\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[3 + \frac{12}{n}i + \frac{12}{n^2}i^2 + 5 + \frac{10}{n}i - 1\right] \\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[7 + \frac{22}{n}i + \frac{12}{n^2}i^2\right] = \lim_{n\to\infty} \frac{2}{n} \left[7n + \frac{22}{n} \cdot \frac{n^2+n}{2} + \frac{12}{n^2} \cdot \frac{2n^3+3n^2+n}{6}\right] \\
&= \lim_{n\to\infty} \left[14 + 22\frac{n^2+n}{n^2} + 4\frac{2n^3+3n^2+n}{n^3}\right] = \lim_{n\to\infty} \left[14 + 22\left(1 + \frac{1}{n}\right) + 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] \\
&= 14 + 22 + 8 = 44
\end{aligned}$$

2. Find the area under the curve $y = x^3 + 5$ from $a = 1$ to $b = 3$. Take x_i^* to be at the left endpoint of the i^{th} interval.

   Solution:

$$\begin{aligned}
A &= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right) = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} f\left(1 + \frac{2}{n}(i-1)\right) \\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[\left(1 + \frac{2}{n}(i-1)\right)^3 + 5\right] \quad \Leftarrow (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[1 + \frac{6}{n}(i-1) + \frac{12}{n^2}(i-1)^2 + \frac{8}{n^3}(i-1)^3 + 5\right] \\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[6 + \frac{6}{n}i - \frac{6}{n} + \frac{12}{n^2}(i^2 - 2i + 1) + \frac{8}{n^3}(i^3 - 3i^2 + 3i - 1)\right] \\
&= \lim_{n\to\infty} \frac{2}{n} \left[6n + \frac{6}{n} \cdot \frac{n^2+n}{2} - \frac{6}{n}n + \frac{12}{n^2} \cdot \frac{2n^3+3n^2+n}{6} - \frac{24}{n^2} \cdot \frac{n^2+n}{2} + \frac{12}{n^2}n \right. \\
&\qquad \left. + \frac{8}{n^3} \cdot \frac{n^4+2n^3+n^2}{4} - \frac{24}{n^3} \cdot \frac{2n^3+3n^2+n}{6} + \frac{24}{n^3} \cdot \frac{n^2+n}{2} - \frac{8}{n^3}n\right] \\
&= 2[6 + 3 - 0 + 2(2) - 0 + 0 + 2(1) - 0 + 0 - 0] \\
&= 2[9 + 4 + 2] \\
&= 30
\end{aligned}$$

Further Questions:

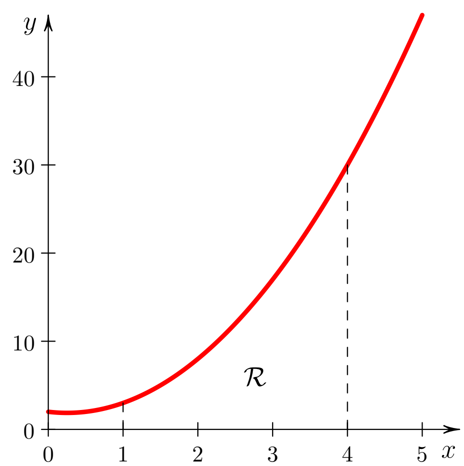
1. Find the area under the curve $y = x^2 + 2$ from $a = 1$ to $b = 4$. Take x_i^* to be the right endpoint of the i^{th} interval.
2. Find the area under the curve $y = x^2 + 3x - 2$ from $a = 1$ to $b = 4$. Take x_i^* to be the left endpoint of the i^{th} interval.

We are assuming that there is no difference in the results of these area formulae, despite the evaluation at different x_i^*. It remains to consider what properties the function must have for this to be true.

Answers: Page 256

Exercise 5-3

1. Let A be the area of the region $\mathcal{R}$ bounded by the x-axis, the lines $x = 1$ and $x = 4$, and the curve $f(x) = 2x^2 - x + 2$ shown below:



   (a) Write a formula for the n^{th} sum S_n approximating A using right endpoints of the approximating rectangles.

   (b) Use $A = \lim_{n\to\infty} S_n$ and your answer from (a) to calculate A.

5.4 The Definite Integral

Our consideration of the evaluation of the area under a curve leads us to define the definite integral.

Definition: Let f be a function defined on the interval $[a, b]$. Divide $[a, b]$ into n equal subintervals of width $\Delta x = \frac{b-a}{n}$ with endpoints $x_0 = a$, x_1, x_2, $\ldots$, x_{n-1}, $x_n = b$. Let x_i^* denote any point in the i^{th} interval $[x_{i-1}, x_i]$. The **definite integral of f from a to b** is:

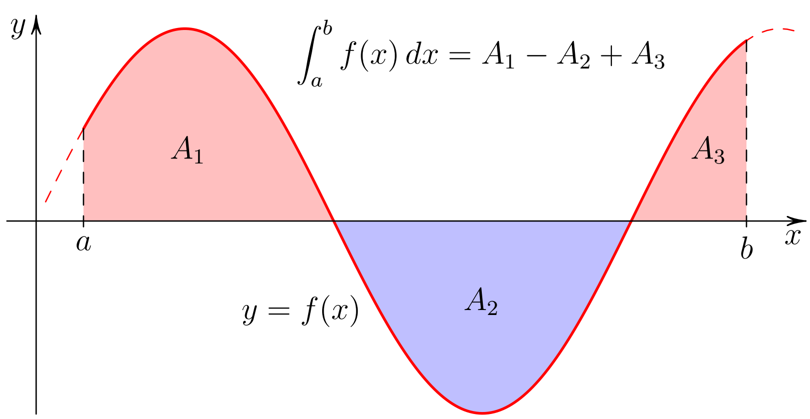
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

if the limit exists and is independent of the choice of the x_i^*. If the limit exists, then f is said to be **integrable** on the interval $[a, b]$.

We introduce the following terminology for the parts of the definite integral:

- $\int$ is the **integral sign**. It is a script S suggesting summation.
- $f(x)$ is the **integrand**.
- x is the **variable of integration**.[1]
- dx is the **differential**.
- b is the **upper limit of integration**.
- a is the **lower limit of integration**.
- The series $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is a **Riemann sum**.

If $f(x) \geq 0$ for all x in $[a, b]$, then $\int_a^b f(x)\,dx$ is the **area under the curve $y = f(x)$** and above the x-axis from $x = a$ to $x = b$ as we have already seen. For arbitrary f, the geometrical meaning of $\int_a^b f(x)\,dx$ is the **net signed area** between the curve and the x-axis on interval $[a, b]$ with areas above the x-axis (where $f(x) > 0$) counted positively and areas below the x-axis (where $f(x) < 0$) counted negatively.



[1]The choice of integration variable, just as the choice of summation index, is arbitrary, i.e.

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(t)\,dt\ .$$

The definite integral, representing a physical area, is therefore just a number.[2]

If the upper limit b is less than the lower limit a we can define, assuming f is integrable on $[b, a]$, that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\,.$$

This definition is consistent with the Riemann sum in the original definition since if $b < a$ there, Δx would be negative. In practice, it is important to remember that if one exchanges limits one must introduce a minus sign.

The following theorem gives a sufficient condition for a function f to be integrable on interval $[a, b]$.

Theorem 5-4: If f is continuous on $[a, b]$ then $\int_a^b f(x)\,dx$ exists.

In fact the weaker condition that f be **piecewise continuous** on $[a, b]$ is sufficient.[3]

The next theorem lists several general properties of the definite integral. While they may be rigorously proven by considering Riemann sums, consideration of their graphical meaning in terms of areas makes them intuitively reasonable.

Theorem 5-5: Let f and g be integrable functions on the given intervals of integration and a, b, c, m and M constants, then the following are true for definite integrals:

1. $\displaystyle\int_a^a f(x)\,dx = 0$
2. $\displaystyle\int_a^b c\,dx = c(b-a)$
3. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
5. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
6. If $f(x) \geq 0$ for all x in $[a, b]$ then $\displaystyle\int_a^b f(x)\,dx \geq 0$.
7. If $f(x) \geq g(x)$ for all x in $[a, b]$ then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
8. If $m \leq f(x) \leq M$ for all x in $[a, b]$ then $\displaystyle m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.
9. $\displaystyle\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

[2]Assuming there are no variables in the limits of integration or non-integration variables in the integrand.

[3]Piecewise continuous means function f is continuous everywhere except at a finite number of removable or jump discontinuities. Vertical asymptotes (infinite discontinuities) are not allowed.

Example 5-15

Evaluate or simplify the following:

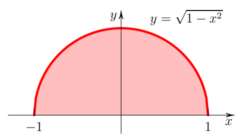
1. $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$

Solution:

The integrand of $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$ is $y = f(x) = \sqrt{1-x^2}$. Squaring both sides yields

$$y^2 = 1 - x^2 \implies x^2 + y^2 = 1$$

and we recognize a circle of radius $r = 1$. Since our $y = \sqrt{1-x^2}$ (and so positive) and our limits are from -1 to 1, the plot is the upper semicircle centred on the origin:



The definite integral is the signed area between the function and the x-axis between the limits -1 and 1. The area is a semicircle of radius $r = 1$. Since our area lies above the x-axis it is counted positively. Therefore

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = +\frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$$

2. $\displaystyle\int_{\pi}^{3\pi} x\sin x\,dx$ if $\displaystyle\int_{\pi}^{4\pi} x\sin x\,dx = -5\pi$ and $\displaystyle\int_{4\pi}^{3\pi} x\sin x\,dx = 7\pi$

Solution:

$$\begin{aligned}
\int_{\pi}^{3\pi} x\sin x\,dx &= \underbrace{\int_{\pi}^{3\pi} x\sin x\,dx + \int_{3\pi}^{4\pi} x\sin x\,dx}_{} - \underbrace{\int_{3\pi}^{4\pi} x\sin x\,dx}_{} \\
&= \int_{\pi}^{4\pi} x\sin x\,dx \quad - \quad \left[-\int_{4\pi}^{3\pi} x\sin x\,dx\right] \\
&= \int_{\pi}^{4\pi} x\sin x\,dx + \int_{4\pi}^{3\pi} x\sin x\,dx \\
&= -5\pi + 7\pi = 2\pi
\end{aligned}$$

Further Questions:

Evaluate or simplify the following:

1. $\displaystyle\int_{-1}^{2} x\,dx$

2. $\displaystyle\int_{9}^{4} \sqrt{t}\,dt$ if $\displaystyle\int_{4}^{9} \sqrt{x}\,dx = \frac{38}{3}$.

3. $\displaystyle\int_{3}^{4} f(x)\,dx + \int_{1}^{3} f(x)\,dx + \int_{4}^{1} f(x)\,dx$

4. $\displaystyle\int_{1}^{3} f(x)\,dx + \int_{3}^{6} f(x)\,dx + \int_{6}^{12} f(x)\,dx$

If a function is integrable we are free to choose the points of evaluation, x_i^*, when evaluating the Riemann sum to be the right endpoints of the intervals.

Theorem 5-6: If f is integrable on $[a, b]$ then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$$

Similar theorems can be given for the left endpoint and midpoint choices of x_i^* as previously discussed.

Example 5-16

Evaluate the following definite integrals using a Riemann sum with the function evaluated at the right endpoints of the interval.

1. $\displaystyle\int_0^2 (x^3 + 2x + 1)dx$
2. $\displaystyle\int_2^4 (2x^2 + 3x + 2)dx$

Solution:

1. $$\begin{aligned}\int_0^2 (x^3+2x+1)dx &= \lim_{n\to\infty} \frac{2-0}{n} \sum_{i=1}^{n} f\left(0 + i\frac{2-0}{n}\right) = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right) \\ &= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[\frac{8i^3}{n^3} + 2\frac{2i}{n} + 1\right] \\ &= \lim_{n\to\infty} \frac{2}{n} \left[\frac{8}{n^3}\frac{n^4+2n^3+n^2}{4} + \frac{4}{n}\frac{n^2+n}{2} + n\right] \\ &= 2[2(1) + 2(1) + 1] = 10\end{aligned}$$

2. $$\begin{aligned}\int_2^4 (2x^2+3x+2)dx &= \lim_{n\to\infty} \frac{4-2}{n} \sum_{n=1}^{\infty} f\left(2 + i\frac{4-2}{n}\right) = \lim_{n\to\infty} \frac{2}{n} \sum_{n=1}^{\infty} f\left(2 + \frac{2i}{n}\right) \\ &= \lim_{n\to\infty} \frac{2}{n} \sum_{n=1}^{\infty} \left[2\left(2+\frac{2i}{n}\right)^2 + 3\left(2 + \frac{2i}{n}\right) + 2\right] \\ &= \lim_{n\to\infty} \frac{2}{n} \sum_{n=1}^{\infty} \left[2\left(4 + \frac{8i}{n} + \frac{4i^2}{n^2}\right) + 3\left(2 + \frac{2i}{n}\right) + 2\right] \\ &= \lim_{n\to\infty} \frac{2}{n} \sum_{n=1}^{\infty} \left[8 + \frac{16}{n}i + \frac{8i^2}{n^2} + 6 + \frac{6i}{n} + 2\right] \\ &= \lim_{n\to\infty} \frac{2}{n} \sum_{n=1}^{\infty} \left[16 + \frac{22}{n}i + \frac{8}{n^2}i^2\right] \\ &= \lim_{n\to\infty} \frac{2}{n} \left[16n + \frac{22}{n}\cdot\frac{n^2+n}{2} + \frac{8}{n^2}\cdot\frac{2n^3+3n^2+n}{6}\right] \\ &= 2\left[16 + 11(1) + \frac{4}{3}(2)\right] = 2\left[16 + 11 + \frac{8}{3}\right] = 2\left[27 + \frac{8}{3}\right] = \frac{178}{3}\end{aligned}$$

Further Questions:

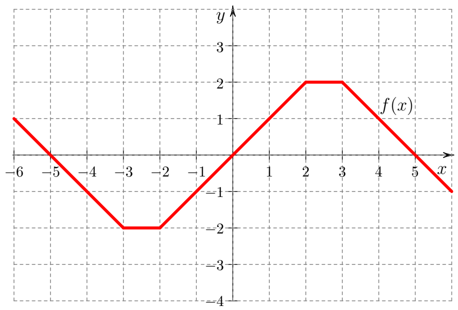
Evaluate the following definite integrals using a Riemann sum with the function evaluated at the right endpoints of the intervals:

1. $\displaystyle\int_0^3 \left(x^3 - 5x\right)\,dx$
2. $\displaystyle\int_1^5 \left(2 + 3x - x^2\right)\,dx$

Answers: Page 256

Exercise 5-4

1. Consider the following graphically defined function $f(x)$:



   (a) Using the interpretation of the definite integral in terms of net signed area between the function and the x-axis, find

      i. $\int_{-5}^{0} f(x)\,dx$

      ii. $\int_{-2}^{5} f(x)\,dx$

   (b) If we define the function $g(t) = \int_{-6}^{t} f(x)\,dx$, on what intervals is

      i. $g(t)$ increasing?

      ii. $g(t)$ decreasing?

2-5: Use the interpretation of the definite integral as the net signed area to find:

2. $\int_{0}^{2} 3x\,dx$

3. $\int_{-1}^{4} 6\,dx$

4. $\int_{0}^{3} (2x-4)\,dx$

5. $\int_{-4}^{3} (-x)\,dx$

---

6. Find $\int_{-r}^{0} \sqrt{r^2 - x^2}\,dx$ by interpreting the integral as an area. Here $r > 0$ is a positive constant.

7. Simplify the following to a single definite integral using the properties of the definite integral.

$$\int_{-1}^{7} f(x)\,dx + \int_{3}^{-1} f(x)\,dx + \int_{7}^{9} f(x)\,dx$$

8-9: Use Riemann sums with right endpoint evaluation to evaluate the following definite integrals.

8. $\int_{0}^{3} \left(x^3 + 1\right) dx$

9. $\int_{0}^{b} x^2\,dx$ where $b > 0$ is constant

5.5 The Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus (FTC)** shows integration and differentiation are inverse processes of each other. We will present two forms of the theorem, one involving derivatives and one antiderivatives. The latter, as we will see, is particularly useful.

Theorem 5-7: The Fundamental Theorem of Calculus (Derivative Form):

If f is continuous on interval $[a, b]$, then the function $g(x)$ defined for all x in $[a, b]$ by

$$g(x) = \int_a^x f(t)\,dt \ ,$$

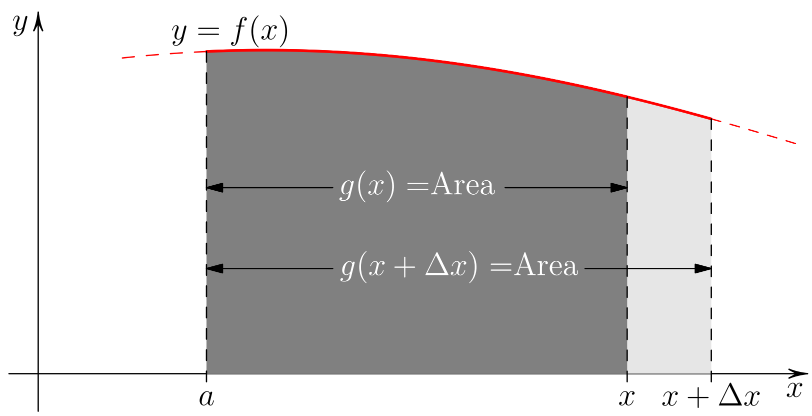
is continuous on $[a, b]$ and differentiable on (a, b) with[4]

$$g'(x) = f(x) \ ,$$

or, equivalently,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \ .$$

That the result of the theorem is plausible can be seen in the following diagram:



The function $g(x) = \int_a^x f(t)\,dt$ associates with each value x the area under the curve $y = f(x)$ from the fixed number a up to x. To find the derivative of the function we must evaluate:

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

In the diagram $g(x)$ is the dark grey area, while $g(x + \Delta x)$ is all of that area plus the light grey shaded area. Therefore the difference $g(x + \Delta x) - g(x)$ is simply the light grey shaded area itself. Now for finite Δx this light grey area is not a rectangle since the left side has height $f(x)$ while the right side has height $f(x + \Delta x)$ and the top side is the curve $y = f(x)$ over $[x, x + \Delta x]$. However, due to the continuity of f, we have $\lim_{\Delta x \to 0} f(x+\Delta x) = f(x)$, in other words the light grey area becomes increasingly like a rectangle of width Δx and height $f(x)$ as $\Delta x \to 0$. Thus

$$g(x + \Delta x) - g(x) \approx f(x)\Delta x \ ,$$

[4]In fact, $g(x)$ is differentiable on $[a, b]$ (i.e. including the endpoints) if we admit right and left-handed derivatives at those points. That is, we use the usual derivative formula with $\lim h \to 0^+$ and $\lim h \to 0^-$ at the endpoints.

and hence

$$\frac{g(x+\Delta x)-g(x)}{\Delta x} \approx f(x)\ .$$

and in the limit we have

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x)-g(x)}{\Delta x} = f(x)$$

exactly. A rigorous proof of the theorem can be made using the Extreme Value Theorem, inequalities of the definite integral, and the Squeeze Theorem.

Example 5-17

Use the derivative form of the Fundamental Theorem of Calculus to find the derivatives of the following functions:

1. $f(x) = \int_0^x (t^2+3t+10)^{\frac{1}{3}}\, dt$
2. $g(x) = \int_0^{x^2} \frac{t^2+1}{t^4+5}\, dt$
3. $H(x) = \int_{\sqrt{x}}^{x^2} \sqrt{t^4+6}\, dt$

Solution:

1. $f(x) = \int_0^x \underbrace{(t^2+3t+10)^{\frac{1}{3}}}_{g(t)}\, dt$

   $f'(x) = g(x) = (x^2+3x+10)^{\frac{1}{3}}$

   So if the lower limit is constant and the upper limit is just the variable then the theorem requires simply its substitution into the integrand.

2. $g(x) = \int_0^{x^2} \frac{t^2+1}{t^4+5}\, dt$

   In this case the upper limit is not simply the variable but a function of the variable. This can be evaluated by recognizing that $g(x)$ is a composition of functions, $g(x) = F(G(x))$, where $F(u) = \int_0^u \frac{t^2+1}{t^4+5}\, dt$ and $u = G(x) = x^2$. Then by the Chain Rule $g'(x) = F'(u(x))G'(x)$. Using the FTC one has $F'(u) = \frac{u^2+1}{u^4+5} \implies F'(u(x)) = \frac{(x^2)^2+1}{(x^2)^4+5} = \frac{x^4+1}{x^8+5}$. Also $G'(x) = (x^2)' = 2x$. Multiplying these gives:

   $$g'(x) = \frac{d}{dx}\int_0^{x^2} \frac{t^2+1}{t^4+5}\, dt = \frac{(x^2)^2+1}{(x^2)^4+5}\cdot(2x) = \frac{x^4+1}{x^8+5}\cdot(2x) = \frac{2x^5+2x}{x^8+5}$$

   This may seem complicated but the pattern is simple. Substitute the upper limit into the integrand as before, but now also multiply by the derivative of the upper limit.

3. $H(x) = \int_{\sqrt{x}}^{x^2} \sqrt{t^4+6}\,dt$

   In this case the lower limit is also a function of the variable. We use our basic properties of integrals to rewrite $H(x)$ into the sum of two integrals to which the FTC may be applied.

$$\begin{aligned} H(x) &= \int_{\sqrt{x}}^{x^2} \sqrt{t^4+6}\,dt = \int_{\sqrt{x}}^{0} \sqrt{t^4+6}\,dt + \int_{0}^{x^2} \sqrt{t^4+6}\,dt \\ &= -\int_{0}^{\sqrt{x}} \sqrt{t^4+6}\,dt + \int_{0}^{x^2} \sqrt{t^4+6}\,dt \\ \implies H'(x) &= -\sqrt{(\sqrt{x})^4+6}\cdot\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + \sqrt{(x^2)^4+6}\cdot(2x) \\ &= -\frac{1}{2\sqrt{x}}\sqrt{x^2+6} + 2x\sqrt{x^8+6} \end{aligned}$$

Further Questions:

Use the derivative form of the Fundamental Theorem of Calculus to find the derivatives of the following functions:

1. $g(x) = \int_{-1}^{x} \sqrt{t^3+1}\,dt$
2. $g(x) = \int_{x}^{4} \left(2+\sqrt{u}\right)^8\,du$
3. $h(x) = \int_{1}^{\sqrt{x}} \frac{s^2}{s^2+1}\,ds$
4. $g(x) = \int_{2}^{x^2+x} \sqrt{1+t^3}\,dt$
5. $g(x) = \int_{0}^{\sin x} \cos\theta\,d\theta$
6. $\mathrm{Si}(2x)$ where $\mathrm{Si}(x)$ is the *sine integral* function $\mathrm{Si}(x) = \int_{0}^{x} \frac{\sin t}{t}\,dt$

Answers: Page 256

Exercise 5-5

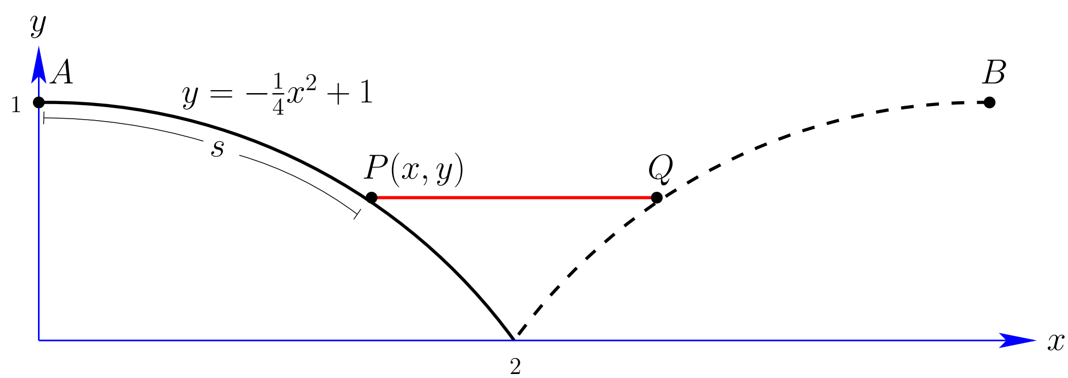
1-4: Compute the derivatives of the following functions using the Fundamental Theorem of Calculus.

1. $F(x) = \int_0^x \sqrt{t^3 + 2t + 1}\, dt$
2. $h(x) = \int_0^{x^4} \sqrt{t^3 + 2t + 1}\, dt$
3. $g(x) = \int_x^1 \left[\cos\left(t^3\right)\right] dt$
4. $H(x) = \int_{2x}^{3x} \sqrt[3]{t^3 + 1}\, dt$

---

5. The *error function*, erf(x), is defined by $\text{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_{-\infty}^{x} e^{-t^2}\, dt$ where e^u is the natural exponential function. If $f(x) = \text{erf}\left(x^3\right)$ find $f'(x)$.

6. The left-side of a symmetrical glacial river valley has an approximate parabolic shape described by the curve $y = -\frac{1}{4}x^2 + 1$ (in km) as shown.



Using calculus techniques the arc length s from point A at the top of the valley to the point $P(x, y)$ can be shown to equal
$$s = \int_0^x \sqrt{1 + \frac{1}{4}t^2}\, dt\ .$$
Engineers wish to build a road connecting points A and B with a bridge spanning the valley at the point P to the corresponding point Q on the opposite side of the valley. If the cost to build the bridge is 25% more per kilometre than the cost of building the road (i.e. it is 5/4 times as much per km), at what point $P(x, y)$ should they start the bridge to minimize the total cost? (Hint: Due to the symmetry of the situation just minimize the cost to build from point A to the middle of the bridge.)

We next turn to the second form of the Fundamental Theorem of Calculus which gives a simple mechanism to evaluate many definite integrals.

Theorem 5-7: The Fundamental Theorem of Calculus (Antiderivative Form):

If f is continuous on $[a,b]$ and F is any antiderivative of f (so $F' = f$) then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}\,.$$

Note we define the shorthand $F(x)|_a^b$ to be the right hand side of the equality, $F(x)|_a^b = F(b) - F(a)$. Unlike the integral sign, the bar is placed on the right.

Proof of the Antiderivative Form of the FTC follows from the fact that the deriative form of the FTC shows that $g(x) = \int_a^x f(t)dt$ is an antiderivative of $f(x)$, since $g'(x) = f(x)$. By Theorem 5-1, any other antiderivative $F(x)$ of $f(x)$ differs by at most a constant from g, $F(x) = g(x) + C$. Hence

$$F(b) - F(a) = [g(b) + C] - [g(a) + C] = g(b) - g(a) = \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{=0} = \int_a^b f(t)\,dt\ .$$

Example 5-18

Use the antiderivative form of the FTC to evaluate the following definite integrals:

1. $\displaystyle\int_0^2 (4x^3 + 3)\,dx$
2. $\displaystyle\int_1^4 \frac{(x+1)^2}{\sqrt{x}}\,dx$
3. $\displaystyle\int_{-1}^2 \frac{1}{(x+1)^2}\,dx$

Solution:

1. $\displaystyle\int_0^2 (4x^3 + 3)\,dx = \left[(4)\left(\frac{1}{4}\right)x^4 + 3x\right|_0^2 = \left[x^4 + 3x\right|_0^2 = [2^4 + 3(2)] - [0 + 0] = 16 + 6 = 22$

2. $$\begin{aligned}\int_1^4 \frac{(x+1)^2}{\sqrt{x}}\,dx &= \int_1^4 \frac{x^2 + 2x + 1}{\sqrt{x}}\,dx = \int_1^4 \left(\frac{x^2}{\sqrt{x}} + \frac{2x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right)dx \\ &= \int_1^4 (x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + x^{-\frac{1}{2}})\,dx = \left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right|_1^4 \\ &= \left[\frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right|_1^4 = \left[\frac{2}{5}(4)^{\frac{5}{2}} + \frac{4}{3}(4)^{\frac{3}{2}} + 2(4)^{\frac{1}{2}}\right] - \left[\frac{2}{5} + \frac{4}{3} + 2\right] \\ &= \left[\frac{2}{5}(2)^5 + \frac{4}{3}(2)^3 + 2(2)\right] - \left[\frac{2}{5} + \frac{4}{3} + 2\right] \\ &= \frac{64}{5} + \frac{32}{3} + 4 - \frac{2}{5} - \frac{4}{3} - 2 = \frac{62}{5} + \frac{28}{3} + 2 = \frac{186 + 140 + 30}{15} = \frac{356}{15}\end{aligned}$$

3. $\displaystyle\int_{-1}^2 \frac{1}{(x+1)^2}\,dx$

   The FTC cannot be applied since $\dfrac{1}{(x+1)^2}$ is not continuous at $x = -1$.

   The integral does not in fact exist.

Further Questions:

Use the antiderivative form of the FTC to evaluate the following definite integrals:

1. $\displaystyle\int_1^2 x^{-2}\,dx$
2. $\displaystyle\int_0^1 \left(y^9 - 2y^5 + 3y\right)\,dy$
3. $\displaystyle\int_0^{\frac{\pi}{2}} (\cos\theta + 2\sin\theta)\,d\theta$
4. $\displaystyle\int_1^4 \frac{\left(\sqrt{x}+2\right)^2}{\sqrt{x}}\,dx$
5. $\displaystyle\int_0^1 \frac{1}{x^3}\,dx$

Exercise 5-6

Answers: Page 257

1-7: Compute the following definite integrals using the Fundamental Theorem of Calculus.

1. $\displaystyle\int_1^3 \left(x^2+3\right)dx$
2. $\displaystyle\int_4^1 \sqrt{x}\,dx$
3. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2\theta\,d\theta$
4. $\displaystyle\int_{-2}^{-1} \frac{4}{x^4}\,dx$
5. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin x + x)\,dx$
6. $\displaystyle\int_{-3}^{2} |x|\,dx$
7. $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$

Indefinite Integrals

Because of the connection between the evaluation of definite integrals with integrand f and the antiderivative of f we define the **indefinite integral** as follows:

Definition: If $F(x)$ is an antiderivative of f, so $F'(x) = f(x)$, then the **indefinite integral** of $f(x)$ is

$$\int f(x)\,dx = F(x) + C$$

The advantage of this notation for the antiderivative (rather than, say $F(x)$) is that $\int f(x)\,dx$ clearly indicates the function f being antidifferentiated, just as $\frac{df}{dx}$ indicates the function differentiated. Note, however, the difference between the definite and indefinite integrals. The definite integral $\int_a^b f(x)\,dx$ is a number while the indefinite integral $\int f(x)\,dx$ is a function.[5]

For indefinite integrals we say, for example, that $\frac{1}{n+1}x^{n+1} + C$ is the **(indefinite) integral** of x^n where x^n is the **integrand**. The **process** of finding the integral is called **integration**.

Using our notation for indefinite integrals and our knowledge of derivatives gives the following.

Table of Indefinite Integrals

1. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \qquad (n \neq -1)$
2. $\displaystyle\int \cos x\,dx = \sin x + C$
3. $\displaystyle\int \sin x\,dx = -\cos x + C$
4. $\displaystyle\int \sec^2 x\,dx = \tan x + C$
5. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$
6. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$
7. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$
8. $\displaystyle\int cf(x)\,dx = c\int f(x)\,dx$
9. $\displaystyle\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

In the last two integration formulae $f(x)$ and $g(x)$ are functions while c is a constant.

[5] Indeed, due to the presence of the constant C, a family of functions.

Example 5-19

Evaluate the following integrals:

1. $\int (x^2+1)^2\,dx$
2. $\int \left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)dt$
3. $\int (3\cos\theta + 5\sec^2\theta)\,d\theta$

Solution:

1. $\int (x^2+1)^2\,dx = \int (x^4+2x^2+1)\,dx = \frac{1}{5}x^5 + \frac{2}{3}x^3 + x + C$
2. $\int \left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)dt = \int (t^{\frac{1}{2}} + t^{-\frac{1}{2}})\,dt = \frac{2}{3}t^{\frac{3}{2}} + 2t^{\frac{1}{2}} + C$
3. $\int (3\cos\theta + 5\sec^2\theta)\,d\theta = 3\sin\theta + 5\tan\theta + C$

Further Questions:

Evaluate the following integrals:

1. $\int \sqrt{x}\left(x^2 - \frac{1}{x}\right)\,dx$
2. $\int \left(\sqrt{x}+3x\right)^2\,dx$
3. $\int \frac{\left(2x^3+3\right)^2}{x^2}\,dx$
4. $\int \left(3\cos t - 2\sin t + \sec t\tan t\right)\,dt$
5. $\int \left(\sqrt[3]{y} + \frac{1}{\sqrt[3]{y}}\right)\,dy$
6. $\int \frac{(x+1)^3}{x^5}\,dx$
7. $\int_1^4 \left(\sqrt{3x}+5x+1\right)\,dx$
8. $\int_0^{\frac{\pi}{4}} \left(2\cos x - 3\sin x + \sec^2 x\right)\,dx$

The last two integrals in the example, definite integrals, illustrate that these just require the further step, after finding the antiderivative, of finding the difference of its evaluation at the endpoints.

Exercise 5-7

Answers: Page 257

1. Explain why we use the indefinite integral symbol, $\int f(x)\,dx$, to represent the general form of the antiderivative of the function $f(x)$.

2-5: Evaluate the following indefinite integrals.

2. $\int \left(x^3 - 3x^4 - 6\right)\,dx$
3. $\int \frac{2+x}{\sqrt{x}}\,dx$
4. $\int \csc\theta\cot\theta\,d\theta$
5. $\int \left(\tan^2 x + 1\right)\,dx$

---

6. Find the general form of the function $y = f(x)$ such that the equation $y' = x^2+9$ is satisfied.

5.6 Integration by Substitution

Consider the indefinite integral
$$\int x^2\sqrt{1+x^3}\,dx\ .$$
Finding an antiderivative is possible if we recall the Chain Rule for differentiating compositions of functions:
$$\frac{d}{dx}f(g(x)) = f'(g(x))\cdot g'(x)\ .$$
In this example consider $\left(1+x^3\right)^{\frac{3}{2}}$ which can be thought of as $f(g(x))$ where $f(u) = u^{\frac{3}{2}}$ and $u = g(x) = 1+x^3$. The Chain Rule gives:
$$\frac{d}{dx}\left(1+x^3\right)^{\frac{3}{2}} = \frac{3}{2}\left(1+x^3\right)^{\frac{1}{2}}\cdot\frac{d}{dx}\left(1+x^3\right) = \frac{3}{2}\sqrt{1+x^3}\cdot\left(0+3x^2\right) = \frac{9}{2}x^2\sqrt{1+x^3}\ .$$
This is almost the integrand. Recalling that $(cf(x))' = cf'(x)$ we can get rid of the $9/2$ by multiplying the original function by $2/9$:
$$\frac{d}{dx}\left[\frac{2}{9}\left(1+x^3\right)^{\frac{3}{2}}\right] = \frac{2}{9}\cdot\frac{9}{2}x^2\sqrt{1+x^3} = x^2\sqrt{1+x^3}\ ,$$
and therefore,
$$\int x^2\sqrt{1+x^3}\,dx = \frac{2}{9}\left(1+x^3\right)^{\frac{3}{2}} + C\ .$$

This example is not very practical because the initial step of considering the derivative of $\left(1+x^3\right)^{\frac{3}{2}}$ came out of nowhere. However it does suggest that the Chain Rule should underlie some useful integration method. It does. In practice we focus on transforming the original integral in x into an integral in terms of a new variable u where $u = g(x)$ is the inner function of a composite function found in the integrand. We have the following:

Theorem 5-8: Substitution Rule (Indefinite Integrals): Suppose $u = g(x)$ is a differentiable function whose range of values is an interval I upon which a further function f is continuous, then
$$\int f(g(x))g'(x)\,dx = \int f(u)\,du\ .$$
where the right hand integral is to be evaluated at $u = g(x)$ after integration.

Note that the du appearing on the right side is the differential:
$$du = g'(x)dx$$
which, recall, can be remembered by thinking $\frac{du}{dx} = g'(x)$ and multiplying both sides by dx.

To integrate using the Substitution Rule for an indefinite integral follow these steps:

1. Selection a substitution u that appears to simplify the integrand. Try to select u so that its derivative is a factor in the integrand.
2. Express the integrand entirely in terms of u and du eliminating the original variable and its differential.
3. Evaluate the new u integral if possible.
4. Express the antiderivative found in the last step (a function of u) in terms of the the original variable.

Consider the original integral again:

$$\int x^2\sqrt{1+x^3}\,dx$$

Let us apply the Substitution Rule steps to solve this integral:

1. Looking at the integrand a useful substitution looks like

$$u = 1 + x^3$$

since this turns $\sqrt{1+x^3}$ into $\sqrt{u}$. Differentiating u we get:

$$\frac{du}{dx} = 0 + 3x^2 = 3x^2\ .$$

Up to the constant 3 this is a factor in the integrand. Our du is:

$$du = 3x^2\,dx$$

2. The original integral becomes, in terms of u and du:

$$\int x^2\sqrt{1+x^3}\,dx = \int \sqrt{1+x^3}(x^2\,dx) = \int \sqrt{u}\,\frac{du}{3}$$

Here we used, from our differential formula that $x^2\,dx = \frac{du}{3}$.

3. Integrating we get:

$$\int \sqrt{u}\,\frac{du}{3} = \frac{1}{3}\int u^{\frac{1}{2}}\,du = \frac{1}{3}\frac{1}{\frac{3}{2}}u^{\frac{3}{2}} + C = \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{9}u^{\frac{3}{2}} + C$$

4. Replacing u with $1+x^3$ gives the final result:

$$\int x^2\sqrt{1+x^3}\,dx = \frac{2}{9}\left(1+x^3\right)^{\frac{3}{2}} + C$$

Note this last step is required as the antiderivative of a function of x must itself be a function of x.

Example 5-20

Evaluate the indefinite integrals:

1. $\int x(3x^2+5)^3\,dx$
2. $\int \sqrt{6+x}\,dx$
3. $\int \left[x(3x^2+5)^3 + \sqrt{6+x} + x^3\right]dx$
4. $\int \sin 2x\,\sqrt[4]{\cos 2x}\,dx$
5. $\int \frac{t^3-2t+4}{t^3}\,dt$
6. $\int \frac{\cos x}{(1+3\sin x)^3}\,dx$
7. $\int x^3\sqrt{2x^2+5}\,dx$
8. $\int \frac{1-x}{(2-6x+3x^2)^4}\,dx$
9. $\int \frac{(\tan\theta+2)^2\sec^2\theta}{\tan^4\theta}\,d\theta$
10. $\int x(2+x^3)^2\,dx$

Solution:

1. $$\int x(3x^2+5)^3\,dx = \int (3x^2+5)^3 x\,dx = \int u^3 \frac{1}{6}\,du = \frac{1}{6}\cdot\frac{u^4}{4} + C = \frac{1}{24}(3x^2+5)^4 + C$$

$$\boxed{\begin{aligned} u &= 3x^2+5 \\ \tfrac{du}{dx} &= 6x \\ du &= 6x\,dx \implies \tfrac{1}{6}du = x\,dx \end{aligned}}$$

2. $$\int \sqrt{6+x}\,dx = \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(6+x)^{\frac{3}{2}} + C$$

$$\boxed{\begin{aligned} u &= 6+x \\ \tfrac{du}{dx} &= 1 \\ du &= dx \end{aligned}}$$

3. $$\begin{aligned} \int \left[x(3x^2+5)^3 + \sqrt{6+x} + x^3\right] dx &= \int x(3x^2+5)^3\,dx \quad + \quad \int \sqrt{6+x}\,dx \quad + \quad \int x^3\,dx \\ &= \begin{pmatrix} \text{sub } u = 3x^2+5 \\ \text{as in Question 1} \end{pmatrix} \begin{pmatrix} \text{sub } w = 6+2 \\ \text{as in Question 2} \end{pmatrix} \begin{pmatrix} \text{no sub} \\ \text{at all} \end{pmatrix} \\ &= \frac{1}{24}(3x^2+5)^4 + \frac{2}{3}(6+x)^{\frac{3}{2}} + \frac{1}{4}x^4 + C \end{aligned}$$

Remember that when integrating we can integrate term by term. One may need a substitution for only one term or potentially different substitutions on different terms. Treat each term as its own integration problem.

4. $$\int \sin 2x \sqrt[4]{\cos 2x}\,dx = -\frac{1}{2}\int \sqrt[4]{u}\,du = -\frac{1}{2}\int u^{\frac{1}{4}}\,du = -\frac{1}{2}\cdot\frac{4}{5}u^{\frac{5}{4}} + C = -\frac{2}{5}(\cos 2x)^{\frac{5}{4}} + C$$

$$\boxed{\begin{aligned} u &= \cos 2x \\ \tfrac{du}{dx} &= (-\sin 2x)(2) = -2\sin 2x \\ du &= -2\sin 2x\,dx \implies -\frac{1}{2}du = \sin 2x\,dx \end{aligned}}$$

5. $$\begin{aligned} \int \frac{t^3-2t+4}{t^3}\,dt &= \int \left[1 - \frac{2}{t^2} + \frac{4}{t^3}\right] dt = \int [1 - 2t^{-2} + 4t^{-3}]\,dt = t - 2\frac{t^{-1}}{-1} + 4\frac{t^{-2}}{-2} + C \\ &= t + \frac{2}{t} - \frac{2}{t^2} + C \end{aligned}$$

(Not every integral is solved using a substitution!)

6. $$\int \frac{\cos x}{(1+3\sin x)^3}\,dx = \int \frac{1}{(1+u)^3}\frac{1}{3}\,du = \frac{1}{3}\int \frac{1}{(1+u)^3}\,du$$

$$\boxed{\begin{aligned} u &= 3\sin x \\ du &= 3\cos x\,dx \implies \frac{1}{3}\,du = \cos x\,dx \end{aligned}}$$

We can try a further substitution at this point.

$$\boxed{\begin{aligned} w &= 1+u \\ dw &= du \end{aligned}} \quad \ldots\frac{1}{3}\int \frac{1}{(1+u)^3}\,du = \frac{1}{3}\int \frac{1}{w^3}\,dw = \frac{1}{3}\int w^{-3}\,dw = \frac{1}{3}\cdot\frac{w^{-2}}{-2} + C$$

$$= -\frac{1}{6w^2} + C = -\frac{1}{6(1+u)^2} + C = -\frac{1}{6(1+3\sin x)^2} + C$$

So with repeated substitutions in indefinite integrals one has to back replace the variable more than once. In this example the reader may show that a more efficient (single) substitution would have been $u = 1 + 3\sin x$ from the beginning.

7. $$\int x^3\sqrt{2x^2+5}\,dx = \int x^2\sqrt{2x^2+5}(x\,dx) = \int \frac{1}{2}(u-5)\sqrt{u}\frac{1}{4}\,du = \frac{1}{8}\int (u^{\frac{3}{2}} - 5u^{\frac{1}{2}})\,du$$

$$\boxed{\begin{aligned} u = 2x^2+5 &\implies x^2 = \frac{1}{2}(u-5) \\ du = 4x\,dx &\implies \frac{1}{4}\,du = x\,dx \end{aligned}}$$

$$= \frac{1}{8}\left[\frac{2}{5}u^{\frac{5}{2}} - 5\frac{2}{3}u^{\frac{3}{2}}\right] + C$$

$$= \frac{1}{20}(2x^2+5)^{\frac{5}{2}} - \frac{5}{12}(2x^2+5)^{\frac{3}{2}} + C$$

This example illustrates that one does not always need to have the differential visibly present when doing a substitution. Since one can, with a substitution $u(x)$, solve for $x(u)$ one can use the latter expression to remove the remaining x's in the integrand in terms of u. Usually one ends up with a more complicated integral to solve, but in some cases, such as this, the integral becomes doable.

8. $$\int \frac{1-x}{(2-6x+3x^2)^4}\,dx = \int \frac{1}{u^4}\cdot\frac{-1}{6}\,du = -\frac{1}{6}\int u^{-4}\,du = -\frac{1}{6}\cdot\frac{u^{-3}}{-3} + C = \frac{1}{18}u^{-3} + C$$

$$= \frac{1}{18(2-6x+3x^2)^3} + C$$

$$\boxed{\begin{aligned} &u = 2-6x+3x^2 \\ &du = (-6+6x)\,dx \implies -\frac{1}{6}\,du = (1-x)\,dx \end{aligned}}$$

9. $$\int \frac{(\tan\theta+2)^2\sec^2\theta}{\tan^4\theta}\,d\theta = \int \frac{(u+2)^2}{u^4}\,du = \int \frac{u^2+4u+4}{u^4}\,du = \int \left(\frac{1}{u^2} + \frac{4}{u^3} + \frac{4}{u^4}\right)\,du$$

$$\boxed{\begin{aligned} u &= \tan\theta \\ du &= \sec^2\theta\,d\theta \end{aligned}}$$

$$= \int (u^{-2} + 4u^{-3} + 4u^{-4})\,du = \frac{u^{-1}}{-1} + 4\frac{u^{-2}}{-2} + 4\frac{u^{-3}}{-3} + C$$

$$= -\frac{1}{u} - \frac{2}{u^2} - \frac{4}{3u^3} + C = -\frac{1}{\tan\theta} - \frac{2}{\tan^2\theta} - \frac{4}{3\tan^3\theta} + C$$

$$= -\cot\theta - 2\cot^2\theta - \frac{4}{3}\cot^3\theta + C$$

10. $$\int x(2+x^3)^2\,dx = \int x(4+4x^3+x^6)\,dx = \int (4x+4x^4+x^7)\,dx = 2x^2 + \frac{4}{5}x^5 + \frac{x^8}{8} + C$$

One may have been tempted to try the substitution $u = 2 + x^3$ so $du = 3x^2\,dx$. However the integrand only has $x\,dx$ outside of the square. When doing a successful substitution all instances of the original variable must be replaced. One can replace the remaining x's in this case by solving $u = 2 + x^3$ for x to get $x = \sqrt[3]{u-2}$, in which case the integral can be rewritten as $\int x(2+x^3)^2\,dx = \int \frac{1}{x}(2+x^3)^2x^2\,dx = \int \frac{u^2}{\sqrt[3]{u-2}}\left(\frac{1}{3}\right)du$. This new integral, however, is no easier to integrate than the original.

Further Questions:

Evaluate the indefinite integrals:

1. $\displaystyle\int \frac{5x^4+3}{(x^5+3x+2)^3}\,dx$
2. $\displaystyle\int \frac{x^2-1}{(x^3-3x+7)^2}\,dx$
3. $\displaystyle\int \tan^2 x\, \sec^2 x\, dx$
4. $\displaystyle\int x\cos\left(x^2+3\right)\,dx$
5. $\displaystyle\int \frac{\cos x}{(1+\sin x)^{\frac{5}{2}}}\,dx$
6. $\displaystyle\int \frac{(x^3+3x)^2}{x^2}\,dx$
7. $\displaystyle\int x^3\sqrt{1+x^2}\,dx$
8. $\displaystyle\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx$
9. $\displaystyle\int x^5\left(2+3x^2\right)^{\frac{3}{2}}\,dx$
10. $\displaystyle\int \sin^2 x\,dx$

Substitution Rule with Definite Integrals

When the substitution method is required for definite integrals the obvious method is to use the substitution to find the antiderivative and then evaluate at the endpoints and subtract as usual.

Example 5-21

Evaluate the definite integral $\displaystyle\int_0^2 x^2\sqrt{1+x^3}\,dx$.

Here we recognize the indefinite integral from before. We solved this using the substitution $u = 1+x^3$ to find the required antiderivative $\frac{2}{9}\left(1+x^3\right)^{\frac{3}{2}}+C$ so the definite integral is just:

$$\int_0^2 x^2\sqrt{1+x^3}\,dx = \ldots = \frac{2}{9}\left(1+x^3\right)^{\frac{3}{2}}\Big|_0^2 = \frac{2}{9}\left[(1+2^3)^{\frac{3}{2}}-(1+0^3)^{\frac{3}{2}}\right] = \frac{2}{9}\left[27-1\right] = \frac{52}{9}$$

Notice here the constant $\frac{2}{9}$ which was a factor of both the upper and lower limit was pulled out to simplify evaluation.

A quicker way to solve the definite integral when doing a substitution is to skip the last step of converting the antiderivative in terms of u back to the original variable. Instead you can convert the limits of the original variable into new limits of the variable u.

Example 5-22

Evaluate the definite integral $\displaystyle\int_0^2 x^2\sqrt{1+x^3}\,dx$ by changing the limits of the integral upon substitution.

As before we have the substitution $u = 1+x^3$ so $du = 3x^2\,dx$. Now x ranges from lower limit 0 to upper limit 2. The new limits of u are:

upper limit: $x = 2 \Rightarrow u = 1+2^3 = 9$

lower limit: $x = 0 \Rightarrow u = 1+0^3 = 1$

The integral becomes:

$$\int_0^2 x^2\sqrt{1+x^3}\,dx = \frac{1}{3}\int_1^9 u^{\frac{1}{2}}\,du = \frac{1}{3}\left[\frac{1}{\frac{3}{2}}u^{\frac{3}{2}}\right|_1^9 = \frac{2}{9}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{2}{9}\left[27-1\right] = \frac{52}{9}$$

Not only is changing the limits to limits in u usually faster it is also cleaner. For instance it would be wrong to write $\int_0^2 x^2\sqrt{1+x^3}\,dx = \frac{1}{3}\int_0^2 u^{\frac{1}{2}}\,du$, even if you ultimately plan on converting u back to x to find the antiderivative. The u is just a dummy variable so $\frac{1}{3}\int_0^2 u^{\frac{1}{2}}\,du$ can be evaluated to equal $2^{\frac{5}{2}}/9$ which is not the answer to the definite integral in question.

Also note that when you change the limits to limits in u it can occur that the upper limit in u may now be lower in value than the lower limit. There is nothing wrong with this. The upper limit in u must correspond to the upper limit of original variable, and similarly for the lower limit.

The whole discussion may be summarized in the following theorem.

Theorem 5-9: Substitution Rule (Definite Integrals): Suppose $u = g(x)$ is a differentiable function whose derivative g' is continuous on $[a, b]$ and a further function f is continuous on the range of $u = g(x)$ (evaluated on $[a, b]$), then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du\ .$$

Example 5-23

Find the definite integrals:

1. $\int_0^1 (y^2+1)^4 y\,dy$
2. $\int_0^{\pi/4} 3\sin(2x-\pi/2)\,dx$
3. $\int_{0.3}^{0.5} \left(x + 4\tan x\sec^2 x\right)dx$

Solution:

1. $$\int_0^1 (y^2+1)^4 y\,dy = \int_1^2 u^4\frac{1}{2}\,du = \left[\frac{1}{2}\cdot\frac{u^5}{5}\right|_{u=1}^{u=2} = \frac{1}{10}\left[u^5\right|_{u=1}^{u=2} = \frac{1}{10}[2^5-1^5] = \frac{1}{10}[32-1] = \frac{31}{10}$$

$$\begin{aligned} u &= y^2+1 \\ du = 2y\,dy &\implies \frac{1}{2}du = y\,dy \\ \text{upper limit: } y = 1 &\implies u = 1^2+1 = 2 \\ \text{lower limit: } y = 0 &\implies u = 0^2+1 = 1 \end{aligned}$$

2. $$\int_0^{\pi/4} 3\sin(2x-\pi/2)\,dx = \int_{-\pi/2}^0 3\sin u\left(\frac{1}{2}\right)du = \frac{3}{2}\int_{-\pi/2}^0 \sin u\,du = \frac{3}{2}\left[-\cos u\right|_{-\pi/2}^0 = -\frac{3}{2}\left[\cos u\right|_{-\pi/2}^0$$
$$= -\frac{3}{2}\left[\cos 0 - \cos(-\pi/2)\right] = -\frac{3}{2}[1-0] = -\frac{3}{2}$$

$$\begin{aligned} u &= 2x - \tfrac{\pi}{2} \\ du = 2\,dx &\implies \tfrac{1}{2}du = dx \\ \text{upper limit: } x = \tfrac{\pi}{4} &\implies u = 2\left(\tfrac{\pi}{4}\right) - \tfrac{\pi}{2} = 0 \\ \text{lower limit: } x = 0 &\implies u = 2(0) - \tfrac{\pi}{2} = -\tfrac{\pi}{2} \end{aligned}$$

Note that as the previous examples illustrate, when doing substitutions with definite integrals, if you change the limits then you never substitute back to the original variable.

3. To integrate $\displaystyle\int_{0.3}^{0.5} \left(x + 4\tan x \sec^2 x\right) dx$ one must integrate term by term, i.e.

$$\int_{0.3}^{0.5} \left(x + 4\tan x \sec^2 x\right) dx = \int_{0.3}^{0.5} x\, dx + \int_{0.3}^{0.5} \tan x \sec^2 x\, dx$$

No substitution is required on the first term and $u = \tan x$ is required on the second. One can do this as above but changing limits only when evaluating the second integral. Alternatively, if one wishes to evaluate an expression only with the initial limits one can just work out the antiderivative (indefinite integral) of the second term on the side and then use this result as follows:

$$\int_{0.3}^{0.5} \left(x + 4\tan x \sec^2 x\right) dx = \left[\frac{1}{2}x^2 + 2\tan^2 x\right|_{0.3}^{0.5} = \left[\frac{(0.5)^2}{2} + 2[\tan(0.5)]^2\right] - \left[\frac{(0.3)^2}{2} + 2[\tan(0.3)]^2\right]$$

$$= 0.72189282 - 0.23637783 = 0.48551499$$

$$\int 4\tan x \sec^2 x\, dx = 4\int u\, du = 2u^2 + C = 2\tan^2 x + C$$

$$u = \tan x \qquad du = \sec^2 x\, dx$$

What one **cannot** do is write $\int_{0.3}^{0.5} \left(x + 4\tan x \sec^2 x\right) dx = \int_{0.3}^{0.5} (x + 4u)\, dx$ which is just wrong. If you do not change your limits then work out the indefinite integrals on the side of the calculation.

Further Questions:

Find the definite integrals:

1. $\displaystyle\int_0^1 (1+x)^5\ dx$
2. $\displaystyle\int_1^4 \frac{(\sqrt{x}+1)^4}{\sqrt{x}}\, dx$
3. $\displaystyle\int_0^{\frac{\pi}{3}} \sin 3\theta\, d\theta$
4. $\displaystyle\int_2^{10} \frac{3}{\sqrt{5x-1}}\, dx$
5. $\displaystyle\int_1^4 \sqrt{5-x}\, dx$

Answers: Page 258

Exercise 5-8

1-6: Evaluate the following indefinite integrals using the Substitution Rule.

1. $\displaystyle\int \frac{x^2+2x}{(x^3+3x^2+4)^5}\,dx$

2. $\displaystyle\int (5x+1)\sqrt{5x^2+2x}\,dx$

3. $\displaystyle\int \left(\frac{\cos\sqrt{t}}{\sqrt{t}}+t^3\right)dt$

4. $\displaystyle\int \cos(\theta)\sqrt{3-\sin\theta}\,d\theta$

5. $\displaystyle\int x\sqrt{4x+1}\,dx$

6. $\displaystyle\int \sec^2\left(2x-\frac{\pi}{3}\right)dx$

7-12: Evaluate the following definite integrals using the Substitution Rule.

7. $\displaystyle\int_0^2 x^3\sqrt{x^4+9}\,dx$

8. $\displaystyle\int_0^{\frac{\pi}{4}} \tan^4\theta\sec^2\theta\,d\theta$

9. $\displaystyle\int_0^1 \left[1+x+(1-x)^5\right]dx$

10. $\displaystyle\int_1^3 \frac{x}{(2x^2+1)^2}\,dx$

11. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin t}{\cos^{\frac{2}{3}} t}\,dt$

12. $\displaystyle\int_0^{\frac{T}{3}} \cos\left(\frac{2\pi t}{T}\right)dt$ ($T>0$ is constant)

5.7 Symmetry and Definite Integrals

Theorem 5-10: The following is true for symmetric functions integrable on $[-a, a]$:

- If $f(x)$ is **even** then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
- If $f(x)$ is **odd** then $\int_{-a}^{a} f(x)\,dx = 0$.

The last theorem is obvious from consideration of the definite integral as a sum of areas for in the even case the areas on each side of the y-axis will be equal. In the odd case the areas are also equal in magnitude but will contribute oppositely in sign to the overall integral (since $f(-x) = -f(x)$). Alternatively one may prove the theorem directly by writing

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \ ,$$

and doing a change of variables of $u = -x$ on the $\int_{-a}^{0}$ integral.

Example 5-24

Integrate the following:

1. $\displaystyle\int_{-2}^{2} (x^4 + 3x^2 + 2)\,dx$
2. $\displaystyle\int_{-2}^{2} (x^3 + 4x)\,dx$
3. $\displaystyle\int_{-5}^{5} \frac{x^5 + 4x^3 + \sin x}{x^2}\,dx$

Solution:

1. If $f(x) = x^4 + 3x^2 + 2$ then $f(-x) = (-x)^4 + 3(-x)^2 + 2 = x^4 + 3x^2 + 2 = f(x)$ implies that $f(x)$ is an even function. Since the limits are $-a = -2$ to $a = 2$ one has

$$\begin{aligned}\int_{-2}^{2} (x^4 + 3x^2 + 2)\,dx &= 2\int_{0}^{2} (x^4 + 3x^2 + 2)\,dx = 2\left[\frac{x^5}{5} + x^3 + 2x\right|_0^2 \\ &= 2\left(\left[\frac{32}{5} + 8 + 4\right] - [0 + 0 + 0]\right) = 2\left[\frac{32}{5} + 12\right] = 2\left(\frac{32 + 60}{5}\right) = \frac{184}{5}\end{aligned}$$

2. If $f(x) = x^3 + 4x$ then $f(-x) = (-x)^3 + 4(-x) = -x^3 - 4x = -(x^3 + 4x) = -f(x)$ implies that $f(x)$ is an odd function. Since the limits are $-a = -2$ to $a = 2$ one has

$$\int_{-2}^{2} (x^3 + 4x)\,dx = 0$$

3. If $f(x) = \dfrac{x^5 + 4x^3 + \sin x}{x^2}$ then

$$\begin{aligned}f(-x) &= \frac{(-x)^5 + 4(-x)^3 + \sin(-x)}{(-x)^2} \quad \Leftarrow \text{Recall sine is an odd function} \\ &= \frac{-x^5 + -4x^3 - \sin(x)}{x^2} = -\frac{x^5 + 4x^3 + \sin x}{x^2} = -f(x)\end{aligned}$$

Thus $f(x)$ is an odd function. Since the limits are $-a = -5$ to $a = 5$ one has

$$\int_{-5}^{5} \frac{x^5 + 4x^3 + \sin x}{x^2}\,dx = 0$$

Note that a critical aspect of applying symmetry in this manner is that the limits be $-a$ and a.

Further Questions:

Integrate the following:

1. $\displaystyle\int_{-2}^{2}(x^2+3)\,dx$

2. $\displaystyle\int_{-\pi/2}^{\pi/2}(\sin x+\sin 2x)\;dx$

Exercise 5-9

Answers: Page 258

1-11: Evaluate the given integrals using any method.

1. $\displaystyle\int\left(x^3+\sqrt{x}-\frac{1}{x^2}+5\right)\,dx$

2. $\displaystyle\int\left(x^3+2\right)^2\,dx$

3. $\displaystyle\int\frac{(2x+\sqrt{x})^2}{\sqrt{x}}\,dx$

4. $\displaystyle\int\left(\cos\theta+\sec^2\theta\right)\,d\theta$

5. $\displaystyle\int 3x^2\left(x^3+4\right)^5\,dx$

6. $\displaystyle\int\cos\theta\,(\sin\theta+3)^{10}\,d\theta$

7. $\displaystyle\int\frac{\left(\sqrt{t}+7\right)^{\frac{4}{3}}}{\sqrt{t}}\,dt$

8. $\displaystyle\int_{-1}^{2}\;(2x+3)^4\,dx$

9. $\displaystyle\int_{0}^{1}\frac{x^2}{\left(x^3+2\right)^3}\,dx$

10. $\displaystyle\int_{0}^{\frac{\pi}{4}}\sin(2\theta)\,d\theta$

11. $\displaystyle\int_{-\frac{3}{2}}^{\frac{3}{2}}\sin(\tan x)\,dx$

5.8 Areas Between Curves

We have already calculated an area between curves. Namely, when $y = f(x) \geq 0$ on $[a, b]$ then

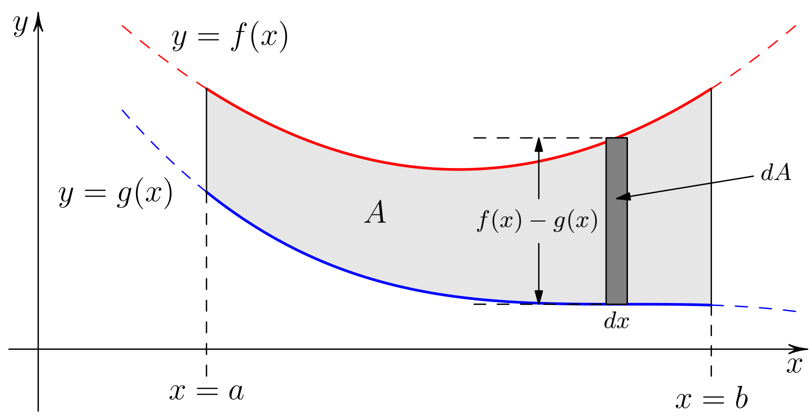
$$A = \int_a^b f(x)\,dx$$

is the area below the curve $y = f(x)$ and above the x-axis. But since the y-axis is just the curve $y = 0$, the above integral is also the area between the curve $y = f(x)$ and the curve $y = 0$. If the lower curve is $y = g(x)$ instead of $y = 0$ we get the following result:

Theorem 5-11: If $y = f(x)$ and $y = g(x)$ are integrable on the x-interval $[a, b]$ with $f(x) \geq g(x)$ on the interval then the **area between the curves** $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$A = \int_a^b \left[f(x) - g(x)\right]\,dx\ .$$

The situation is illustrated in the following diagram:



A simple way to remember the theorem is the differential notation. We are finding the sum $A = \int dA$ of the infinitesimal rectangle areas each of area dA where

$$dA = \underbrace{\left[f(x) - g(x)\right]}_{\text{height}} \cdot \underbrace{dx}_{\text{width}}\ .$$

Since $f(x) \geq g(x)$ the height of the infinitesimal rectangular area, $f(x) - g(x)$, is positive, even if one (or both) of the curves lies below the x-axis. Since $a < b$ for an interval $[a, b]$ the width dx is also positive, thereby ensuring a positive area element dA.

Another useful way to remember the result is that the area between the curves $y = f(x)$ and $y = g(x)$ is just the area under the curve $y = f(x)$ minus the area under the curve $y = g(x)$. This is verified by using the properties of the definite integral to get:

$$A = \int_a^b \left[f(x) - g(x)\right]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

This interpretation gives the proper area even when one or both of the curves lie below the y-axis. (Think about it!)

Example 5-25

Find the area of the region bounded by $y = x^2 + 2$ and $y = 4 - x^2$.

Solution:

First solve to find the x-coordinates of the points of intersection:

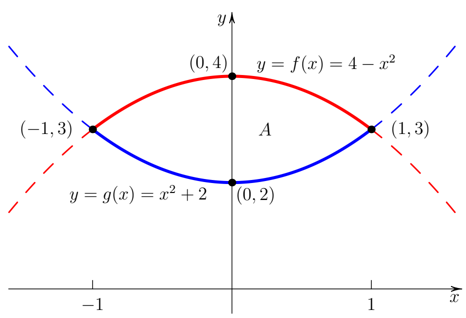
$x^2 + 2 = 4 - x^2 \implies 2x^2 = 2 \implies x^2 = 1 \implies x = \pm 1$

So our integration interval is $[a, b] = [-1, 1]$. The y-coordinates of the endpoints are:

$$y(-1) = (-1)^2 + 2 = 4 - (-1)^2 = 3$$
$$y(1) = (1)^2 + 2 = 4 - (1)^2 = 3$$

To find which function is higher on the interval we can evaluate them at a point it contains, say $x = 0$. Then $0^2 + 2 = 2$ and $4 - 0^2 = 4$ show that $f(x) = 4 - x^2$ is the upper function and $g(x) = x^2 + 2$ is the lower function on the interval.

A sketch of the region is as follows:



The area between the curves is:

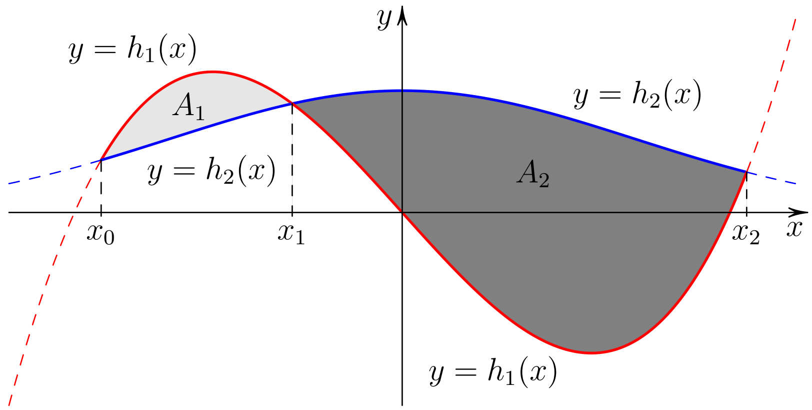
$$A = \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{1} [(4 - x^2) - (x^2 + 2)] = \int_{-1}^{1} (2 - 2x^2)\,dx \quad \text{(Even integrand!)}$$

$$= 2\int_0^1 (2 - 2x^2)\,dx = 2\left[2x - \frac{2}{3}x^3\right|_0^1 = 2\left[\left(2 - \frac{2}{3}\right) - (0 - 0)\right] = 2\frac{6-2}{3} = \frac{8}{3}$$

Further Question:

Find the area of the region bounded by $y = 2x + 4$ and $y = x^2 + 2x + 3$ between $x = 0$ and $x = \frac{1}{2}$.

Sometimes one is asked to find the area between two curves $y = h_1(x)$ and $y = h_2(x)$ with no endpoints a, b given. In that case the curves in question must intersect at least twice to form one (or more) regions bounded by the two curves. The problem is illustrated below.



Here the two curves intersect at three points with x-coordinates x_0, x_1, and x_2. The total area between the curves $A = A_1 + A_2$ where the light grey area A_1 is over $[x_0, x_1]$ and has $h_1(x) \geq h_2(x)$, while the dark grey area A_2 is over $[x_1, x_2]$ and has $h_2(x) \geq h_1(x)$. In general proceed as follows.

1. Find the x-coordinates of the intersection points of the two curves. These will determine where each bounded subregion begins and ends. Since an intersection point must lie on both curves this requires that $h_1(x) = y = h_2(x)$. In other words we must solve:

$$h_1(x) = h_2(x)$$

   for x to get the solutions x_i.

2. For each interval corresponding to a bounded subregion one must determine which curve is higher than the other, i.e. which is $f(x)$ and which $g(x)$ in the area formula. This can be achieved by a sketch of the curve or consideration of the values of $h_1(x)$ and $h_2(x)$ of a test point x in the interval. If two or more regions are bounded by the curves then $h_1(x)$ and $h_2(x)$ will typically alternate being the higher curve.

3. Calculate the area of each subregion using the area formula and add them together. (If any subregion area results in a negative value you have misidentified the higher curve.)

Example 5-26

Further Questions:

Find the area of the region bounded by the following curves:

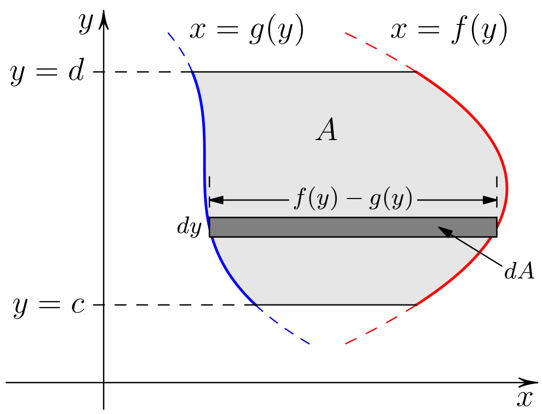
1. $y = 2x + 4$ and $y = x^2 + 2x + 3$
2. $y = x^2$ and $y = 18 - x^2$
3. $y = x^2$ and $y = (x - 2)^2$ from $x = 0$ to $x = 3$

Sometimes the region for which one wants an area is better described by functions $x = f(y)$ and $x = g(y)$ denoting the right and left boundaries of the region respectively. One then has the result:

Theorem 5-12: If $x = f(y)$ and $x = g(y)$ are integrable on the y-interval $[c, d]$ with $f(y) \geq g(y)$ on the interval then the **area between the curves** $x = f(y)$ and $x = g(y)$ and the lines $y = c$ and $y = d$ is

$$A = \int_c^d [f(y) - g(y)]\, dy \; .$$

The situation is depicted below:



In this case the integral represents the addition of horizontal area elements dA of area

$$dA = \underbrace{[f(y) - g(y)]}_{\text{width}} \cdot \underbrace{dy}_{\text{height}} \quad .$$

When only the region bounded by $x = h_1(y)$ and $x = h_2(y)$ is requested with no interval $[c, d]$ given one must solve $h_1(y) = h_2(y)$ to find the y-values y_i which enclose the bounded regions and identify the greater curve ($x = f(y)$) and the lesser curve ($x = g(y)$) for each interval.

Example 5-27

Find the area of the region bounded by $x = 3 - y^2$ and $y + x = 1$.

Solution:

In this case the y^2 prevents us from writing y as a function of x for the first curve. It is already written as $x(y)$ however. Writing the second equation as a function of y as well gives $y + x - 1 \implies x = 1 - y$.

Next solve for the y-coordinates of the points of intersection of the curves:

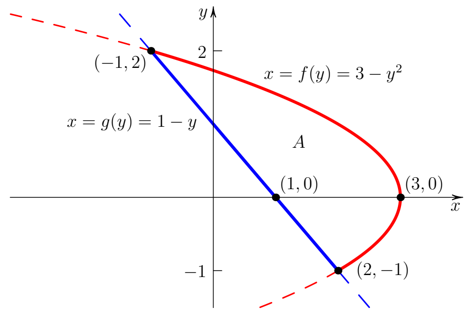
$1 - y = 3 - y^2 \implies y^2 - y - 2 = 0 \implies (y - 2)(y + 1) = 0 \implies y = 2$ or $y = -1$

So our integration interval is $[c, d] = [-1, 2]$. The x-coordinates of the endpoints are:

$$x(2) = 1 - 2 = 3 - 2^2 = -1$$
$$x(-1) = 1 - (-1) = 3 - (-1)^2 = 2$$

To find which curve is further to the right we can evaluate our functions and some point in the interval, say $y = 0$ to get $1 - 0 = 1$ and $3 - 0^2 = 3$. This shows that $f(y) = 3 - y^2$ is further to the right than $g(y) = 1 - y$.

A sketch of the region is as follows:



The area is:

$$A = \int_c^d [f(y) - g(y)]\, dy = \int_{-1}^{2} \left[(3 - y^2) - (1 - y)\right]\, dy = \int_{-1}^{2} (2 + y - y^2)\, dy = \left[2y + \frac{1}{2}y^2 - \frac{1}{3}y^3\right|_{-1}^{2}$$

$$= \left[4 + 2 - \frac{8}{3}\right] - \left[-2 + \frac{1}{2} + \frac{1}{3}\right] = 6 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3} = 8 - 3 - \frac{1}{2} = 5 - \frac{1}{2} = \frac{9}{2}$$

Further Questions:

Find the area of the region bounded by the following curves:

1. $y^2 = 2x + 10$ and $y = x + 1$
2. $2y^2 = x + 4$ and $x = y^2$

If given a choice of doing a vertical or horizontal area analysis of a bounded region one should consider:

- Whether the enclosing curves can be written as **functions** of x or y.
- Whether the area is comprised of any horizontal or vertical lines.
- Whether the area would require piecewise defined functions along either direction.

Example 5-28

Further Questions:

Find the area of the region bounded by the following curves:

1. $x^2 + y = 1$ and $x - y = 1$
2. $y^2 = 4 + x$ and $y^2 + x = 2$
3. $y = x + 6$, $y = x^3$, and $2y + x = 0$

Exercise 5-10

Answers: Page 259

1-7: Find the area of the region bounded by the given curves.

1. $y = x^2 - 3x + 8$ and $y = 4x - x^2$ over the closed interval $[-1, 2]$.
2. $y = x^2 - 5x - 1$ and $y = x - 6$ over the closed interval $[1, 6]$.
3. $y = x^2 + 6, \;\; y = 2x^2 + 2$
4. $x = 2y^2, \;\; x = y^2 + 4$
5. $y = x^2 - 2x, \;\; y = x - 2$
6. $y = 0, \;\; y = 2x, \;\; x + y = 3$
7. $y = \dfrac{8}{x^2}$, $y = x$, $y = 4x^2 + 4x$ and lying in the first quadrant $(x > 0$ and $y > 0)$.

5.9 Net Change

If a variable y is a function f of x, then as x changes from x_1 to x_2, and so undergoes a change $\Delta x = x_2 - x_1$, there is a corresponding change in y of $\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$. Assuming f has continuous first derivative f' on $[x_1, x_2]$ it follows by the Fundamental Theorem of Calculus that the net change in y is

$$\Delta y = \int_{x_1}^{x_2} f'(x)\,dx\ .$$

In other words, if we know the derivative $\frac{dy}{dx} = f'(x)$, we can calculate the change in y from its initial value y_1 at x_1 as x changes to x_2. In our formula, one notes that since the differential dy is $dy = f'(x)\,dx$ the total change in y, Δy, is just the sum of infinitesimal changes in y, i.e. $\Delta y = \int_{y_1}^{y_2} dy$ as one expects.

Often the independent variable x is time t. For instance, an object undergoing motion in one spatial dimension with velocity $v = \frac{ds}{dt}$ will have a total change in displacement $s(t)$ given by

$$\Delta s = \int_{t_1}^{t_2} \frac{ds}{dt}\,dt = \int_{t_1}^{t_2} v\,dt$$

between times t_1 and t_2.

Example 5-29

Using ten years of data a biologist estimates that the rate at which a population of mice changes in a particular region oscillates with the approximate behaviour:

$$\frac{dN}{dt} = a\cos(2\pi t/b + c) \quad \text{[mice/year]}$$

with constants $a = 4500\ \frac{\text{mice}}{\text{yr}}$, $b = 2.6\,\text{yr}$ and $c = 0.4$. Here $t = 0$ yr is the time at which measurement began.

(a) What is the expected change of the mouse population from $t = 0$ to $t = 15$ years?

(b) If the initial population was $N_0 = 14000$ mice, what is the expected population at $t = 15$ years?

Solution:

1. We will retain the literal constants a, b, and c to make the calculation cleaner.

$$\Delta N = \int_0^{15} \frac{dN}{dt}\,dt = a\int_0^{15} \cos(2\pi t/b + c)\,dt = a\int_c^{\frac{30\pi}{b}+c} (\cos u)\cdot\left(\frac{b}{2\pi}\right)du$$

$$\begin{aligned} u &= 2\pi t/b + c = \tfrac{2\pi}{b}t + c \\ du &= \tfrac{2\pi}{b}\,dt \implies \tfrac{b}{2\pi}du = dt \\ \text{upper limit: } t = 15 &\implies u = 2\pi(15)/b + c \\ &\qquad\quad = 30\pi/b + c \\ \text{lower limit: } t = 0 &\implies u = 2\pi(0)/b + c = c \end{aligned}$$

$$\begin{aligned} &= \frac{ab}{2\pi}\left[\sin u\Big|_c^{\frac{30\pi}{b}+c}\right] \\ &= \frac{ab}{2\pi}\left[\sin\left(\frac{30\pi}{b} + c\right) - \sin c\right] \\ &= \frac{(4500)(2.6)}{2\pi}\left[\sin\left(\frac{30\pi}{2.6} + 0.4\right) - \sin(0.4)\right] \\ &= -2340 \text{ mice} \end{aligned}$$

2. The population at $t = 15$ years is just the initial population plus the change:

$$N = N_0 + \Delta N = 14000 + (-2340) = 11660\,\text{mice}$$

Further Questions:

1. A ball thrown vertically upward has velocity given by $v(t) = -10t + 18$ in m/s. What is the total change in displacement of the ball between $t = 1$ and $t = 2$ seconds?

2. Over the first 20 years of its existence a factory had a production rate of units given by

$$\frac{dU}{dt} = 1000 + 150t \text{ [units/year]} .$$

How many units did the factory produce in its first 20 years?

3. In the Further Question 7 of Example 3-31 (page 113), the rate of change of height of the pyramid frustum with time was

$$\frac{dh}{dt} = \frac{1}{B^2\left(1 - \frac{h}{H}\right)^2} R ,$$

where $B = 230$ m was the length of the base of the pyramid, $H = 150$ m was its final height, and $R = 360$ m^3/day was the constant rate of change of volume with time. The rate of change of time with respect to height is the reciprocal:

$$\frac{dt}{dh} = \frac{B^2}{R}\left(1 - \frac{h}{H}\right)^2 .$$

How long did it take to build the top third of the pyramid? (i.e. from 2/3 of its final height to its final height.)[6]

Exercise 5-11

Answers: Page 259

1. A particle oscillates in a straight line with velocity $v(t) = \sin(\pi t)$ centimetres per second. Compute the particle's displacement over the following time intervals.

   (a) $t = 0$ to $t = 1$ seconds.

   (b) $t = 1$ to $t = 2$ seconds.

   (c) $t = 0$ to $t = 2$ seconds.

2. The flow rate at a particular location for a large river over the month of May was approximately $f(t) = -\frac{1}{3}(t - 15)^2 + 100$ in gigalitres per day. Here the time t is measured in days from the beginning of the month. How much water flowed past that location between the times $t = 10$ and $t = 20$ days?

[6]Note you can check your answer directly by working out the volume of the top third of the pyramid and dividing by R.

5.10 Differential Equations

We are accustomed to solving equations like $x^3 - 4x = 0$ to find those values of x that make the equation true (i.e. the solutions, here $x = -2$, $x = 0$, and $x = 2$.). We now consider equations where the unknown is not a value x, but rather a function $y = f(x)$.

Definition: An equation which contains an unknown function and at least one of its derivatives is called at **differential equation**.

Example 5-30

$y'' + \sin(xy) + 2 = \dfrac{1}{x}$ is a differential equation involving unknown function $y = f(x)$.

Definition: The highest order derivative that occurs in a differential equation is called the **order** of the differential equation.

Example 5-31

The order of the following differential equations is as follows:

1. $y' + 3xy = \sin x$ (1^{st} order)
2. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 5y = \cos x$ (2^{nd} order)
3. $xy' + 3y\sin x = x^2 + 5$ (1^{st} order)

Definition: If a function $y = f(x)$ and its derivatives when substituted into a differential equation satisfies that equation then $f(x)$ is called a **solution** of the differential equation.

Example 5-32

Show that the given function is a solution of the given differential equation.

1. $y = \dfrac{c}{x} + x$ in $y' = 2 - \dfrac{y}{x}$
2. $y = \tan x$ in $y'' = 2yy'$

Solution:

1. $y = \dfrac{c}{x} + x = cx^{-1} + x \implies y' = -cx^{-2} + 1 = -\dfrac{c}{x^2} + 1$

   Substitution into the differential equation gives:

$$\begin{aligned} y' &\stackrel{?}{=} 2 - \frac{y}{x} \\ -\frac{c}{x^2} + 1 &\stackrel{?}{=} 2 - \frac{1}{x}\left(\frac{c}{x} + x\right) \\ -\frac{c}{x^2} + 1 &\stackrel{?}{=} 2 - \frac{c}{x^2} - 1 \\ -\frac{c}{x^2} + 1 &= -\frac{c}{x^2} + 1 \end{aligned}$$

2. $y = \tan x \implies y' = \sec^2 x \implies y'' = 2(\sec x)(\sec x \tan x) = 2\sec^2 x \tan x$

   Substitution into the differential equation gives:

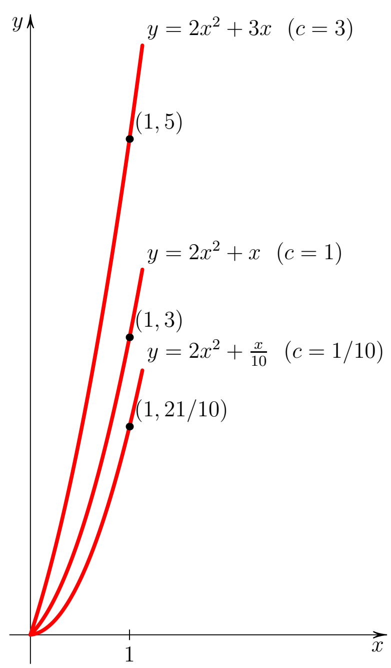
$$\begin{aligned} y'' &\stackrel{?}{=} 2yy' \\ 2\sec^2 x \tan x &\stackrel{?}{=} 2(\tan x)(\sec^2 x) \\ 2\sec^2 x \tan x &= 2\sec^2 x \tan x \end{aligned}$$

Further Questions:

Show that the given function is a solution of the given differential equation.

1. $y = 2x^2 + cx$ in $xy' - y = 2x^2$
2. $y = x \sin x$ in $y'' + y - 2\cos x = 0$

The first solution in Example 5-32, containing a constant, is called a **general solution** to the differential equation.[7] It can be shown that n different constants are required for a general solution to a differential equation of order n. Choosing different values for the constant(s) in a general solution generates a family of solutions to the differential equation. Assigning specific values to the constants in a general solution yields a **particular solution** to the differential equation. The second solution in Example 5-32 is a particular solution to that differential equation. The next graph shows particular solutions generated by using the constants $c = 1/10$, $c = 1$, and $c = 3$ respectively in the general solution of the first problem in Example 5-32.



Physical problems often require finding a function that is not only a solution to a differential equation but also satisfies **initial conditions** at some value of the independent variable. For example one may require the function satisfy the constraint $y(t_0) = y_0$ for given constants t_0 and y_0. Here the

[7] Other solutions called **singular solutions** may also arise for certain differential equations.

terminology *initial* arises from problems for which the independent variable is time. An **initial value problem** is one for which one seeks a solution to a differential equation that satisfies a given set of initial conditions. The initial conditions of the problem determine the constants of the general solution to the differential equation. The resulting particular solution is the physical solution to the problem. Higher order differential equations require more initial conditions, which may involve either the function or its derivatives having specified values at time t_0.

Example 5-33

Find the value of c that makes $y = \frac{c}{x} + x$ as solution to the initial value problem $y' = 2 - \frac{y}{x}$, $y(1) = 3$.

Solution:

In Question 1 of Example 5-32 it was shown that $y = \frac{c}{x} + x$ is a solution of $y' = 2 - \frac{y}{x}$. Set $y(1) = 3$ to find c:

$$3 = y(1) = \frac{c}{(1)} + 1 \implies 3 - 1 = c \implies c = 2$$

Hence $y = \frac{2}{x} + x$ is a solution to the initial value problem.

Further Questions:

1. Find the value of c that makes $y = 2x^2 + cx$ a solution to the initial value problem $xy' - y = 2x^2$, $y(1) = 5$.
2. For the initial value problem $y' + y\tan t = \sec t$, $y(0) = 2$, show that $y = \sin t + C\cos t$ is a general solution to the differential equation and find the value C that makes it satisfy the initial condition.

As the previous example shows, solving an initial value problem given a general solution to the differential equation amounts to solving one (or more) equations for the unknown constant(s). An equation is generated using each of the initial conditions substituted into the general solution, or potentially its derivatives. So the central problem becomes how to find the general solution to a given differential equation. Books and courses are devoted to that problem. However there is a class of such differential equations we can solve at this point, namely those of the form

$$y^{(n)} = g(x) ,$$

where $y^{(n)}$ is the n^{th} derivative of y with respect to x and $g(x)$ is a function of x. When $n = 1$ we have

$$y' = g(x) .$$

In Section 5.1 we have already seen that the solution to such a problem is just the *antiderivative* of $g(x)$. Since the indefinite integral is just the antiderivative, it follows that the solution of

$$y' = g(x)$$

is just

$$y = \int g(x)\,dx + C .$$

Here the constant of integration has been written explicitly. It is this constant that make the indefinite integral the general solution to the differential equation. The constant C will be determined by the given initial condition in an initial value problem.

For the higher order differential equation

$$y^{(n)} = g(x)$$

one can, in principle, integrate n times with respect to x. Each such integration will introduce a new constant of integration so that our general solution $y(x)$ will contain n constants as expected for this order n differential equation. These, in turn, must be determined by n initial conditions to find the physical solution to the initial value problem.

Example 5-34

Solve the following differential equations and initial value problems.

1. $y'' = \dfrac{1}{\sqrt{x}}$
2. $y' \tan x = \sin x \ , \ y(\pi/2) = 4$

Solution:

1. Since the second derivative of y is isolated and y itself does not appear on the right hand side we just need to integrate twice:

$$y'' = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \implies y' = \int x^{-\frac{1}{2}}\, dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = 2x^{\frac{1}{2}} + C$$

$$\implies y = \int \left(2x^{\frac{1}{2}} + C\right) dx = 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + Cx + D$$

$$\implies y = \frac{4}{3}x^{\frac{3}{2}} + Cx + D$$

2. First solve the differential equation by isolating y' and integrating:

$$y' \tan x = \sin x \implies y' = \tfrac{\sin x}{\tan x} = \tfrac{\sin x}{\frac{\sin x}{\cos x}} \implies y' = \cos x \implies y = \textstyle\int \cos x\, dx \implies y = \sin x + C$$

Next find C by using the initial value $y(\pi/2) = 4$:

$4 = y(\pi/2) = \sin(\pi/2) + C = 1 + C \implies C = 4 - 1 = 3$

Thus $y = \sin x + 3$ is a solution to the initial value problem.

Further Questions:

Solve the following differential equations and initial value problems.

1. $\dfrac{dy}{dx} = \sin^2 x \cos x$
2. $y'' = x^2 + \cos x$
3. $\sqrt{1 - x^2} + \dfrac{x}{y'} = 0$, $y(0) = 3$
4. $\dfrac{d^2y}{dt^2} = -9.8 \text{ m/s}^2$, $y(0) = 3$ m, $\left.\dfrac{dy}{dt}\right|_{t=0} = 2$ m/s

Answers: Page 259

Exercise 5-12

1. Show $y = x^3 + 2x + 2$ is a solution of the differential equation $2y'' + 3xy' - 9y + 18 = 0$.
2. Show $y(t) = A\cos(2t) + B\sin(2t)$ is a general solution to the differential equation $y'' + 4y = 0$. If as well $y(0) = 3$ and $y'(0) = 5$, find the constants A and B for the particular solution to that initial value problem.

3-5: Solve the following differential equations and initial value problems.

3. $\dfrac{dy}{dx} = \sqrt{x} + \csc x \cot x$
4. $\dfrac{d^2x}{dt^2} = (2t+3)^5$
5. $y' = 2x + \sin x$, $y(0) = 1$

Answers: Page 260

Chapter 5 Review Exercises

1-4: Find the antiderivative of the given functions.

1. $f(x) = \sqrt[5]{x^3} + \dfrac{4}{\sqrt{x}} + x^3 + 10$

2. $g(x) = \dfrac{\sqrt{x^3} + 5x + 1}{2x^3}$

3. $f(\theta) = 3\sin\theta + 5\cos\theta + \theta^3 + 1$

4. $g(\theta) = 2\tan\theta\sec\theta - 2\cos\theta + \dfrac{1}{\cos^2\theta}$

5-7: Find function f satisfying the given conditions.

5. $f''(x) = \sqrt{x} + x^2 - 6$

6. $f''(t) = 20\sqrt[3]{x^2} - 3x - 5,\ f(1) = 1,\ f'(1) = -1$

7. $f''(\theta) = 5\sin\theta - 4\cos\theta + 10,\ f(0) = -13,\ f'(0) = 2$

8-13: Evaluate the given integrals.

8. $\displaystyle\int x^4\left(x^5 + 3\right)^8\, dx$

9. $\displaystyle\int \sec^2(2\theta)\left[\tan(2\theta) + 1\right]^5\, d\theta$

10. $\displaystyle\int \frac{\left(\sqrt[5]{t^2} - 4\right)^{\frac{1}{3}}}{\sqrt[5]{t^3}}\, dt$

11. $\displaystyle\int_{-1}^{1} x(x-1)^6\, dx$

12. $\displaystyle\int_0^{\pi/4} \cos^4(3x)\sin(3x)\, dx$

13. $\displaystyle\int_0^{\frac{\pi}{3}} \cos^2(3\theta)\, d\theta$

14-16: Find the area of the region bounded by the given curves.

14. $y = x^2 + 1$ and $y = 5$.

15. $y = x$, $y = 4x$, and $x + y = 3$.

16. $x = 10 - y^2$ and $x = 2 + y^2$.

17-18: Solve the following differential equations and initial value problems.

17. $y' = \dfrac{3 + x^5}{x^2}$

18. $y' \sec x = 2,\ y(0) = 3$

Appendix A: Inequalities

The *inequality* $-5 < 2$ is true because -5 is to the left of 2 on the number line, or equivalently the difference $2 - (-5) = 7$ is *positive* (strictly greater than zero).[1] We can exchange both sides of the inequality if we flip the direction of the inequality. So, using a *greater than* sign ($>$) instead of the *less than* sign we have $2 > -5$ which is also true. In either case the pointy side of the inequality points to the smaller number and the open side points to the larger number. Note that smaller and larger mean with respect to the position of the numbers on the number line, not the *magnitude* of the numbers. Thus $-5 < 2$ is true despite the fact that the magnitude of -5 (which is $|-5| = 5$) is greater than the magnitude of 2 (which is $|2| = 2$).

Just as in equations we often introduce a variable in an inequality, such as $x < 5$. The set of all x that satisfy this inequality is the interval $(-\infty, 5)$. Inequalities can either be *strict* ($<$ or $>$) or otherwise ($\leq$ or $\geq$). Here $\leq$ means "less than or equal to". So writing $x \leq 5$ is logically equivalent to

$$x < 5 \;\text{ or }\; x = 5 \;.$$

If desired one can find those values satisfying the inequality $x \leq 5$ by solving the strict inequality and the equality separately and combining (taking the union of) the results. In this simple example the set of x satisfying the inequality is clearly $(-\infty, 5]$.

When the variable is not already isolated on one side of the inequality one will want to *solve* the inequality to find those values that make it true. One often has to manipulate inequalities to isolate the variable x, to find those solutions. When it comes to addition or subtraction, inequalities can be manipulated like equations. So $x - 5 < 0$ can be solved to get $x < 5$ by adding 5 to both sides of the inequality. This similar behaviour is also the same when we multiply or divide both sides of an inequality with a *positive* number. However when solving an inequality one has to be careful when multiplying or dividing by *negative* numbers. In that case one must remember to flip the direction of the inequality. To see that this is necessary consider $x - 5 < 0$ again. We can legitimately subtract x from both sides to get the equivalent inequality $-5 < -x$. Now, however if we multiply (or divide) both sides by -1 without flipping the inequality direction we would get $5 < x$ which is wrong as we saw above that $x < 5$ is the correct answer. Thus when multiplying (or dividing) both sides of $-5 < -x$ by the negative number -1 we must flip the inequality to get $5 > x$ which is equivalent to the correct inequality $x < 5$ we found above.

When reciprocating both sides of inequality care must also be taken. If $2 < 5$ then we see that one must flip the inequality when reciprocating since $\frac{1}{2} > \frac{1}{5}$. In general for $0 < a < b$ (which means "$0 < a$ and $a < b$", so a and b are both positive)[2] one must flip the inequality:

$$0 < a < b \quad \Rightarrow \quad 0 < \frac{1}{b} < \frac{1}{a} \;.$$

This flipping also occurs if a and b are both negative ($a < b < 0$). If one of a or b is positive and the other is negative however then the inequality does not flip, since the sign of the reciprocal numbers will be the same sign as the original numbers and the initial inequality ordering is therefore maintained.

If our original inequality does not have a simple linear form for the variable (like $ax + b < 0$) then solving the inequality is more work. Consider solving the inequality involving a rational function such as

$$\frac{x^2 + 5x + 6}{5 - x} < 0 \;.$$

We can factor this into linear expressions:

$$\frac{(x+3)(x+2)}{5 - x} < 0$$

[1] If one wishes to say a number is positive or zero (i.e. $x \geq 0$) one can use the term *non-negative*.

[2] The notation $a < b < c$ reflects the *transitive* nature of inequalities, that is if $a < b$ and $b < c$ then $a < c$ is also true.

We are looking for those values of x which make the left hand side of the inequality negative. For a rational expression such as this, this means we must have an odd number of negative factors in the expression as a whole (numerator and denominator), which can happen in several ways, namely

$$\frac{(+)(+)}{(-)} \text{ or } \frac{(+)(-)}{(+)} \text{ or } \frac{(-)(+)}{(+)} \text{ or } \frac{(-)(-)}{(-)} .$$

Each one of these expressions can contribute to the solution (since they are logically connected by "or"). Looking at the first possibility, $\frac{(+)(+)}{(-)}$, we require

$$x+3>0 \text{ and } x+2>0 \text{ and } 5-x<0 .$$

Solving each inequality shows this is equivalent to

$$x>-3 \text{ and } x>-2 \text{ and } x>5 .$$

Now because these must all be true for a given solution x (due to the "and") we can clearly simplify this logic. We have the intersection of three intervals, namely $(-3,\infty)\cap(-2,\infty)\cap(5,\infty)$ which just equals (by considering, for instance, the overlap of these intervals on the number line) the interval $(5,\infty)$. In other words, if logically x must be greater than -3 and -2 and 5 then it just must be greater than 5. Sign analysis of the second possibility, $\frac{(+)(-)}{(+)}$, implies

$$x+3>0 \text{ and } x+2<0 \text{ and } 5-x>0 ,$$

or

$$x>-3 \text{ and } x<-2 \text{ and } x<5 .$$

The overlap of these intervals is

$$(-3,\infty)\cap(-\infty,-2)\cap(-\infty,5)=(-3,-2) ,$$

so $-3<x<-2$. The third option, $\frac{(-)(+)}{(+)}$, yields

$$x+3<0 \text{ and } x+2>0 \text{ and } 5-x>0 ,$$

or

$$x<-3 \text{ and } x>-2 \text{ and } x<5 .$$

The overlap of these intervals,

$$(-\infty,-3)\cap(-2,\infty)\cap(-\infty,5) ,$$

contains no points (since the first two intervals have no overlap). We can write this "solution" using the empty set $\varnothing$ which we recall is defined to be $\{\ \}$. The final possibility, $\frac{(-)(-)}{(-)}$ implies

$$x+3<0 \text{ and } x+2<0 \text{ and } 5-x<0 ,$$

or

$$x<-3 \text{ and } x<-2 \text{ and } x>5 .$$

These intervals again have no common solution,

$$(-3,\infty)\cap(-\infty,-2)\cap(5,\infty)=\varnothing .$$

Putting together the four possibilities we get the complete solution to the inequality which is the union of the intervals (since they are logically connected by "or") namely

$$(5,\infty)\cup(-3,-2)\cup\varnothing\cup\varnothing$$

which simplifies to

$$(-3,-2) \cup (5,\infty) \ .$$

Alternatively, using set theory notation, the solution is

$$\{x \in \mathbb{R} \,|\, -3 < x < -2 \ \text{ or } \ 5 < x\} \ .$$

The above sign analysis can be expedited by first tabulating where each linear factor is positive and negative:

Factor	Positive on	Negative on
$x+3$	$(-3,\infty)$	$(-\infty,-3)$
$x+2$	$(-2,\infty)$	$(-\infty,-2)$
$5-x$	$(-\infty,5)$	$(5,\infty)$

and then intersecting the intervals required for each sign combination.

If our original inequality had instead been

$$\frac{x^2+5x+6}{5-x} \leq 0 \ ,$$

then we would have also had to include the solutions to

$$\frac{(x+3)(x+2)}{5-x} = 0 \ ,$$

namely where the numerator vanishes, $x = -3$ or $x = -2$. Our solution would then have to include these interval endpoints, namely

$$[-3,-2] \cup (5,\infty) \ ,$$

or

$$\{x \in \mathbb{R} \,|\, -3 \leq x \leq -2 \ \text{ or } \ 5 < x\} \ .$$

Note that $x = 5$ is not included; the left hand side of the inequality is undefined there.

If our inequality had been

$$\frac{x^2+5x+6}{5-x} > 0 \ ,$$

then we could again do sign analysis, this time requiring an even number of negative factors to produce a positive expression. In our example above this is:

$$\frac{(+)(+)}{(+)} \ \text{ or } \ \frac{(-)(-)}{(+)} \ \text{ or } \ \frac{(-)(+)}{(-)} \ \text{ or } \ \frac{(+)(-)}{(-)} \ .$$

Using our earlier linear factor table we have immediately that the solution is

$$[(-3,\infty)\cap(-2,\infty)\cap(-\infty,5)] \cup [(-\infty,-3)\cap(-\infty,-2)\cap(-\infty,5)] \\ \cup [(-\infty,-3)\cap(-2,\infty)\cap(5,\infty)] \cup [(-3,\infty)\cap(-\infty,-2)\cap(5,\infty)]$$

which simplifies to

$$(-2,5) \cup (-\infty,-3) \cup \varnothing \cup \varnothing = (-\infty,-3) \cup (-2,5) \ ,$$

or

$$\{x \in \mathbb{R} \,|\, x < -3 \ \text{ or } \ -2 < x < 5\} \ .$$

Since being greater than zero is the complement[3] of being less than or equal to zero we can confirm this result, by taking the complement of our previous $\leq$ solution,

$$[-3,-2] \cup (5,\infty) \ ,$$

to get

$$(-\infty,-3) \cup (-2,5)$$

for our greater than solution. Once again, $x=5$ must be excluded when taking the complement as that value is not in the domain of the rational function.

As an alternative approach to the above procedure, one can analyze the inequality

$$\frac{x^2+5x+6}{5-x} < 0$$

by looking at those locations where the rational function $f(x) = \frac{x^2+5x+6}{5-x}$ can potentially change sign. In places where $f(x)$ is continuous this can only occur only if $f(x)=0$. Graphically speaking, the curve $y=f(x)$ crosses the x-axis ($y=0$) to go from negative to positive or vice versa. If there is a discontinuity in the curve at a point x, say where the function $f(x)$ is undefined, then a change in sign of $f(x)$ can potentially occur there as well. So in the above example we have that

$$f(x) = \frac{x^2+5x+6}{5-x} = \frac{(x+3)(x+2)}{5-x}$$

equals zero when the numerator vanishes, namely when factors $x+3=0$ or $x+2=0$, or simply when $x=-3$ or $x=-2$. The function is undefined when the denominator vanishes, so $5-x=0$ (or $x=5$). These three values of x partition the real axis into four open intervals, namely $(-\infty,-3)$,$(-3,-2)$, $(-2,5)$, and $(5,\infty)$. On each of these intervals, by our previous argument, the function must be either positive ($f(x)>0$) or negative ($f(x)<0$). To determine which, we can take a convenient *test value* within each interval and find the sign of f at that point. So, for instance, on $(-\infty,-3)$ we find that $f(-4)=\frac{2}{9}>0$ so the function is positive on that interval. A summary of the analysis of all the intervals is on the following table.

Interval	Test Point	$f(x)$	Sign $f(x)$	Interpretation
$(-\infty,-3)$	-4	$\frac{2}{9}$	$+$	f positive on interval
$(-3,-2)$	$-\frac{5}{2}$	$-\frac{1}{30}$	$-$	f negative on interval
$(-2,5)$	0	$\frac{6}{5}$	$+$	f positive on interval
$(5,\infty)$	6	-72	$-$	f negative on interval

From this we see that $\frac{x^2+5x+6}{5-x}<0$ has solution $(-3,-2)\cup(5,\infty)$ as before while $\frac{x^2+5x+6}{5-x}>0$ has solution $(-\infty,-3)\cup(-2,5)$. Finally if the inequality is not strict (so $\leq$ or $\geq$) we have to include solutions to the equation $f(x)=0$ (here $x=-3$ or $x=-2$) in our solution, which, as before, are endpoints which close some of these intervals.

Either of the previous procedures rely on writing the expression $f(x)$ as a rational expression so that we have a product and/or quotient of factors. To do this one might need to get a common denominator. For instance to solve the inequality

$$\frac{2x^2+6}{5-x} + x < 0$$

[3]The **complement** of a set A is the set $U-A$ where U is the *universal set* of elements under consideration. Usually U will be $\mathbb{R}$ if we are considering real numbers, but in this case U is those values for which the inequality can be evaluated, namely the domain of the rational function. The complement of a set A is sometimes denoted A', A^c, $\bar{A}$ or $\tilde{A}$.

we can multiply the second term by $\frac{5-x}{5-x}$ to rewrite the left hand side as

$$\frac{2x^2+6}{5-x} + \frac{x(5-x)}{5-x} < 0$$

which simplifies, upon expanding and combining the numerators, to

$$\frac{x^2+5x+6}{5-x} < 0 \ ,$$

which can be solved as before to get x in $(-3,-2) \cup (5,\infty)$.

In general if we do not have zero on both sides of the inequality, we can use the rules given previously to rewrite the inequality with $f(x)$ on one side and 0 on the other. So for, instance, if we had to solve

$$x < \frac{2x^2+6}{x-5}$$

we could subtract the right side from both sides to get

$$-\frac{2x^2+6}{x-5} + x < 0$$

Bringing the minus sign into the denominator of the first term,

$$-1 \cdot \frac{2x^2+6}{x-5} = \frac{1}{-1} \cdot \frac{2x^2+6}{x-5} = \frac{1(2x^2+6)}{-1(x-5)} = \frac{2x^2+6}{5-x} \ ,$$

our inequality becomes

$$\frac{2x^2+6}{5-x} + x < 0$$

which we solved above to get $(-3,-2) \cup (5,\infty)$.

Note however, that while adding and subtracting an expression involving x does not modify the inequality, multiplying or dividing by such an expression requires care. For instance if we wanted to solve the inequality

$$x(x-5) < 2x^2+6$$

then the easiest way to do so is to expand the left hand side and subtract it from both sides to get the equivalent inequality

$$0 < 2x^2+6-(x^2-5x)$$

or more simply

$$0 < x^2+5x+6 \ .$$

Factoring the right hand side gives

$$0 < (x+3)(x+2)$$

which, upon sign analysis (so $(-)(-)$ or $(+)(+)$) results in the solution $(-\infty,-3) \cup (-2,\infty)$. But suppose instead we started again with

$$x(x-5) < 2x^2+6$$

but wanted to divide both sides by $(x-5)$. Since $(x-5)$ could be positive or negative this would now require us to solve two problems, as we can have solutions from

$$x < \frac{2x^2+6}{x-5} \quad \text{and } x-5 > 0 \ ,$$

as well as ("or")

$$x > \frac{2x^2+6}{x-5} \quad \text{and } x-5<0\ .$$

The first inequality, $x < \frac{2x^2+6}{x-5}$, has, as we have seen, the solutions $(-3,-2) \cup (5,\infty)$. Requiring $x-5>0$ (so $x>5$) means intersecting this with $(5,\infty)$ to get only $(5,\infty)$. The second inequality, $x > \frac{2x^2+6}{x-5}$, has solution $(-\infty,-3) \cup (-2,5)$. Requiring $x-5<0$ (so $x<5$) means intersecting this with $(-\infty,5)$ to get $(-\infty,-3) \cup (-2,5)$. Putting both possibilities together we get

$$(-\infty,-3) \cup (-2,5) \cup (5,\infty)\ .$$

Finally, we should notice that $x=5$ was also a solution to the original inequality. (It got lost when we divided both sides by $x-5$.) If we add this to our solution we do, finally, get

$$(-\infty,-3) \cup (-2,\infty)$$

as we found the easy way. The point of this last example is that multiplying or dividing both sides of an inequality by expressions involving a variable is non-trivial and requires some bookkeeping. Expressions having a constant sign (like x^2+1 which is always positive) would be an exception to this.

Once an inequality is solved it should be checked by choosing some test values from the solution (to see that they satisfy the inequality) and some values that are not part of the solution (to see that they do not). Graphing the non-zero side of the inequality can be done to check many values at once.

Exercise A-1

Answers: Page 260

1-3: Solve the inequalities.

1. $x^2-6x-7>0$
2. $\dfrac{1-2x}{x^2+4x+3} \le 0$
3. $\dfrac{10}{7-x} < x$

Appendix B: Trigonometric Equations

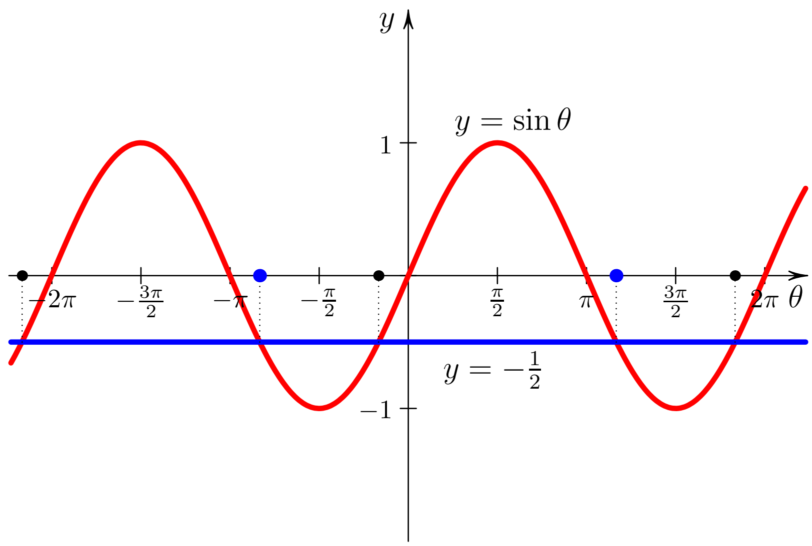
To solve the trigonometric equation

$$\sin\theta = -\frac{1}{2}$$

we need to find the angles θ that make the equation true. One way to get an answer is to apply the inverse sine function $\sin^{-1}(x)$, also written $\arcsin(x)$, to both sides of the equation to get the solution

$$\theta = \sin^{-1}\left(-\frac{1}{2}\right) = -0.523598875 \text{ (radians)}\,,$$

assuming your calculator is in *radian mode*. Otherwise if your calculator is in *degree mode* it would return $-30°$. The problem, however, is that the sine function defined on its usual domain $D = \mathbb{R}$ is not invertible and the above procedure only returns the solution to the original equation that lies in $[-\pi/2, \pi/2]$. There are many more, in fact infinitely many more, solutions to the original equation. To see this, plot the sine function and look to see those values of θ which make sine equal -1/2 :



The circles indicate all the solutions with the first small circle to the left of the origin being the one provided by the calculator. Other solutions (the other smaller dots) can be found by adding integer multiples of 2π ($\pm 2\pi$, $\pm 4\pi, \ldots$) to this value. This occurs due to the *periodicity* of sine, that is, because

$$\sin(\theta + 2\pi) = \sin(\theta)\,.$$

However, as shown in the diagram, there is an infinite number of other possible solutions labelled by the larger dots; these also are separated from each other by multiples of 2π. Note that one of these solutions also lies in $[0, 2\pi)$.

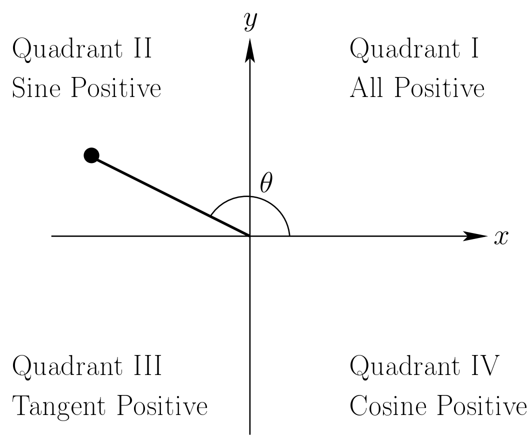
As a general strategy to solving the equation

$$\text{trig}(\theta) = \#\,,$$

where trig represents one of sine, cosine, and tangent, and # is some numerical value, we first find the "physical" angle solutions (usually two of them) in the interval $[0, 2\pi)$. In a geometrical problem, the other angle solutions will overlap with these two angles as they just are adding multiples of 360° (2π) to them. For problems where the trigonometric equation is not arising from a geometrical problem the other solutions may be meaningful as well.

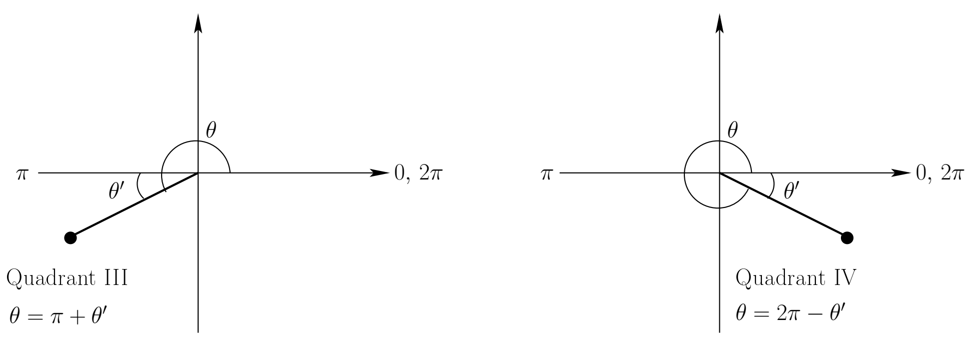
Follow these steps to find the solutions of $\text{trig}(\theta) = \#$:

Step 1: Consult the following CAST diagram to determine in which quadrants your two solutions lie.



In our case $\# = -1/2$ is negative and our trigonometric function is sine. The CAST diagram shows sine is positive in the first quadrant and the second quadrant. Since our value is negative our solutions lie in quadrant III and quadrant IV. (If the value is exactly zero you can consider it a small positive number for purposes of this step.)

Step 2: Draw two diagrams showing your angle solutions in each quadrant. Also label the *reference angle* θ' which is the acute angle from your terminal ray to the x-axis. (The reference angle θ' will be the same for each of your solutions.) Determine the value of θ in terms of your reference angle. In our case we have[1]



Step 3: Next we need to solve for the reference angle θ'. The reference angle satisfies the identical equation as the original equation except we need to take the absolute value of the number:

$$\text{trig}(\theta') = |\#|$$

In our example, since our value is negative we need to make it positive:

$$\sin(\theta') = \frac{1}{2}$$

(If the value in our θ equation had been positive then the θ' equation would have been identical to the θ equation.) Since the reference angle by definition is acute it satisfies $0 \leq \theta' \leq \frac{\pi}{2}$. From

[1]The relation between θ and θ' are the same for any problem, namely $\theta = \theta'$ for quadrant I, $\theta = \pi - \theta'$ for quadrant II, $\theta = \pi + \theta'$ for quadrant III, and $\theta = 2\pi - \theta'$ for quadrant IV. The student should verify each of these with a diagram.

our basic triangles one sees that the angle that has a sine of $1/2$ is $\theta' = \frac{\pi}{6}$. Alternatively the calculator's inverse sine function will give us the correct answer in decimal (assuming radian mode):

$$\theta' = \sin^{-1}(1/2) = 0.523598775\,.$$

Obviously it is better to use the exact answer $\pi/6$ obtained by thinking about the relevant 30-60-90 triangle.

Step 4: Next we get our two solutions for θ by using the relationships we derived in Step 2. For our quadrant III solution we have

$$\theta = \pi + \theta' = \pi + \frac{\pi}{6} = \frac{6\pi}{6} + \frac{\pi}{6} = \frac{7\pi}{6}\ ,$$

while for our quadrant IV solution we get

$$\theta = 2\pi - \theta' = 2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}\ .$$

Step 5: Finally since all the trig functions are periodic, $\text{trig}(\theta + 2\pi) = \text{trig}(\theta)$, we need to add $2n\pi$ for all possible integers n $(0, \pm 1, \pm 2, \ldots)$ to each of the solutions from Step 4 to get the remaining solutions. In our case we have for the complete solution set:

$$\left\{\frac{7\pi}{6} + 2n\pi \mid n \text{ an integer}\right\} \cup \left\{\frac{11\pi}{6} + 2n\pi \mid n \text{ an integer}\right\}\ .$$

We have shown how to solve a trigonometric equation of the form $\text{trig}(\theta) = \#$, for trigonometric functions sine, cosine, and tangent. If your equation is not in this standard form try to solve for the trigonometric function. Thus, for example, $2\sin\theta + 1 = 0$ is the same equation as $\sin\theta = -1/2$ once you solve for the trigonometric function.

If you have a trigonometric equation involving a single cosecant, secant, or cotangent, try to solve for the trigonometric function as before and then reciprocate both sides. Thus to solve

$$\csc(\theta) = -2$$

we reciprocate both sides to get

$$\frac{1}{\csc(\theta)} = -\frac{1}{2}\,.$$

But now we use the identity $\csc(\theta) = 1/\sin(\theta)$ to get

$$\sin(\theta) = -\frac{1}{2}\,.$$

which we have already solved. Similarly a secant equation can be turned into an equation involving cosine and a cotangent equation an equation involving tangent.

Next suppose the argument of our trigonometric function is more complicated than θ. Suppose we wish to solve[2]

$$\sin(3\theta) = -\frac{1}{2}\,.$$

In that case just define a new variable α equal to the argument, here $\alpha = 3\theta$. Then we have that

$$\sin(\alpha) = -\frac{1}{2}\,.$$

[2]Recall that $\sin(3\theta) \neq 3\sin\theta$ in general so we cannot "pull out the 3" from the argument of sine to simplify our equation. That $\sin(3\theta) \neq 3\sin\theta$ in general can be verified by plugging a few values of θ into your calculator.

The solutions for this, as we found above, are:

$$\alpha \in \left\{\frac{7\pi}{6} + 2n\pi \mid n \text{ an integer}\right\} \cup \left\{\frac{11\pi}{6} + 2n\pi \mid n \text{ an integer}\right\}.$$

But $\alpha = 3\theta$ implies $\theta = \frac{\alpha}{3}$ so we need to divide each of our α solutions by 3 to get our θ solutions:

$$\theta \in \left\{\frac{7\pi}{18} + \frac{2n\pi}{3} \mid n \text{ an integer}\right\} \cup \left\{\frac{11\pi}{18} + \frac{2n\pi}{3} \mid n \text{ an integer}\right\}.$$

If you list out the solutions you will see that there are now 6 different "physical" solutions in $[0, 2\pi)$, namely

$$\left\{\frac{7\pi}{18}, \frac{19\pi}{18}, \frac{31\pi}{18}, \frac{11\pi}{18}, \frac{23\pi}{18}, \frac{35\pi}{18}\right\}.$$

If the original equation we were solving arose from a geometrical problem where there had to be a single answer for some reason, further information would need to be used to identify which angle was the unique solution to the problem.

What if your trigonometric function cannot be isolated? Suppose we wish to solve the trigonometric equation

$$\sin^2\theta - \frac{1}{4} = 0\,.$$

Recognizing that $\sin^2\theta = (\sin\theta)^2$ by definition, note that if we let $x = \sin\theta$ our original equation becomes:

$$x^2 - \frac{1}{4} = 0\,.$$

By making the substitution we have turned our problem into solving a simple quadratic equation. Recognizing a difference of squares (or using the quadratic formula) we see there are two solutions, namely $x = -1/2$ or $x = 1/2$. How does this help? Well $x = \sin\theta$ so now we just have to find all the solutions of[3]

$$\sin\theta = -\frac{1}{2} \quad \text{or} \quad \sin\theta = \frac{1}{2}\,.$$

The first equation we have already solved. For the second we note that our two solutions lie in quadrant I and II because that is where sine is positive. In terms of the reference angle (draw the diagrams) we see that the quadrant I solution is just $\theta = \theta'$ while the quadrant II solution is $\theta = \pi - \theta'$. In this case our reference angle satisfies $\sin\theta' = |1/2| = 1/2$ which we already solved before to get $\theta' = \pi/6$. Our quadrant I solution is therefore:

$$\theta = \theta' = \frac{\pi}{6}$$

while our quadrant II solution is

$$\theta = \theta' = \pi - \frac{\pi}{6} = \frac{5\pi}{6}\,.$$

(At this stage it would be prudent to check on your calculator that the sine of these angles really is $1/2$.) Next our solutions for the second equation are, by periodicity:

$$\left\{\frac{\pi}{6} + 2n\pi \mid n \text{ an integer}\right\} \cup \left\{\frac{5\pi}{6} + 2n\pi \mid n \text{ an integer}\right\}$$

Combining these with the solutions to the $\sin\theta = -1/2$ equation gives for the complete solution set:

$$\left\{\frac{7\pi}{6} + 2n\pi \mid n \in \mathbb{Z}\right\} \cup \left\{\frac{11\pi}{6} + 2n\pi \mid n \in \mathbb{Z}\right\} \cup \left\{\frac{\pi}{6} + 2n\pi \mid n \in \mathbb{Z}\right\} \cup \left\{\frac{5\pi}{6} + 2n\pi \mid n \in \mathbb{Z}\right\},$$

[3]Note that we logically want *or* and not *and* in this equation as either $x = -1/2$ *or* $x = 1/2$ was a solution to the quadratic. If the situation arose where we really did want to find those θ that solved $\sin\theta = -\frac{1}{2}$ *and* $\sin\theta = \frac{1}{2}$ we would have to take the set intersection ($\cap$) not the set union ($\cup$) below of the solution sets of each equation and we would, in fact, see there is no θ satisfying both equations. (Which of course there could not be as an angle cannot have a sine that is both $1/2$ and simultaneously $-1/2$.) So be careful to use *or* and *and* appropriately.

where here we have used $\in$ for "element of" (in) and $\mathbb{Z}$, the traditional symbol for the set of integers ($\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$).

Looking back at the last two examples we see that in both equations we were recognizing function composition to simplify the problem. In the first we had $\text{trig}(g(\theta)) = \#$ while in the second case we had $f(\text{trig}(\theta)) = \#$ for some functions f and g. As discussed in Section 1.2.10, when composition like this occurs, solving the problem becomes a two step process, namely solving the outer equation and then the inner equation equalling the solutions of the former.

Returning to the trigonometric equations, what do you do if the problem involves more than one trigonometric function? Suppose we wish to solve:

$$\sin\theta - \tan\theta\cos^3\theta + \frac{1}{8} = 0\,.$$

One strategy is to use *trigonometric identities* to try to write the expression in terms of a single trigonometric function.[4] Since all trigonometric functions can be written in terms of sine and cosine, for instance, we could try writing the above expression using the identity $\tan\theta = \sin\theta/\cos\theta$ to get an expression in terms of just sine and cosine:

$$\sin\theta - \sin\theta\cos^2\theta + \frac{1}{8} = 0\,.$$

Next since only even powers of cosine appear here we can use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ (so $\cos^2\theta = 1 - \sin^2\theta$) to get

$$\sin^3\theta + \frac{1}{8} = 0\,.$$

Now the expression only involves $\sin\theta$. Isolate $\sin\theta$ to get

$$\sin^3\theta = -\frac{1}{8}\,.$$

Since cubing is an invertible function we can cube root both sides to get

$$\sin\theta = -\frac{1}{2}\,.$$

This is the same problem we solved originally so the solution to this problem is the same as that one. In summary, trigonometric identities are useful for transforming complicated trigonometric equations into ones that can be solved.

Suppose you cannot find a way to transform your trigonometric equation into one involving a single trigonometric function. A further technique which can be used is to rearrange your equation so that zero is on the right and then try to see if you can factor the expression on the left so that each factor involves only a single trigonometric function (or can be transformed to such a factor with identities). For instance, suppose we wish to solve:

$$2\sin\theta\tan\theta + 2\sin\theta + \tan\theta = -1\,.$$

Putting all terms on the left hand side this becomes

$$2\sin\theta\tan\theta + 2\sin\theta + \tan\theta + 1 = 0\,.$$

Notice the left-hand side factors to give the new equation:[5]

$$(2\sin\theta + 1)(\tan\theta + 1) = 0\,.$$

[4] Recall a trigonometric identity is a trigonometric equation that is true for all values of the variable. It thus can be used to convert one expression in an equation into a simpler expression, thereby making the equation easier to solve.

[5] To see the factoring it may be easier to set $x = \sin\theta$ and $y = \tan\theta$ and notice that we have to then only factor the algebraic expression $2xy + 2x + y + 1 = 2x(y+1) + 1(y+1) = (2x+1)(y+1)$ by grouping and then reintroduce the trigonometric values of x and y at the end.

Now since the right hand side is zero, and recalling that the only way a product can equal zero is if one of the factors is zero, solving the original equation is equivalent to solving

$$2\sin\theta + 1 = 0 \quad \text{or} \quad \tan\theta + 1 = 0\,,$$

or equivalently, upon isolating the trigonometric function,

$$\sin\theta = -\frac{1}{2} \quad \text{or} \quad \tan\theta = -1\,.$$

The first equation was previously found to have solutions:

$$\left\{\frac{7\pi}{6} + 2n\pi \mid n \text{ an integer}\right\} \cup \left\{\frac{11\pi}{6} + 2n\pi \mid n \text{ an integer}\right\}\,.$$

Following the usual steps for $\tan\theta = -1$ we see by the CAST diagram that tangent is positive in quadrants I and III. Since our right hand side is negative that means our solutions in $[0, 2\pi)$ lie in quadrants II and IV. Drawing the diagrams we see that $\theta = \pi - \theta'$ for the quadrant II solution and $\theta = 2\pi - \theta'$ for the quadrant IV solution where the reference angle θ' satisfies $\tan\theta' = 1$. From our 45-45-90 triangle we recognize that the tangent of 45° is indeed 1 and so our reference angle is $\theta' = \pi/4$. (Alternatively use $\theta' = \tan^{-1}(1)$ to get this.) Our quadrant II solution is therefore $\theta = \pi - \pi/4 = 3\pi/4$ and our quadrant IV solution is $\theta = 2\pi - \pi/4 = 7\pi/4$. By periodicity our complete solution to $\tan\theta = -1$ is

$$\left\{\frac{3\pi}{4} + 2n\pi \mid n \text{ an integer}\right\} \cup \left\{\frac{7\pi}{4} + 2n\pi \mid n \text{ an integer}\right\}\,.$$

Listing out these values one realizes this may be simplified to[6]

$$\left\{\frac{3\pi}{4} + n\pi \mid n \text{ an integer}\right\}\,.$$

Finally putting the solutions to both equations together we have the solution set for the original equation:

$$\left\{\frac{7\pi}{6} + 2n\pi \mid n \in \mathbb{Z}\right\} \cup \left\{\frac{11\pi}{6} + 2n\pi \mid n \in \mathbb{Z}\right\} \cup \left\{\frac{3\pi}{4} + n\pi \mid n \in \mathbb{Z}\right\}\,.$$

As a final note, if your trigonometric equation cannot be solved using any of the techniques outlined here, remember that trigonometric functions are just functions and you can apply any numerical solution-finding method, such as the Bisection Method, to them as you would any other equation. In that case you should seek all the distinct solutions in the interval $[0, 2\pi)$ first. Assuming you have written your equation as $f(\theta) = 0$, plotting $f(\theta)$ will help here to see where the solutions lie and how many of them there are. If your equation involves only trigonometric functions and the arguments of your trigonometric functions are θ, or more generally $m\theta$ where m is an integer, adding multiples of 2π to your solutions in $[0, 2\pi)$ will also be solutions to your equation.

Exercise B-1

Answers: Page 261

1-4: Solve the following trigonometric equations.

1. $2\cos\theta = \sqrt{3}$
2. $\sec\theta = -2$
3. $\cos(4\alpha) = -\frac{1}{2}$
4. $\tan^2 x = \tan x$

[6] This simplification results from the fact that tangent actually satisfies the simpler periodic relation $\tan(\theta + \pi) = \tan\theta$.

Answers

Exercise 1-1 (page 5)

1. $x = 3$
2. $x = 3$ or $x = -\frac{1}{2}$. Written as a *solution set* it is $\{3, -1/2\}$
3. $x = \dfrac{-3 \pm \sqrt{-7}}{8}$, so no real solution.
4. $x = 2$
5. $\{-2, 2, 4\}$
6. $x = 1$
7. $\left\{0, \pm\sqrt{\dfrac{3}{2}}\right\}$
8. $\left\{\pm 1, \pm\sqrt{2}\right\}$
9. $\left\{-1, \pm\sqrt{\dfrac{2}{3}}\right\}$
10. $\left\{-1, 0, \dfrac{-3 \pm \sqrt{33}}{4}\right\}$

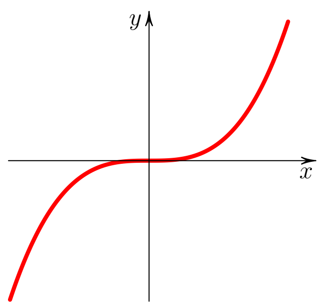
Exercise 1-2 (page 15)

1. $D = \mathbb{R} = (-\infty, \infty)$; x-int=0; y-int=0
2. $D = [6, \infty)$; x-int= 6; No y-int
3. $D = \mathbb{R} = (-\infty, \infty)$; x-int=-2; y-int=4
4. $D = \mathbb{R} = (-\infty, \infty)$; x-int=-5, 0 ,1; y-int=0
5. $D = \mathbb{R} - \{1\} = (-\infty, 1) \cup (1, \infty)$; No x-int; y-int= -1
6. $D = \mathbb{R} - \{-2\} = (-\infty, -2) \cup (-2, \infty)$; No x-int; y-int= $\dfrac{1}{4}$
7. $D = [-2, 2]$; x-int=-2, 2; y-int=2
8. $D = \mathbb{R} = (-\infty, \infty)$; x-int= $-\dfrac{2}{3}, -1$; y-int= 2
9. $D = \mathbb{R} = (-\infty, \infty)$; x-int= 1; y-int= -4
10. $D = \mathbb{R} - \{-7\} = (-\infty, -7) \cup (-7, \infty)$; x-int= -5; y-int= $\dfrac{5}{7}$

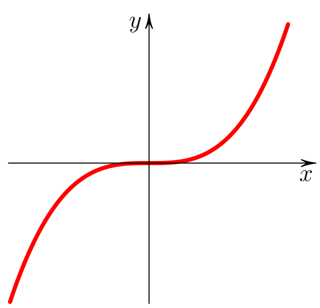
11. $D = \mathbb{R} - \{-1/2, 3\} = (-\infty, -1/2) \cup (-1/2, 3) \cup (3, \infty)$; No x-int; y-int$= -\dfrac{10}{3}$
12. $D = \{x \in \mathbb{R} | x \leq 4\} = (-\infty, 4]$; x-int$= 4$; y-int$= 2$
13. $D = \left(-\infty, -\sqrt{10}\right] \cup \left[\sqrt{10}, \infty\right)$; x-int$= \pm\sqrt{10}$; No y-int
14. $D = (-\infty, -6] \cup (3/2, \infty)$; x-int$= -6$; No y-int
15. $D = \left(-\infty, -\sqrt{10}\right] \cup \left[\sqrt{10}, \infty\right)$; x-int$= \pm\sqrt{10}$; No y-int
16. $D = (-2, 0) \cup (0, \infty)$; x-int=2; No y-int

17. Even
18. Odd
19. Neither
20. Odd
21. Neither
22. Odd
23. Even
24. Even
25. Even
26. Odd
27. Neither
28. Even
29. Odd
30. Neither
31. Even
32. Even

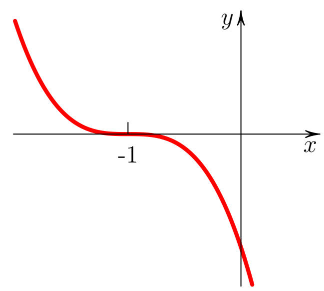
33. (a)



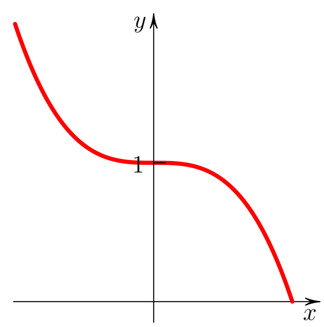
(b)



(c)



(d)



(e) $f(-x) = -f(x)$, and so f is odd.

Exercise 1-3 (page 22)

1. $\dfrac{\sqrt{3}}{2}$
2. $-\dfrac{1}{\sqrt{2}}$
3. $\dfrac{1}{2}$
4. 0
5. -$\sqrt{2}$
6. $\sqrt{3}$
7. -1
8. 1

Exercise 1-4 (page 26)

1. (a) $f(x+2) = \dfrac{1}{x+4}$ (b) $f(f(x)) = \dfrac{x+2}{2x+5}$

2. $f \circ g(x) = f(g(x)) = 2\left(x^2+3\right)^{\frac{3}{2}}$ with $D = \mathbb{R}$, $g \circ f(x) = g(f(x)) = \sqrt{4x^6+3}$ with $D = \mathbb{R}$

3. $f \circ g(x) = f(g(x)) = 3(3x-2)^2 + 6(3x-2) + 4 = 27x^2 - 18x + 4$ with $D = \mathbb{R}$, $g \circ f(x) = g(f(x)) = 3(3x^2+6x+4) - 2 = 9x^2 + 18x + 10$ with $D = \mathbb{R}$

4. $f \circ g(z) = f(g(z)) = \sqrt{\left(\dfrac{z}{z+1}\right)^2 + 5} = \sqrt{\dfrac{6z^2+10z+5}{(z+1)^2}}$ with $D = \mathbb{R} - \{-1\}$, $g \circ f(z) = g(f(z)) = \dfrac{\sqrt{z^2+5}}{\sqrt{z^2+5}+1}$ with $D = \mathbb{R}$

5. $f \circ g(x) = f(g(x)) = \dfrac{2\left(x^2+3\right)+5}{(x^2+3)-4} = \dfrac{2x^2+11}{x^2-1}$ with $D = \mathbb{R} - \{-1, 1\}$, $g \circ f(x) = g(f(x)) = \left(\dfrac{2x+5}{x-4}\right)^2 + 3 = \dfrac{7x^2-4x+73}{(x-4)^2}$ with $D = \mathbb{R} - \{4\}$

6. $f(x) = \sqrt{x} - 3$, $g(x) = x^2 + 1$.

7. $\left\{\pm 1, \pm\sqrt{2}\right\}$

Chapter 1 Review Exercises (page 27)

1. $x = 1/2$, $x = -2$. Written as a solution set: $\{1/2, -2\}$

2. $\{-5, 4\}$

3. $\{-2, -1, 1\}$

4. $\{\pm 1, \pm 2\}$

5. $\{-2, 1, 2\}$

6. $D = \mathbb{R} - \left\{-\dfrac{4}{5}\right\}$, x-int$= \dfrac{3}{2}$, y-int$= -\dfrac{3}{4}$

7. $D = (-\infty, -2] \cup [2, \infty)$, x-int=2, −2, y-int does not exist

8. $D = (-\infty, -5) \cup \left[-\dfrac{1}{2}, \infty\right)$, x-int$= -\dfrac{1}{2}$, y-int$= \sqrt{\dfrac{1}{5}}$

9. $D = [-8, 3) \cup (3, \infty)$, x-int$= -8$, y-int$= -\dfrac{\sqrt{8}}{3}$

10. Even
11. Even
12. Odd
13. Odd

14. $f \circ g(x) = f(g(x)) = x^2 + 6$ with $D = \mathbb{R}$, $g \circ f(x) = g(f(x)) = \sqrt[3]{\left(x^3+6\right)^2}$ with $D = \mathbb{R}$

15. $f \circ g(t) = f(g(t)) = \dfrac{2t^2+11}{t^2-1}$ with $D = \mathbb{R} - \{1, -1\}$, $g \circ f(t) = g(f(t)) = \dfrac{4t^2+20t+25}{(t-4)^2} + 3$ with $D = \mathbb{R} - \{4\}$

16. $f \circ g(x) = f(g(x)) = \sqrt{\dfrac{2}{x+3}}$ with $D = (-3, \infty)$, $g \circ f(x) = g(f(x)) = \dfrac{\sqrt{x-1}+5}{\sqrt{x-1}+3}$ with $D = [1, \infty)$

Exercise 2-1 (page 35)

1. (a) $(-1)^3 + (-1)^2 - 2(-1) + 3 = 5$, $(0)^3 + (0)^2 - 2(0) + 3 = 3$
   (b) $\dfrac{3-5}{0-(-1)} = -2$
   (c) $m(x) = \dfrac{x^3 + x^2 - 2x + 3 - 5}{x - (-1)} = x^2 - 2 \quad (x \neq -1)$
   (d) $m_t = \lim\limits_{x \to -1} m(x) = -1$

Exercise 2-2 (page 44)

1. $\frac{13}{5}$
2. 0
3. $\frac{1}{4}$
4. 0
5. $\frac{7}{4}$
6. $-\frac{1}{9}$
7. 1
8. 1
9. $\frac{1}{4}$
10. $-\frac{3}{10}$
11. -6
12. $-\frac{1}{25}$
13. -10
14. 12
15. $\frac{7}{3}$

Exercise 2-3 (page 47)

1. 1
2. 1
3. 0
4. $\frac{7}{5}$
5. 0
6. 1
7. $-\frac{1}{\pi}$
8. $\frac{4}{\pi}$
9. 0
10. 1
11. $\frac{1}{2}$

Exercise 2-4 (page 52)

1. $\lim\limits_{x \to 0^-} f = 4$, $\lim\limits_{x \to 0^+} f = 4$, $\lim\limits_{x \to 2^-} f = 3$, $\lim\limits_{x \to 2^+} f = 2$, $\lim\limits_{x \to 4^-} f = \infty$ (limit does not exist but approaches infinity), $\lim\limits_{x \to 4^+} f = 2$

2. $\lim\limits_{x \to 0^-} f = -1$, $\lim\limits_{x \to 0^+} f = -1$, $\lim\limits_{x \to 2^-} f = 1$, $\lim\limits_{x \to 2^+} f = -2$, $\lim\limits_{x \to 4^-} f = 2$, $\lim\limits_{x \to 4^+} f = 2$, $\lim\limits_{x \to 0} f = -1$, $\lim\limits_{x \to 2} f$ does not exist, $\lim\limits_{x \to 4} f = 2$

3. $\lim\limits_{x \to 2^-} f(x) = \dfrac{2}{3}$, $\lim\limits_{x \to 2^+} f(x) = 5$, $\lim\limits_{x \to 2} f(x)$ does not exist as the left and right-handed limits are not equal at $x = 2$.

4. $c = \dfrac{2}{7}$

Exercise 2-5 (page 58)

1. ∞
2. $-\infty$
3. $\frac{6}{7}$
4. ∞
5. Vertical asymptote: $x = 2$
6. No vertical asymptotes
7. Vertical asymptote: $t = 2$
8. Vertical asymptotes: $x = -2$, $x = 3$
9. Vertical asymptote: $x = 0$
10. Vertical asymptotes: $x = -4$, $x = 4$
11. Vertical asymptote: $x = 0$
12. No vertical asymptotes

Exercise 2-6 (page 68)

1. "f is continuous at $x = a$" if (a) a is in the domain of f (b) $\lim_{x\to a} f(x)$ exists (c) $\lim_{x\to a} f(x) = f(a)$.
2. Continuous
3. Continuous
4. Removable Discontinuity
5. Continuous
6. Infinite Discontinuity
7. Jump Discontinuity
8. (a) $\lim_{x\to 0^-} f(x) = 1$ and $\lim_{x\to 0^+} f(x) = 4$. Therefore $\lim_{x\to 0} f(x)$ does not exist.

   (b) $c = \pm 1$
9. $\mathbb{R} - \{-1, 1\}$. Note the limit actually exists at $x = -1$ but the function is not defined there.
10. Let $f(x) = x^3 + 2x^2$. Then notice that $f(1) = 3$ and $f(3) = 33$. Then since $f(1) < 10 < f(3)$ and f is continuous, there exists a $c \in (1, 3)$ such that $f(c) = 10$ by the IVT. Since $f(c) = c^3 + 2c^2$ the result follows.
11. $f(x) = x^2 + \cos x - 2$ is continuous on $[0, 2]$, $f(0) = -1 < 0$, $f(2) \approx 1.58 > 0$, so by the Intermediate Value Theorem there is a c in $(0, 2)$ with $f(c) = 0$. i.e. $c^2 + \cos c - 2 = 0$ and c is therefore a solution to the equation.

Chapter 2 Review Exercises (page 69)

1. 1
2. $\frac{1}{24}$
3. $-\frac{3}{4}$
4. $-\frac{3}{32}$
5. $-\frac{1}{4}$
6. $\frac{4}{5}$
7. $\frac{3}{4}$
8. 4
9. 0
10. Continuous
11. Not continuous
12. Not continuous

Exercise 3-1 (page 76)

1. (a) 8 (b) $m \approx 13$ (c) Point-slope form: $y = 13(x-2)+12$, Slope-intercept form: $y = 13x - 14$
2. (a) 1 cm (b) 17 cm (c) 8 cm/s (d) $v \approx 20$ cm/s
3. (a) $\dfrac{\Delta V}{\Delta P} \approx -7.47$ L/atm (b) -2.49 L/atm

Exercise 3-2 (page 81)

1. $f'(2) = \lim_{h\to 0} \dfrac{\frac{1}{(2+h)+1} - \frac{1}{2+1}}{h} = -\dfrac{1}{9}$
2. $g'(4) = \lim_{h\to 0} \dfrac{\sqrt{2(4+h)} - \sqrt{2(4)}}{h} = \dfrac{1}{2\sqrt{2}}$
3. $f'(x) = \lim_{h\to 0} \dfrac{(x+h)^2 + 3 - (x^2+3)}{h} = 2x$
4. $f'(x) = \lim_{h\to 0} \dfrac{\frac{1}{3(x+h)} - \frac{1}{3x}}{h} = -\dfrac{1}{3x^2}$
5. $f'(x) = \lim_{h\to 0} \dfrac{(x+2+h)^2 - (x+2)^2}{h} = 2x+4$
6. $f'(x) = \lim_{h\to 0} \dfrac{\sqrt{x+h+2} - \sqrt{x+2}}{h} = \dfrac{1}{2\sqrt{x+2}}$
7. $f'(x) = \lim_{h\to 0} \dfrac{\frac{3(x+h)+2}{x+h+1} - \frac{3x+2}{x+1}}{h} = \dfrac{1}{(x+1)^2}$
8. $f'(x) = \lim_{h\to 0} \dfrac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = -\dfrac{1}{2x\sqrt{x}} = -\dfrac{1}{2}x^{-\frac{3}{2}}$
9. $\lim_{h\to 0^-} \dfrac{\sqrt{(2+h-2)^2}}{h} = \lim_{h\to 0^-} \dfrac{\sqrt{h^2}}{h} = \lim_{h\to 0^-} \dfrac{|h|}{h} = \lim_{h\to 0^-} \dfrac{-h}{h} = -1$

   $\lim_{h\to 0^+} \dfrac{\sqrt{(2+h-2)^2}}{h} = \lim_{h\to 0^+} \dfrac{\sqrt{h^2}}{h} = \lim_{h\to 0^+} \dfrac{|h|}{h} = \lim_{h\to 0^+} \dfrac{h}{h} = 1$
10. (a) Differentiable
    (b) Differentiable
    (c) Not Differentiable. Discontinuous at $x = 0$.

(d) Not differentiable. At $x = 0$ the left and right hand limits for the derivative are not equal:

$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \neq \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h},$$

therefore the limit itself (the derivative) does not exist. Geometrically no tangent line may be drawn at the point so there can be no derivative as that is the tangent slope.

Exercise 3-3 (page 85)

1. $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 12x^{11} = \frac{1}{2\sqrt{x}} - 12x^{11}$
2. $g'(x) = -\frac{5}{2}x^{-\frac{7}{2}} = \frac{-5}{2\sqrt{x^7}}$
3. $\frac{dy}{dx} = \frac{12}{5}x^2$
4. $f'(u) = -4u^{-5} + 4u^3$; $f'(1) = 0$
5. $f'(x) = 6x^5 + 8x^3 + 2x$
6. $f'(x) = \sqrt{3}\cdot\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{\sqrt[5]{3}}\cdot\frac{1}{5}x^{-\frac{4}{5}} = \frac{\sqrt{3}}{2\sqrt{x}} + \frac{1}{5\sqrt[5]{3x^4}}$
7. $f'(x) = 2a\sin(\pi/15)x^{2a-1}$
8. $\frac{ds}{dt} = -gt + v_0$
9. 1(b): $\left.\frac{dy}{dx}\right|_{x=2} = \left[3x^2+1\right|_{x=2} = 13$

   2(d): $v(2) = \left.\frac{ds}{dt}\right|_{t=2} = \left[3t^2+4t\right|_{t=2} = 20$ cm/s

   3(b): $\left.\frac{dV}{dP}\right|_{P=3} = -(22.4)\cdot P^{-2}\big|_{P=3} = -2.49$ L/atm
10. $y = \frac{3}{2}x + \frac{1}{2}$
11. $x = 0,\ \frac{4}{3}$
12. (a) $C(2000) = 368.6732 \approx 369$ ppm

    (b) $C'(2000) = 1.7948 \approx 1.79$ ppm/year

    (c) $\frac{C(2005)-C(2000)}{C(2000)} \times 100 = 2.53\%$.
13. (a) The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Use similar triangles to show that the radius of the surface of the liquid is $r = \frac{2}{5}y$.

    (b) $\left.\frac{dV}{dy}\right|_{y=4\text{ m}} = \frac{64\pi}{25} \approx 8.04\ \frac{\text{m}^3}{\text{m}}$

Exercise 3-4 (page 89)

1. $f'(x) = \left(4x^3-6x\right)\left(x^{\frac{1}{3}}-x\right) + \left(x^4-3x^2+2\right)\left(\frac{1}{3}x^{-\frac{2}{3}}-1\right)$
2. $f'(x) = \left(3x^2+\pi\right)\left(2+x^{-3}\right) + \left(x^3+\pi x+2\right)\left(-3x^{-4}\right)$
3. $\frac{dy}{dx} = (2x)\left(x^3+2\right)\left(2x^2+\sqrt{x}\right) + \left(x^2-1\right)\left(3x^2\right)\left(2x^2+\sqrt{x}\right) + \left(x^2-1\right)\left(x^3+2\right)\left(4x+\frac{1}{2}x^{-\frac{1}{2}}\right)$
4. $\frac{df}{dx} = -\frac{2}{(x-6)^2}$

5. $f'(\theta) = \dfrac{(2\theta+3)\left(\theta^2-7\right) - \left(\theta^2+3\theta-4\right)(2\theta)}{\left(\theta^2-7\right)^2} = -\dfrac{3\theta^2+6\theta+21}{(\theta^2-7)^2}$

6. $g'(x) = -\dfrac{1}{2}x^{-\frac{3}{2}} + \dfrac{3}{2}x^{\frac{1}{2}}$; $g'(4) = \frac{47}{16}$

7. $f'(v) = \frac{[(2)(v+4/v)+(2v+3)(1-4/v^2)](v^2+v)-(2v+3)(v+4/v)(2v+1)}{(v^2+v)^2}$

8. $h'(x) = 2cx + \dfrac{3}{2\sqrt{x}}\left(2x^2+x\right) + \left(3\sqrt{x}+2\right)(4x+1)$

Exercise 3-5 (page 91)

1. Point-slope form: $y = 8(x-2)+1$, Slope-intercept form: $y = 8x - 15$

2. Point-slope form: $y = -\dfrac{2}{5}(x-1)+2$, Slope-intercept form: $y = -\dfrac{2}{5}x + \dfrac{12}{5}$

3. $(-2,-12)$, $(2,4)$

Exercise 3-6 (page 93)

1. $f'(x) = 18x\left(x^2+3\right)^8$

2. $\dfrac{dg}{dx} = -\dfrac{\frac{1}{2\sqrt{x}}+1}{\left(x+\sqrt{x}\right)^2} = -\dfrac{2\sqrt{x}+1}{2\sqrt{x}\,(x^2+x)+4x^2}$

3. $f'(t) = -\dfrac{7(4t+3)}{2\left(2t^2+3t+4\right)^{\frac{3}{2}}}$

4. $y' = \dfrac{1}{7}\left(\dfrac{4x+3}{x^2+x}\right)^{-\frac{8}{7}} \dfrac{4x^2+6x+3}{\left(x^2+x\right)^2}$

5. $h'(x) = \dfrac{5nx^{n-1}}{3\left(5x^n+4c\right)^{\frac{2}{3}}}$

6. $f'(x) = 5\left[\left(2\,x+\sqrt{x}\right)^4 + 3\,x\right]^4 \left[4\left(2\,x+\sqrt{x}\right)^3\left(2+\dfrac{1}{2\sqrt{x}}\right)+3\right]$

Exercise 3-7 (page 96)

1. Use the sine addition identity followed by the fundamental limits involving sine and cosine to get $f'(x) = 4\cos 4x$.

2. $f'(x) = 2x\cos x - x^2\sin x$

3. $f'(t) = \dfrac{3t^2(\sin t+\tan t) - t^3\left(\cos t+\sec^2 t\right)}{(\sin t+\tan t)^2}$

4. $\dfrac{dH}{d\theta} = -\csc\theta\cot^2\theta - \csc^3\theta$; $H'(\pi/3) = -\dfrac{10}{3\sqrt{3}}$

5. $f'(x) = (\cos x - \sin x)(\sec x - \cot x) + (\sin x + \cos x)(\sec x\tan x + \csc^2 x)$

6. $\dfrac{df}{d\theta} = 0$

7. (a) Since $\tan(\pi/4) = 1$, $x = \pi/4$ and $y = \pi/4$ indeed satisfy the equation.

(b) Point-slope form: $y = (1+\pi/2)(x-\pi/4)+\pi/4$, Slope-intercept form: $y = (1+\pi/2)x-\pi^2/8$

Exercise 3-8 (page 99)

1. $f'(x) = 12\left(x^8-3x^4+2\right)^{11}\left(8x^7-12x^3\right)$
2. $g'(x) = \dfrac{1}{2}\left(3x^2+2\right)^{-\frac{1}{2}}(6x) = \dfrac{3x}{\sqrt{3x^2+2}}$; $g'(2) = \frac{6}{\sqrt{14}}$
3. $f'(\theta) = 2\theta\cos(\theta^2)$
4. $h'(\theta) = -2\cot\theta\csc^2\theta$
5. $f'(x) = \sec\left[\left(x^3+3\right)\left(\sqrt{x}+x\right)\right]\tan\left[\left(x^3+3\right)\left(\sqrt{x}+x\right)\right]\left[\left(3x^2\right)\left(\sqrt{x}+x\right)+\left(x^3+3\right)\left(\dfrac{1}{2\sqrt{x}}+1\right)\right]$
6. $y' = \left(-4\sin\sqrt[3]{x}\right)\cdot\left(\dfrac{1}{3}x^{-\frac{2}{3}}\right) = -\dfrac{4\sin\sqrt[3]{x}}{3\sqrt[3]{x^2}}$
7. $f'(x) = (-1)\left(3+\sin^2 x\right)^{-2}(2\sin x\cos x) = -\dfrac{2\sin x\cos x}{\left(3+\sin^2 x\right)^2}$
8. $\dfrac{dy}{dx} = -5(\csc x+2)^4\csc x\cot x+2x+1$
9. $y' = \pi\sec^2\theta+\pi\sec^2(\pi\theta)$
10. $g'(x) = \dfrac{1}{2}\left(x+\sqrt{x}\right)^{-\frac{1}{2}}\left(1+\dfrac{1}{2}x^{-\frac{1}{2}}\right)\left(x^4-1\right)^7+\left(\sqrt{x+\sqrt{x}}\right)(7)\left(x^4-1\right)^6\left(4x^3\right)$
$= \dfrac{\left(2\sqrt{x}+1\right)\left(x^4-1\right)^7}{4\sqrt{x}\sqrt{x+\sqrt{x}}}+28x^3\left(x^4-1\right)^6\sqrt{x+\sqrt{x}}$
11. $f'(x) = 3\left(\dfrac{x-3}{x+1}\right)^2\cdot\dfrac{4}{(x+1)^2} = 12\dfrac{(x-3)^2}{(x+1)^4}$
12. $\dfrac{dA}{dt} = -\omega\sin(\omega t+\phi)$; $\left.\dfrac{dA}{dt}\right|_{t=0} = -\omega\sin(\phi)$
13. $f'(x) = \left(\cos\left[\cos(x^2+x)\right]\right)\cdot\left(-\sin\left(x^2+x\right)\right)\cdot(2x+1)$
14. (a) $-2 = 5(0)+3(0)-2(1)$
    (b) $y = 11x-2$
15. $\left\{\dfrac{7\pi}{6}+2n\pi \mid n \text{ an integer}\right\}\bigcup\left\{\dfrac{11\pi}{6}+2n\pi \mid n \text{ an integer}\right\}\bigcup\left\{\dfrac{\pi}{2}+n\pi \mid n \text{ an integer}\right\}$

Exercise 3-9 (page 103)

1. $y' = \dfrac{3-2x}{2y}$
2. $y' = \dfrac{6x^2y-3x^2-y^2}{2xy-2x^3}$; $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,2)} = \dfrac{5}{2}$
3. $y' = -\dfrac{b^2x}{a^2y}$
4. $y' = \dfrac{y\cos(xy)}{1-x\cos(xy)}$

5. $y' = \dfrac{2x - \sin(x+y)}{\cos y + \sin(x+y)}$

6. $y' = \dfrac{x^2 \cos x + \sec y}{x \sec y \tan y}$

7. (a) Noting that $x^{\frac{2}{3}} = (\sqrt[3]{x})^2$ it follows that the values $x = -1$ and $y = 3\sqrt{3} = (\sqrt{3})^3$ simultaneously satisfy the equation.

   (b) Point-slope form: $y = \sqrt{3}(x+1) + 3\sqrt{3}$, Slope-intercept form: $y = \sqrt{3}\,x + 4\sqrt{3}$

Exercise 3-10 (page 103)

1. $f'(x) = 20x^4 + 6x + 1$

2. $\dfrac{dy}{dx} = 8x^7 - \dfrac{6}{x^4} - \dfrac{1}{2}x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}}$

3. $g'(t) = \dfrac{2}{3}t^{-\frac{1}{3}} + 4t^3 - \dfrac{12}{t^3}$

4. $f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}} - \dfrac{5}{2}x^{-\frac{3}{2}}$

5. $g'(x) = \left(\dfrac{1}{2}x^{-\frac{1}{2}} + 3\right)(x+\pi) + \left(\sqrt{x} + 3x + 1\right)$

6. $h'(y) = \dfrac{3(y+4)^2(1+0)(y+5) - (y+4)^3(1+0)}{(y+5)^2} = \dfrac{(y+4)^2(2y+11)}{(y+5)^2}$

7. $y' = \dfrac{1}{4}\left(x^3 + 2x + 5\right)^{-\frac{3}{4}}\left(3x^2 + 2\right)$

8. $f'(\theta) = -3\sin(3\theta) + 2\sin\theta\cos\theta$

9. $g'(x) = 2x\sec^2\left(x^2+1\right)\cos(x) - \tan\left(x^2+1\right)\sin(x)$

10. $\dfrac{dy}{dt} = \dfrac{3}{4}(\sin t + 5)^{-\frac{1}{4}}\cos t$

11. $\dfrac{df}{dx} = \dfrac{1}{3}\left(\dfrac{x^4+5x-1}{x^2-3}\right)^{-\frac{2}{3}} \dfrac{\left(4x^3+5\right)\left(x^2-3\right) - \left(x^4+5x-1\right)(2x)}{\left(x^2-3\right)^2}$

    $= \dfrac{1}{3}\left(\dfrac{x^4+5x-1}{x^2-3}\right)^{-\frac{2}{3}} \dfrac{2x^5 - 12x^3 - 5x^2 + 2x - 15}{\left(x^2-3\right)^2}$

12. $y' = \dfrac{y - 3x^2y^4}{4x^3y^3 + 2y - x}$

Exercise 3-11 (page 107)

1. $\dfrac{d^2f}{dx^2} = 2\csc^2 x \cot x$

2. $f''(x) = 90(x-2)^8$; $f''(3) = 90$

3. $y'' = 6x\sec x + 6x^2\sec x\tan x + x^3\sec x\tan^2 x + x^3\sec^3 x$

4. $y'' = \dfrac{y - xy'}{y^2} = \dfrac{1}{y} - \dfrac{x^2}{y^3}$

Exercise 3-12 (page 113)

1. $\dfrac{5}{\sqrt{18\pi}}$ km/h
2. $\dfrac{6}{25\pi}$ cm/sec
3. $\dfrac{3}{2}$ km/sec
4. $\dfrac{5}{3\pi}$ m/min
5. $\dfrac{1840}{29}$ km/h
6. $\dfrac{4}{5}$ m/sec
7. $-\dfrac{5}{4}$ cm/sec

Exercise 3-13 (page 117)

1. $V \pm \Delta V = 125 \pm 15\ \text{cm}^3$
2. $V \pm \Delta V = 288\pi \pm 72\pi\ \text{cm}^3$
3. $L(x) = 3 + 4(x-2)$
4. $L(x) = 1 + 2(x - \pi/4)$

Chapter 3 Review Exercises (page 118)

1. $f'(x) = 3x^2$
2. $f'(x) = \dfrac{11}{(x+2)^2}$
3. $f'(x) = \dfrac{1}{\sqrt{2x+1}}$
4. $y' = 12x^3 + \dfrac{\sqrt[3]{2}}{3}x^{-2/3} + \dfrac{5}{2}x^{-3/2}$
5. $g'(x) = \left(\dfrac{1}{\sqrt{2x}} - 4\right)(3x + \sin x) + \left(\sqrt{2x} - 4x + 3\right)(3 + \cos x)$
6. $h'(y) = -\dfrac{3y+28}{2\sqrt{y+5}\,(3y+2)^2}$
7. $f'(\theta) = -2\cos\theta\sin\theta - 8\theta\sin\left(\theta^2\right)$
8. $g'(x) = 3x^2\sec\left(x^3+4\right)\tan\left(x^3+4\right)\cos(2x) - 2\sec\left(x^3+4\right)\sin(2x)$
9. $f'(x) = \dfrac{1}{5}\left(\dfrac{x^3-4x+10}{4x^2+5}\right)^{-4/5}\dfrac{4x^4+31x^2-80x-20}{\left(4x^2+5\right)^2}$
10. $\dfrac{dy}{dx} = \dfrac{y - 4x^3y^3}{3x^4y^2 + 8y - x - \cos y}$

11. Point-slope form: $y = -6(x - \pi/2) + 3$, Slope-intercept form: $y = -6x + 3\pi + 3$
12. $x = 1$, $x = -3$
13. $\theta = \dfrac{\pi}{3} + 2n\pi$, $\theta = \dfrac{5\pi}{3} + 2n\pi$, $\theta = n\pi$ (n an integer)
14. 57π cm^3/sec
15. $-\dfrac{1}{10}$ rad/sec
16. $\dfrac{1}{2(3)^{1/4}}$ cm/sec

Exercise 4-1 (page 126)

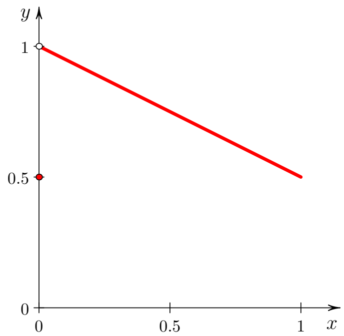
1. Note: It is assumed that the functions extend beyond the graph of the plot with the same trend unless they are explicitly terminated with a point. The values below are approximate.
   (a) Relative minima: $f(-1.4) = -0.8$ and $f(1.9) = 0.5$; Relative maxima: $f(-2.0) = 2.0$, $f(0.4) = 1.0$, and $f(1.3) = 2.6$. No absolute minimum nor absolute maximum.
   (b) Absolute maximum value of $f(0) = 0.5$. This is also a relative maximum.
   (c) Relative minimum: $f(-1) = 2$; Relative maximum: $f(-3) = 6$, $f(0) = 3$. Absolute maximum: $f(-3) = 6$, No absolute minimum.
2. $x = 2, 4$
3. $x = 0$
4. $s = \sqrt{6}$
5. $x = -\dfrac{1}{3},\ -3$
6. $t = 1$
7. No critical numbers
8. $t = -2,\ 2$
9. $x = -\sqrt{5},\ 0,\ \sqrt{5}$
10. θ in $\left\{\dfrac{\pi}{3} + 2n\pi \mid n \text{ an integer}\right\} \bigcup \left\{\dfrac{5\pi}{3} + 2n\pi \mid n \text{ an integer}\right\}$
11. Absolute maximum: $f(-2) = f(2) = 13$, Absolute minimum: $f\left(\pm\dfrac{1}{\sqrt{2}}\right) = \dfrac{3}{4}$
12. Absolute maximum: $f(-1) = 2$, Absolute minimum: $f\left(-1/3\right) = \dfrac{14}{27}$
13. Absolute maximum: $g(0) = 0$, Absolute minimum: $g\left(2/3\right) = -\dfrac{4}{3}\sqrt{\dfrac{2}{3}}$
14. Absolute maximum: $H(8) = -1$, Absolute minimum: $H\left(-1\right) = -4$
15. Absolute maximum: $f(\pi/4) = f(5\pi/4) = \dfrac{1}{2}$, Absolute minimum: $f(3\pi/4) = f(7\pi/4) = -\dfrac{1}{2}$

Exercise 4-2 (page 130)

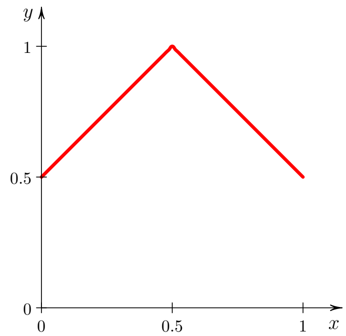
1. f is continuous on $[-1, 4]$, differentiable on $(-1, 4)$, and $f(-1) = f(4) = 2$ so Rolle's Theorem applies. Solving $f'(c) = 0$ shows $c = 7/3$ or $c = -1$, however only $c = 7/3$ is in the open interval $(-1, 4)$.

2. Each of the following graphs of f give one possible counterexample. There is no point in the interval $[0, 1]$ where the function has a horizontal tangent and hence the conclusion to Rolle's theorem is invalid.

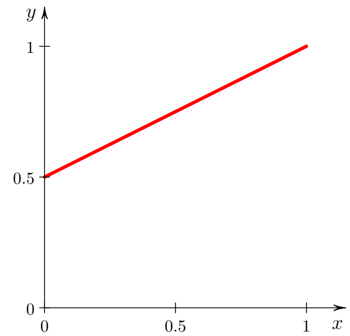
   (a) Continuity fails:



   (b) Differentiability fails:



   (c) $f(0) \neq f(1)$:



3. Use the Mean Value Theorem.

4. $c = 1$ in $(-1, 2)$ has $f'(c) = \dfrac{10 - (-5)}{2 - (-1)} = 5\,.$

Exercise 4-3 (page 135)

1. Decreasing on: $(-\infty, -1)$; Increasing on: $(-1, \infty)$; No relative maxima; Relative minimum: $f(-1) = 0$

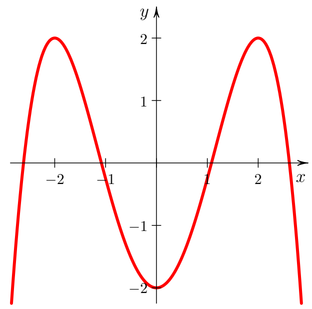
2. Increasing on: $(-\infty, 1) \cup (1, \infty)$; No relative maxima or minima

3. Decreasing on: $\left(0, \dfrac{7\pi}{6}\right) \cup \left(\dfrac{11\pi}{6}, 2\pi\right)$; Increasing on: $\left(\dfrac{7\pi}{6}, \dfrac{11\pi}{6}\right)$;
   Relative maximum: $f\left(\dfrac{11\pi}{6}\right) = 2\sqrt{3} - \dfrac{11\pi}{3}$; Relative minimum: $f\left(\dfrac{7\pi}{6}\right) = -2\sqrt{3} - \dfrac{7\pi}{3}$

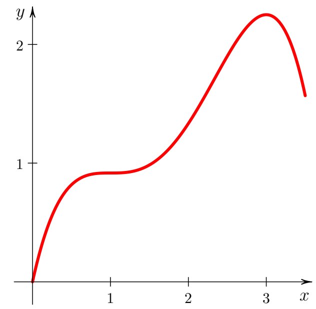
4. Decreasing on: $(-\infty, 0)$; Increasing on: $(0, \infty)$; Relative minimum: $f(0) = 0$

5. Decreasing on: $(-\infty, 0)$; Increasing on: $(0, \infty)$; Relative maximum: $f(0) = 2$. Note that the First Derivative Test cannot be applied here because $f(x)$ is discontinuous at $x = 0$. One must return to the definition of relative maximum to evaluate the critical number $x = 0\,.$

6.



7.



Note that your graphs should be equivalent up to vertical shift by a constant.

Exercise 4-4 (page 144)

1. Notice that $f''(x) = 24x^2$ and $24x^2$ is positive for all values of x except 0. The only potential inflection point would therefore occur at $x = 0$ but the concavity is positive on both sides of $x = 0$ and hence does not change at that value.

2. Notice that $f''(x) = \frac{2}{x^3}$ which is negative for $x < 0$ and positive for $x > 0$. Therefore the concavity does, in fact, change at $x = 0$. However the function is not defined at 0 so there is no point on the curve there (it is a vertical asymptote) and hence no inflection point exists.

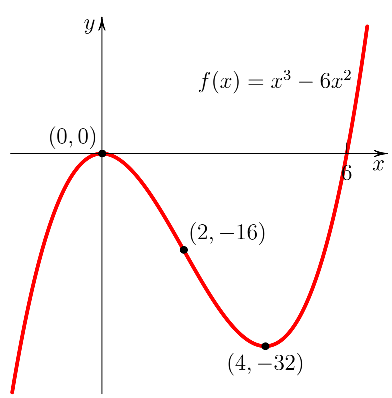
3. Notice that $f'(x) = 5x^2(x-3)(x+3)$, $f''(x) = 10x\left(\sqrt{2}x-3\right)\left(\sqrt{2}x+3\right)$. Relative maximum: $f(-3) = 163$; Relative minimum: $f(3) = -161$; Inflection points: $\left(-\frac{3}{\sqrt{2}}, -\frac{243}{2^{\frac{5}{2}}} + \frac{405}{2^{\frac{3}{2}}} + 1\right)$, $(0,1)$, $\left(\frac{3}{\sqrt{2}}, \frac{243}{2^{\frac{5}{2}}} - \frac{405}{2^{\frac{3}{2}}} + 1\right)$; Concave upward on: $\left(-3/\sqrt{2}, 0\right) \cup \left(3/\sqrt{2}, \infty\right)$; Concave downward on: $\left(-\infty, -3/\sqrt{2}\right) \cup \left(0, 3/\sqrt{2}\right)$;

4. Relative maximum: $f(-2) = 17$; Relative mininum: $f(2) = -15$

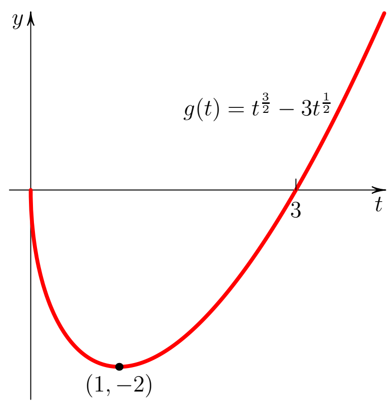
5. Relative maximum at $f\left(-\frac{\pi}{2}\right) = 3$; Relative minimum: $f\left(\frac{\pi}{2}\right) = -5$;

6. No since $f''(x) = 12\cos^2 x \sin^2 x - 4\sin^4 x$ vanishes at $x = 0$ so the test is inconclusive. The first derivative test shows that $x = 0$ is a the location of a local minimum of f.

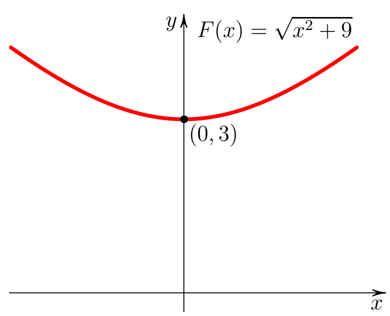
7. $D = \mathbb{R} = (-\infty, \infty)$; x-int= 0, 6; y-int= 0; Increasing on: $(-\infty, 0) \cup (4, \infty)$; Decreasing on: $(0, 4)$; Relative maximum: $f(0) = 0$; Relative minimum: $f(4) = -32$; Concave upward on: $(2, \infty)$; Concave downward on: $(-\infty, 2)$; Inflection point: $(2, -16)$; Graph:



8. $D = [0, \infty)$; t-int$=$ 0, 3; y-int$=$ 0; Increasing on: $(1, \infty)$; Decreasing on: $(0, 1)$; No relative maxima; Relative minimum: $g(1) = -2$; Concave upward on: $(0, \infty)$; Not concave downward anywhere; No inflection points; Graph:



9. $D = \mathbb{R} = (-\infty, \infty)$; No x-int; y-int$=$ 3; Increasing on: $(0, \infty)$; Decreasing on: $(-\infty, 0)$; No relative maxima; Relative minimum: $F(0) = 3$; Concave upward on: $(-\infty, \infty)$; Not concave downward anywhere; No inflection points; Graph:



Exercise 4-5 (page 150)

1. 0
2. ∞
3. $-\dfrac{1}{6}$
4. 2
5. $-\dfrac{3}{2}$
6. $\dfrac{1}{3}$
7. ∞
8. 0
9. 1
10. $-\sqrt{2}$
11. 4
12. 2
13. 3
14. Horizontal asymptote: $y = \dfrac{3}{2}$
15. No horizontal asymptotes
16. Horizontal asymptotes: $y = -1$, $y = 1$
17. Horizontal asymptote: $y = \dfrac{1}{2}$

18. Horizontal asymptote: $y = 0$ (Hint: The Squeeze Theorem, generalized to a limit at infinity, can be used here to evaluate the limits.)

19. Horizontal asymptote: $y = 5$

20. Horizontal asymptote: $y = 1$

21. Horizontal asymptotes: $y = -\dfrac{1}{2}$, $y = \dfrac{1}{2}$

Exercise 4-6 (page 152)

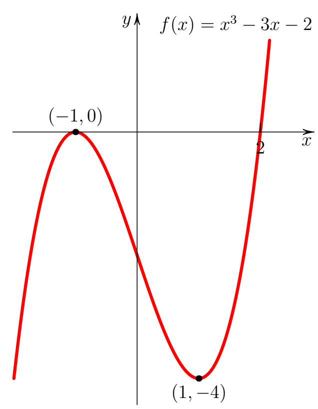
1. Slant asymptote: $y = 3x - 10$

2. No slant asymptotes

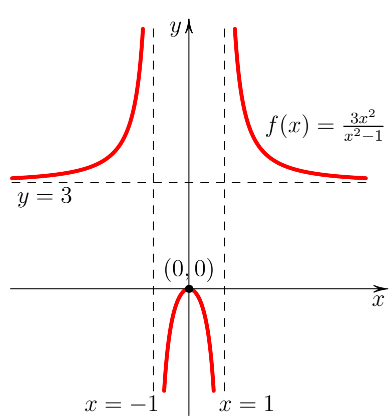
3. Slant asymptote: $y = x - 2$

Exercise 4-7 (page 157)

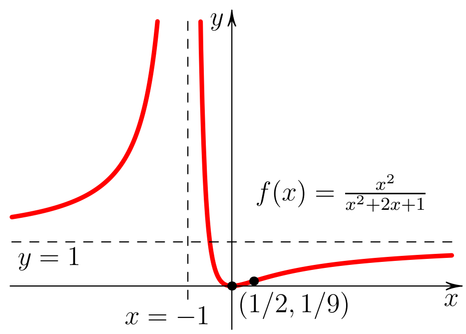
1. Notice $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$ and $f''(x) = 6x$. $D = \mathbb{R} = (-\infty, \infty)$; x-int$= -1,\ 2$; y-int$= -2$; No asymptotes; No symmetry; Increasing on: $(-\infty, -1) \cup (1, \infty)$; Decreasing on: $(-1, 1)$; Relative maxima: $f(-1) = 0$; Relative minimum: $f(1) = -4$; Concave upward on: $(0, \infty)$; Concave downward on $(-\infty, 0)$; Inflection point: $(0, -2)$; Graph:



2. Notice that the first and second derivatives simplify to $y'(x) = -\dfrac{6x}{(x^2-1)^2}$ and $y'' = \dfrac{6(3x^2+1)}{(x^2-1)^3}$. $D = \mathbb{R} - \{-1, 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$; x-int$= 0$; y-int$= 0$; Horizontal asymptote: $y = 3$; Vertical asymptotes: $x = -1$, $x = 1$; Symmetric about the y-axis; Increasing on: $(-\infty, -1) \cup (-1, 0)$; Decreasing on: $(0, 1) \cup (1, \infty)$; Relative maxima: $f(0) = 0$; No relative minimum; Concave upward on: $(-\infty, -1) \cup (1, \infty)$; Concave downward on $(-1, 1)$; No inflection points; Graph:



3. Notice that the derivatives simplify to $f'(x) = \dfrac{2x}{(x+1)^3}$ and $f''(x) = \dfrac{2-4x}{(x+1)^4}$. $D = \mathbb{R} - \{-1\} = (-\infty, -1) \cup (-1, \infty)$; x-int$=0$; y-int$=0$; Horizontal asymptote: $y = 1$; Vertical asymptote: $x = -1$; No symmetry; Increasing on: $(-\infty, -1) \cup (0, \infty)$; Decreasing on: $(-1, 0)$; No relative maxima; Relative minimum: $f(0) = 0$; Concave upward on: $(-\infty, -1) \cup (-1, 1/2)$; Concave downward on $(1/2, \infty)$; Inflection point: $(1/2, 1/9)$; Graph:



Exercise 4-8 (page 162)

1. Base length$=\frac{5}{2}$ m, Height$=\frac{5}{2}$ m, Area$=\frac{25}{8}$ m^2

2. $(3/5, 16/5)$. For part (a) note it is easier to minimize the distance-squared than the distance. For part (b) the line perpendicular is $y = -\frac{1}{2}(x-3) + 2 = -\frac{1}{2}x + \frac{1}{2}$. To find the intersection of this and the original line we solve the two equations simultaneously since the point of interest must lie on both lines.

3. First number=10, Second number=5

4. Radius$=\sqrt[3]{\dfrac{5}{\pi}} \approx 1.17$ cm, Height$=2\sqrt[3]{\dfrac{5}{\pi}} \approx 2.34$ cm

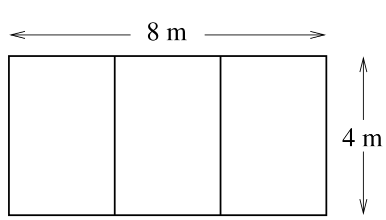
5. (a) $x = \dfrac{2}{3\sqrt{7}} \approx 0.252$ km

   (b) $x = \dfrac{2}{\sqrt{15}} \approx 0.516$ km

   (c) $x = \dfrac{b}{\sqrt{\left(\frac{w}{v}\right)^2 - 1}}$

   (d) Although the total time taken t does depend on a the value for x does not. As expected, it does not matter how far upstream you start, you would still turn off at the same location. That said, a does play a role in the solution because valid values of x must lie in the interval $[0, a]$. If a had been smaller than the solution for x, say in part (a) had the starting distance been 0.2 km, then the critical number would no longer be in the interval and that solution would be invalid. One would consider the endpoints of the interval to see which was optimal.

6. $x = 40$ m, $h = 20$ m

7. 

8. $10\left(2-\sqrt{2}\right) \approx 5.9$ km

9. $x = 100$ m, $y = \dfrac{200}{\pi}$ m

10. $r = 4$ cm, $\theta = 2$ rad $\approx 115°$, $A = 16$ cm^2

Chapter 4 Review Exercises (page 165)

1. $x = -1$, $x = -3$

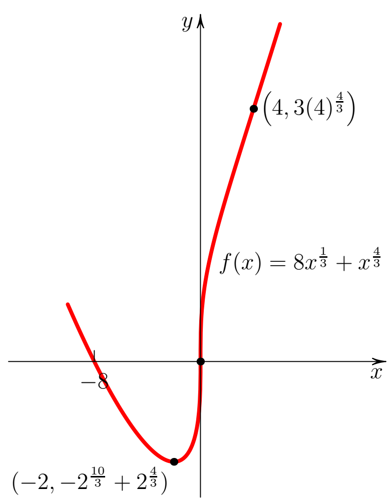
2. $t = 0$, $t = 3$

3. θ in $\left\{\dfrac{\pi}{6} + 2n\pi\right\} \cup \left\{\dfrac{5\pi}{6} + 2n\pi\right\} \cup \left\{\dfrac{\pi}{2} + 2n\pi\right\} \cup \left\{\dfrac{3\pi}{2} + 2n\pi\right\}$ (n an integer)

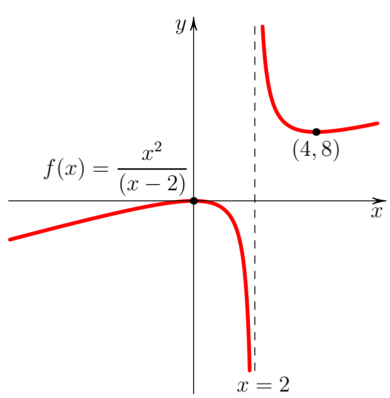
4. Absolute maximum: $f(1) = \dfrac{1}{17}$, Absolute mininimum: $f(-1) = -\dfrac{1}{17}$

5. Absolute maximum: $g(1) = \sqrt{7}$, Absolute minimum: $g(0) = 0$

6. $D = \mathbb{R} = (-\infty, \infty)$; x-int=0, -8; y-int=0; No asymptotes; Increasing on: $(-2, \infty)$; Decreasing on: $(-\infty, -2)$; No relative maxima; Relative mininimum: $f(-2) = -2^{\frac{10}{3}} + 2^{\frac{4}{3}}$; Concave upward on $(-\infty, 0) \cup (4, \infty)$; Concave downward on $(0, 4)$; Inflection points: $(0, 0)$, $\left(4, 3(4)^{\frac{4}{3}}\right)$; Graph:



7. $D = \mathbb{R} - \{2\} = (-\infty, 2) \cup (2, \infty)$; x-int=0; y-int=0; Vertical asymptote: $x = 2$; Increasing on: $(-\infty, 0) \cup (4, \infty)$; Decreasing on $(0, 2) \cup (2, \infty)$; Relative maximum: $f(0) = 0$; Relative minimum: $f(4) = 8$, , Concave upward on $(2, \infty)$; Concave downward on $(-\infty, 2)$; No inflection points; Graph:



8. 0
9. 5
10. $-\dfrac{\sqrt{5}}{2}$
11. $-\infty$
12. Vertical asymptote: $x = 2$, Horizontal asymptote: $y = 2$
13. Vertical asymptote: $t = -\dfrac{3}{2}$, Horizontal asymptotes: $y = -1$, $y = 1$
14. Vertical length in diagram is 30 m and horizontal length is $\dfrac{100}{3}$ m; lot area=2560 m^2
15. Brick side is 10 m and exclusively fence side is 25 m
16. Maximal area occurs when Width=Circle diameter=$\dfrac{20}{4+\pi}$ m and Height=$\dfrac{10}{4+\pi}$ m (i.e. half the width).

Exercise 5-1 (page 171)

1. Both F_1 and F_2 differentiate to x^3. As this problem suggests, any two antiderivatives of a function differ at most by a constant.
2. $F(x) = x^3 - \dfrac{5}{2}x^2 + 6x + C$
3. $F(x) = \dfrac{1}{2}x^2 - \dfrac{4}{x} + C$
4. $G(t) = \dfrac{2}{3}t^{\frac{3}{2}} + 4t^{\frac{1}{2}} + C$
5. $H(x) = \dfrac{3}{5}x^{\frac{5}{3}} - \dfrac{4}{7}x^7 + \pi x + C$
6. $F(\theta) = 2\sin\theta + \cos\theta + \tan\theta + C$
7. $f(x) = \dfrac{1}{10}x^5 - \dfrac{5}{3}x^3 + \dfrac{3}{2}x^2 + Cx + D$
8. $f(t) = \dfrac{4}{15}t^{\frac{5}{2}} + t^3 - \dfrac{5}{3}t + \dfrac{7}{5}$
9. $f(\theta) = -3\sin\theta - \cos\theta + \dfrac{5}{2}\theta^2 + 2\theta + 4$
10. If $f'''(x) = 0$, then $f''(x) = C$ where C is a real constant. If C is non-zero then f has the same concavity everywhere, while if $C = 0$ then $f''(x) = 0$ implies $f(x) = Dx + E$ so f is linear and hence has no point of inflection.

Exercise 5-2 (page 176)

1. $\dfrac{41}{4}$
2. 60
3. $\dfrac{1}{n^3}\left(\dfrac{2n^3+3n^2+n}{6} + n\right) = \dfrac{2n^2+3n+7}{6n^2}$
4. $\dfrac{2n^3+3n^2+n}{6} - \dfrac{3n^2+3n}{2} = \dfrac{n^3-3n^2-4n}{3}$

Exercise 5-3 (page 182)

1. (a) $S_n = \frac{3}{n}\sum_{i=1}^{n} f\left(1+\frac{3i}{n}\right) = \sum_{i=1}^{n}\left(\frac{9}{n}+\frac{27i}{n^2}+\frac{54i^2}{n^3}\right) = \frac{81}{2}+\frac{81}{2n}+\frac{9}{n^2}$

   (b) $A = \lim_{n\to\infty} S_n = \frac{81}{2}$

Exercise 5-4 (page 187)

1. (a) i. -6
      ii. 4
   (b) i. Increasing on $(-6,-5)\cup(0,5)$
      ii. Decreasing on $(-5,0)\cup(5,6)$

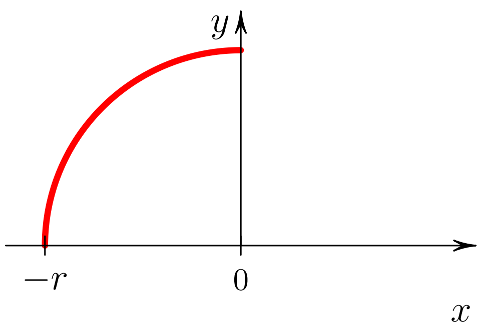
2. 6

3. 30

4. -3

5. $\frac{7}{2}$

6. Notice that the area can be viewed as the quarter of a circle:



   Then the integral is $\frac{1}{4}\left(\pi r^2\right)$. (Since the area is above the x-axis the integral is positive.)

7. $\int_3^9 f(x)\,dx$

8. $S_n = \frac{3}{n}\left(\frac{27n^4+54n^3+27n^2}{4n^3}+n\right) = \frac{93n^2+162n+81}{4n^2}$
   and so $\int_0^3 \left(x^3+1\right)dx = \lim_{n\to\infty} S_n = \frac{93}{4}$.

9. $S_n = b^3\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)$ and so $\int_0^b x^2\,dx = \lim_{n\to\infty} S_n = \frac{b^3}{3}$.

Exercise 5-5 (page 191)

1. $\frac{dF}{dx} = \sqrt{x^3+2x+1}$

2. $h'(x) = 4x^3\sqrt{x^{12}+2x^4+1}$

3. $g'(x) = -\left[\cos\left(x^3\right)\right]$

4. Noting that $H(x) = \int_{2x}^{0} \sqrt[3]{t^3+1}\,dt + \int_0^{3x} \sqrt[3]{t^3+1}\,dt = -\int_0^{2x} \sqrt[3]{t^3+1}\,dt + \int_0^{3x} \sqrt[3]{t^3+1}\,dt$, one gets $H'(x) = -2\sqrt[3]{8x^3+1}+3\sqrt[3]{27x^3+1}$

5. $f'(x) = \frac{2}{\sqrt{\pi}} e^{-(x^3)^2} \left(3x^2\right) = \frac{6x^2}{\sqrt{\pi}} e^{-x^6}$ (Use the Chain Rule and the Fundamental Theorem of Calculus.)

6. $(x, y) = \left(\frac{3}{2} \text{ km}, \frac{7}{16} \text{ km}\right)$

Exercise 5-6 (page 193)

1. $\frac{44}{3}$

2. $-\frac{14}{3}$

3. 2

4. $\frac{7}{6}$

5. 0

6. $\frac{13}{2}$

7. The Fundamental Theorem of Calculus is not applicable here because the integrand $\frac{1}{x^2}$ is discontinuous on $[-1, 1]$. The area under the curve (i.e. the integral) in fact diverges to $+\infty$. To show that requires a consideration of improper integrals.

Exercise 5-7 (page 195)

1. The Fundamental Theorem of Calculus shows that the definite integral $\int_a^b f(x)\,dx$ is, assuming the conditions of the theorem are met, intimately connected to the antiderivative of f by the relation $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is an antiderivative of f. Thus in many cases finding a definite integral is a two-step process where first one finds an antiderivative of f and then secondly takes the difference of that function evaluated at the limits of integration. It is natural, therefore, to generally write the answer to the first step, namely the antiderivative of f, symbolically as $\int f(x)\,dx$. (The notation is further convenient because it embeds the function we are antidifferentiating directly in the symbol in the same way we write $\frac{df}{dx}$ abstractly for the derivative of f.)

2. $\frac{x^4}{4} - \frac{3x^5}{5} - 6x + C$

3. $4x^{\frac{1}{2}} + \frac{2}{3}x^{\frac{3}{2}} + C$

4. $-\csc\theta + C$

5. $\int \left(\tan^2 x + 1\right)\,dx = \int \sec^2 x\,dx = \tan x + C$

6. The equation states that the derivative of y is $x^2 + 9$ so y must be the general form of the antiderivative of $x^2 + 9$ which is just its indefinite integral:

   $y = \int \left(x^2 + 9\right)\,dx = \frac{1}{3}x^3 + 9x + C$

   The general solution for any differential equation of the form $y' = f(x)$ is similarly $y = \int f(x)\,dx$.

Exercise 5-8 (page 203)

1. $-\dfrac{1}{12}\left(x^3+3x^2+4\right)^{-4}+C$
2. $\dfrac{1}{3}\left(5x^2+2x\right)^{\frac{3}{2}}+C$
3. $2\sin(\sqrt{t})+\frac{1}{4}t^4+C$
4. $-\dfrac{2}{3}(3-\sin\theta)^{\frac{3}{2}}+C$
5. $\dfrac{1}{40}(4x+1)^{\frac{5}{2}}-\dfrac{1}{24}(4x+1)^{\frac{3}{2}}+C$
6. $\dfrac{1}{2}\tan\left(2x-\dfrac{\pi}{3}\right)+C$
7. Using $u=x^4+9$, integral is $=\displaystyle\int_9^{25}u^{\frac{1}{2}}\frac{du}{4}=\frac{49}{3}$
8. Using $u=\tan\theta$, integral is $=\displaystyle\int_0^1 u^4\,du=\frac{1}{5}$
9. Breaking the integral into three separate integrals (one per term) and using $u=1-x$ on the last one that integral equals $=\int_1^0 u^5(-du)=\frac{1}{6}$. Combining this with the definite integral of the first two terms gives the final answer $\frac{5}{3}$. Alternatively, one can find the indefinite integral of the third term (i.e. substitute back to x) to get the antiderivative of the entire integrand as $x+\frac{1}{2}x^2-\frac{1}{6}(1-x)^6+C$ and evaluate that at the original limits of x.
10. Using $u=2x^2+1$, integral is $=\displaystyle\int_3^{19}u^{-2}\frac{du}{4}=\frac{4}{57}$
11. Using $u=\cos t$, integral is $=\displaystyle\int_{\frac{1}{\sqrt{2}}}^{\frac{1}{2}}u^{-\frac{2}{3}}(-du)=3\left(\frac{1}{\sqrt[6]{2}}-\frac{1}{\sqrt[3]{2}}\right)$
12. Using $u=\dfrac{2\pi t}{T}$, integral is $=\displaystyle\int_0^{\frac{\pi}{3}}\cos(u)\frac{T\,du}{2\pi}=\frac{\sqrt{3}\,T}{4\pi}$

Exercise 5-9 (page 205)

1. $\dfrac{1}{4}x^4+\dfrac{2}{3}x^{\frac{3}{2}}+\dfrac{1}{x}+5x+C$
2. $\dfrac{1}{7}x^7+x^4+4x+C$
3. $\dfrac{8}{5}x^{\frac{5}{2}}+2x^2+\dfrac{2}{3}x^{\frac{3}{2}}+C$
4. $\sin\theta+\tan\theta+C$
5. $\dfrac{1}{6}\left(x^3+4\right)^6+C$
6. $\dfrac{1}{11}(\sin\theta+3)^{11}+C$
7. $\dfrac{6}{7}\left(\sqrt{t}+7\right)^{\frac{7}{3}}+C$

8. $\dfrac{16806}{10}$
9. $\dfrac{5}{216}$
10. $\dfrac{1}{2}$
11. 0 Note that since sine and tangent are odd, the integrand itself is odd. Since the limits are $\pm a$, the integral vanishes.

Exercise 5-10 (page 211)

1. $\dfrac{39}{2}$ units^2
2. 13 units^2
3. $\dfrac{32}{3}$ unit^2
4. $\dfrac{32}{3}$ unit^2
5. $\dfrac{1}{6}$ unit^2
6. 3 unit^2
7. $\dfrac{16}{3}$ units^2

Exercise 5-11 (page 213)

1. (a) $\displaystyle\int_0^1 v(t)\,dt = \frac{2}{\pi}$ cm
   (b) $\displaystyle\int_1^2 v(t)\,dt = -\frac{2}{\pi}$ cm
   (c) 0 cm (The particle is back where it started after 2 seconds.)
2. $\dfrac{8750}{9}$ gigalitres

Exercise 5-12 (page 218)

2. $y = 3\cos(2t) + \dfrac{5}{2}\sin(2t)$
3. $y = \dfrac{2}{3}x^{\frac{3}{2}} - \csc x + C$
4. $x(t) = \dfrac{1}{168}(2t+3)^7 + Ct + D$
5. $y = x^2 - \cos x + 2$

Chapter 5 Review Exercises (page 219)

1. $F(x) = \frac{5}{8}x^{8/5} + 8x^{1/2} + \frac{1}{4}x^4 + 10x + C$
2. $G(x) = -x^{-1/2} - \frac{5}{2x} - \frac{1}{4x^2} + C$
3. $F(\theta) = -3\cos\theta + 5\sin\theta + \frac{1}{4}\theta^4 + \theta + C$
4. $G(\theta) = 2\sec\theta - 2\sin\theta + \tan\theta + C$
5. $f(x) = \frac{4}{15}x^{5/2} + \frac{1}{12}x^4 - 3x^2 + Cx + D$
6. $f(t) = \frac{9}{2}t^{8/3} - \frac{1}{2}t^3 - \frac{5}{2}t^2 - \frac{13}{2}t + 6$
7. $f(\theta) = -5\sin\theta + 4\cos\theta + 5\theta^2 + 7\theta - 17$
8. $\frac{1}{45}\left(x^5+3\right)^9 + C$
9. $\frac{1}{12}\left[\tan(2\theta)+1\right]^6 + C$
10. $\frac{15}{8}\left(t^{2/5}-4\right)^{4/3} + C$
11. $-\frac{96}{7}$
12. $\frac{1}{15}\left(\frac{\sqrt{2}}{8}+1\right)$
13. $\frac{\pi}{6}$
14. $\frac{32}{3}$
15. $\frac{27}{20}$
16. $\frac{64}{3}$
17. $y = -\frac{3}{x} + \frac{1}{4}x^4 + C$
18. $y = 2\sin x + 3$

Exercise A-1 (page 227)

1. $(-\infty,-1)\cup(7,\infty) = \{x\in\mathbb{R}\,|\,x<-1 \text{ or } x>7\}$
2. $(-3,-1)\cup[1/2,\infty) = \{x\in\mathbb{R}\,|\,-3<x<-1 \text{ or } x\geq 1/2\}$
3. $(2,5)\cup(7,\infty) = \{x\in\mathbb{R}\,|\,2<x<5 \text{ or } x>7\}$

Exercise B-1 (page 235)

1. $\left\{\frac{\pi}{6} + 2n\pi \mid n \text{ an integer}\right\} \cup \left\{\frac{11\pi}{6} + 2n\pi \mid n \text{ an integer}\right\}$
2. $\left\{\frac{2\pi}{3} + 2n\pi \mid n \text{ an integer}\right\} \cup \left\{\frac{4\pi}{3} + 2n\pi \mid n \text{ an integer}\right\}$
3. $\left\{\frac{\pi}{6} + \frac{n\pi}{2} \mid n \text{ an integer}\right\} \cup \left\{\frac{\pi}{3} + \frac{n\pi}{2} \mid n \text{ an integer}\right\}$
   Note that in $[0, 2\pi)$ there are eight angle solutions, namely $\left\{\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}\right\}$.
4. $\left\{\frac{\pi}{4} + 2n\pi \mid n \in \mathbb{Z}\right\} \cup \left\{\frac{5\pi}{4} + 2n\pi \mid n \in \mathbb{Z}\right\} \cup \{0 + 2n\pi \mid n \in \mathbb{Z}\} \cup \{\pi + 2n\pi \mid n \in \mathbb{Z}\}$
   $= \left\{\frac{\pi}{4} + n\pi \mid n \in \mathbb{Z}\right\} \cup \{n\pi \mid n \in \mathbb{Z}\}$

Differentiation

Definition: For a given function $f(x)$ the **derivative function** is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \ .$$

Table of Derivatives

1. $\dfrac{d}{dx}(c) = 0$
2. $\dfrac{d}{dx}(x^n) = nx^{n-1}$ (**Power Rule**)
3. $\dfrac{d}{dx}(\sin x) = \cos x$
4. $\dfrac{d}{dx}(\cos x) = -\sin x$
5. $\dfrac{d}{dx}(\tan x) = \sec^2 x$
6. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$
7. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
8. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$
9. $\dfrac{d}{dx}(cf) = c\dfrac{df}{dx}$
10. $\dfrac{d}{dx}(f \pm g) = \dfrac{df}{dx} \pm \dfrac{dg}{dx}$
11. $\dfrac{d}{dx}(fg) = \dfrac{df}{dx}g + f\dfrac{dg}{dx}$ (**Product Rule**)
12. $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$ (**Quotient Rule**)
13. $\dfrac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$ (**General Power Rule**)
14. $\dfrac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$ (**Chain Rule**)

Here c and n are constants, f and g are functions, and primes (f', g') denote differentiation.

Integration

The Fundamental Theorem of Calculus (Antiderivative Form):

If f is continuous on $[a, b]$ and F is any antiderivative of f (so $F' = f$) then

$$\int_a^b f(x)\,dx = F(b) - F(a)\ .$$

Definition: If $F(x)$ is an antiderivative of f, so $F'(x) = f(x)$, then the **indefinite integral** of $f(x)$ is

$$\int f(x)\,dx = F(x) + C$$

Table of Indefinite Integrals

1. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \qquad (n \neq -1)$
2. $\displaystyle\int \cos x\,dx = \sin x + C$
3. $\displaystyle\int \sin x\,dx = -\cos x + C$
4. $\displaystyle\int \sec^2 x\,dx = \tan x + C$
5. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$
6. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$
7. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$
8. $\displaystyle\int cf(x)\,dx = c\int f(x)\,dx$
9. $\displaystyle\int [f(x) \pm g(x)]\ dx = \int f(x)\,dx \pm \int g(x)\,dx$

Substitution Rule (Indefinite Integrals): Suppose $u = g(x)$ is a differentiable function whose range of values is an interval I upon which a further function f is continuous, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du\ ,$$

where $du = g'(x)dx$ and the right hand integral is to be evaluated at $u = g(x)$ after integration.

Substitution Rule (Definite Integrals): Suppose $u = g(x)$ is a differentiable function whose derivative g' is continuous on $[a, b]$ and a further function f is continuous on the range of $u = g(x)$ (evaluated on $[a, b]$), then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du\ .$$

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